# Minimal Enclosings of Triple Systems I: Adding One Point

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**Abstract:** We solve the problem of existence of minimal enclosings for triple systems with $1 \le \lambda \le 6$ and any v, i.e., an inclusion of BIBD(v, 3, $\lambda$) into BIBD(v+1, 3, $\lambda$+m) for minimal positive m. A new necessary general condition is derived and some general results are obtained for larger $\lambda$ values.

*Key Words*: enclosings, embeddings, BIBD, Steiner System, PBIBD, GDD, resolvable.

## 1. Introduction

In the study of combinatorial designs, as in any area of algebraic or combinatorial structures, one of the most basic questions is this: under what circumstances can a given design or near design be enclosed or embedded in some larger design. This problem is strongly related to that of whether a design has contained within it a proper subdesign or is contained in a natural way in some larger design, or whether it can be completed in some special way. Curiously, although all these topics have been studied, surprisingly little is known about the topic that seems to be the most basic of them and subject of the present paper, enclosings of designs. Before considering this further, we need to discuss terminology and background information.

A balanced incomplete block design, or BIBD(v, b, r, k, λ), is a collection B of b subsets or blocks of a set V of order v such that all blocks have size k, each element appears in r blocks, and each pair of elements appears together in λ blocks. The conditions imply that $bk = vr$ and $\lambda(v-1) = r(k-1)$, and we usually write BIBD(v, k, λ) for short. When $k = 3$, the design is often called a triple system, a TS(λ, 3; v) or TS(λ, v). We use the more generic BIBD since we hope the results here will lead to further research on similar questions for other values of k. We refer the reader to [8, 14, 16] for background about BIBD's or triple systems.

A design injection of X = BIBD(v, $b_1$, $r_1$, k, λ) into Y = BIBD(v+s, $b_2$, $r_2$, k, λ+m) is a mapping φ such that, φ is a one-to-one map from $V_1$ to $V_2$, and for each block $B_i$ of X, $\varphi(B_i)$ is a block of Y. The injection is an *embedding* if $m = 0$ and an *enclosing* if $m > 0$. We will always regard the injection as the inclusion map. Therefore, the enclosing design Y will always be based on the points of X and some s new points.

In this paper, except in Section 8, k will be 3 and s will be 1. An enclosing of X = BIBD(v, 3, λ) into Y = BIBD(v+s, 3, λ+m) is *minimal* if $s = 1$ and no design BIBD(v+1, 3, λ+n) exists with $0 < n < m$. There is a large body of literature on embeddings, and we refer the reader to [5, 7, 10, 11, 13]. However, enclosings, the subject of this note, have been studied less extensively, and the reader is referred to papers by (in various combinations) Bigelow, Colburn, Hamm, and Rosa [1, 2, 4, 6] mostly for $\lambda = 1$. Much of this earlier work has been accumulated and presented in [8].

Simple designs have no repeated blocks, and the enclosings of [1, 2] are faithful, i.e. each new block of Y has at least one new point. These are both severe restrictions and we do not consider them here. We are further motivated by the comments in [8], p. 155-156: "Enclosing of partial systems have not been seriously studied; in fact, as we see next, even enclosings of triple systems themselves have not been determined. ... Both the enclosing and the faithful enclosing problems appear to be far from a solution at this point. The situation is worse for simple enclosings." In this paper we begin the systematic general study of enclosings of designs.

We point out that Bigelow and Colbourn [1,2] proved that a BIBD(v, 3, λ) can be faithfully enclosed in a triple system BIBD(w, 3, μ) whenever $w \geq 2v + 1$ and $\mu \equiv 0 \pmod{\gcd(w-2, 6)}$. Although this is a wide ranging general result, in view of the necessary condition which we derive in Section 8 and in view of its implication that there is a range of s-values for which no enclosing is possible, it is important to examine *both* ends of this s-interval. To that end, in this paper in Sections 2 to 7 we solve a specific problem: we show, *when $1 \leq \lambda \leq 6$, that the necessary conditions are sufficient for the existence for an enclosing of X = BIBD(v, 3, λ) into some Y = BIBD(v+1, 3, λ+m) for a minimal positive m.* We mention that the new necessary condition referred to just above is for arbitrary enclosing of any BIBD(v, k, λ) into BIBD(v+s, k, λ+m), k not necessarily 3.

We will frequently refer to Table 1 below [14, p.50] which provides necessary and sufficient conditions for the existence of a λ-fold triple system of order v. We will always use the variable x to denote the element added to the set V. A *group divisible design, a GDD,* for our purposes here is a design in which the points are partitioned into disjoint sets of equal size called groups. Points within a group will said to have index zero with each other, i.e., they will not appear together in any common block. All pairs of points not in a common group will have the same index λ. We use the superscript notation GDD($g^u$) to denote a GDD with group size g, index one, and block size 3, with v/g = u. Existence questions are addressed in [9] and [15].

Table 1: The λ-v Spectrum of Triple Systems

| | |
|---|---|
| $\lambda \equiv 0 \pmod 6$ | All $v \neq 2$ |
| $\lambda \equiv 1,5 \pmod 6$ | All $v \equiv 1, 3 \pmod 6$ |
| $\lambda \equiv 2, 4 \pmod 6$ | All $v \equiv 0, 1 \pmod 3$ |
| $\lambda \equiv 3 \pmod 6$ | All odd v |

A *partially balanced incomplete block design, a PBIBD,* on v elements is a collection if blocks such that every element say u appears in $\lambda_1$ blocks with some set of $n_1$ elements, called first associates of u, and u appears in $\lambda_2$ blocks with $n_2$ elements, called second associates of u. Note $n_1$ and $n_2$ are constants for all points of the design and $n_1 + n_2 = v - 1$. In the present paper we will always have $\lambda_2 = 0$. Thus, for us, GDD's are a special type of PBIBD in which the condition of being a second associate is transitive and in which the number of second associates is g – 1.

Our first result is a simple construction that will be applied several times.

**Theorem 1.1** *Suppose a PBIBD(v, b, r, 3,* $n_1 = v-(\lambda+2)$, $n_2 = \lambda + 1$, $\lambda_1 = 1$, $\lambda_2 = 0$*) exists. Then a BIBD(v, 3, λ) may be enclosed in a BIBD(v+1, 3, λ+1).*

**Proof:** Suppose X = BIBD(v, 3, λ) and Z is the PBIBD. We identify elements of Z with the elements of X, arbitrarily, so that X and Z are both designs on the set V. The design Y is based on the set V ∪ {x}. We make the blocks of X and of Z the blocks of Y = BIBD(v+1, 3, λ+1), and we will add certain other blocks to complete the design. The blocks so far make the index equal to λ + 1 for elements of Z which are first associates. For any u in V there exist λ + 1 elements $a_1, a_2, \ldots, a_{\lambda+1}$ which are second associates of u. For this u we add the following blocks to Y: {x, u, $a_1$}, {x, u, $a_2$}, ..., {x, u, $a_{\lambda+1}$}. We do this for every u in V taking care not to repeat a block. Clearly the index is $\lambda+1$ for every pair of elements of $V \cup \{x\}$.

When we employ this useful construction, we automatically forfeit faithfulness since none of the blocks of Z contain any new points. On the other hand, in order to apply the theorem, we have constructed many families of PBIBDs with block size 3 which may be new as the first members are not in [3] and are not familiar to the authors as known families.

**Example 1.2** We apply Theorem 1.1 to show X = BIBD(12, 3, 6) can be enclosed into Y = BIBD(13, 3, 7). We first construct Z = PBIBD(12, 3, $n_1$ = 4, $n_2$ = 7, $\lambda_1$ = 1, $\lambda_2$ = 0):

{1, 2, 3}, {4, 5, 6}, {7, 8, 9}, {10, 11, 12},
{1, 5, 9}, {4, 8, 12}, {7, 11, 5}, {2, 6, 10}.

Each point has 7 second associates. The enclosing comes from applying Theorem 1.1. Now suppose A = BIBD(6, 3, 6). Since a BIBD(7, 3, 7) exists, by Table 1, we would expect that there exists a minimal enclosing of A into some B = BIBD(7, 3, 7). However, this is not possible, as we now show. Suppose A is based on the points 1, 2, ..., 6 and B is based on the points 1, 2, ..., 7. As the new index is 7, we must have the points 1 and 7 together in seven blocks. By the pigeon hole principle, some point, say 2, must be in two of these blocks. But 1 and 2 are already together in 6 blocks from A. So the index is 8 for points 1 and 2, a contradiction.

Section 8 explains in a more general way why the presumed enclosing in the example was not possible. Sections 2 to 7 consider $\lambda$ from 1 to 6 and all possible v, and in all these sections we prove the minimal enclosing occurs for the smallest allowable parameters.

## 2. $\lambda = 1$.

With $\lambda = 1$, we need only consider $v \equiv 1, 3 \pmod 6$, according to Table 1. A *parallel class* or *a resolution class* is a set of blocks in a design which contain every point once and only once. A design is *resolvable* if its blocks can be partitioned into parallel classes. The following result is well-known [14].

**Lemma 2.1** *For all $v \equiv 1, 3 \pmod 6$ and $v \neq 9$, there exists a cyclic BIBD(v, 3, 1). For all $v \equiv 3 \pmod 6$, $v \geq 9$, there exists a resolvable BIBD(v, 3, 1) (called a Kirkman triple system).*

**Theorem 2.2** *For every X = BIBD(v, 3, 1) with $v \equiv 3 \pmod 6$, there is a minimal enclosing Y = BIBD(v+1, 3, 2).*

**Proof:** Suppose $v \equiv 3 \pmod 6$, and X = BIBD(v, 3, 1). Let Z = BIBD(v, 3, 1) be a Kirkman triple system. We identify the v elements of Z with the elements of X, and we define a block system Y on the set $V \cup \{x\}$ such that blocks of Y are the blocks of X plus the blocks of Z, with the following modification. For every block {a, b, c} in one of the resolution classes of Z, we replace {a, b, c} with the set of blocks {x, a, b}, {x, a, c}, {x, b, c}. One notes that this does not increase the index for elements a, b, and c, but x now appears with each u in V exactly twice. (We call this expanding the resolution class with new point x.) The result is a BIBD(v+1, 3, 2).

**Theorem 2.3** *Suppose $v \equiv 1 \pmod 6$, and X = BIBD(v, 3, 1) is an arbitrary Steiner triple system. Then there exists a minimal enclosing Y = BIBD(v+1, 3, 6).*

**Proof:** Since $v + 1 \equiv 2 \pmod 6$, there is no smaller index possible than 6 by Table 1. Suppose now that Z is a cyclic Steiner triple system on v elements. We next create the blocks for Y as follows. The blocks of X are blocks for Y, and we use five copies of the blocks of Z for Y with the following modification. Suppose {a, b, c} is one of the difference sets for Z. Replace {a, b, c} with three blocks {x, a, b}, {x, a, c} and {x, b, c}. Similarly replace each block developed from {a, b, c} by 3 new blocks. As in the previous proof, one notes that this does not increase the index between any elements of Z. As {a, b, c} is developed to create Z, element a appears in three blocks. Thus a now appears six times with x in the blocks developed for Y. This shows the index for Y is 6. (We call this construction developing the difference set with new point x.)

The types of constructions in Theorem 2.2 and 2.3 are both well-known.

## 3. $\lambda = 2$.

Since $v \equiv 0, 1 \pmod 3$ in this case by Table 1, there are four subcases to consider ($\lambda$+m = $\Lambda$):

a. $v \equiv 1 \pmod 6$ and $\Lambda = 6$;
b. $v \equiv 3 \pmod 6$ and $\Lambda = 4$;
c. $v \equiv 4 \pmod 6$ and $\Lambda = 3$; and
d. $v \equiv 0 \pmod 6$ and $\Lambda = 3$.

The values for the smallest possible $\Lambda$ just above depend on v+1 (mod 6) and Table 1. For example, consider Case b just above. Since $v + 1 \equiv 1 \pmod 3$, the smallest index greater than 2, from Table 1 again, is $\lambda$+m = 4.

**Theorem 3.1** *Suppose there exists X = BIBD(v, 3, 2) for $v \equiv 1 \pmod 6$. Then there exists a minimal enclosing Y = BIBD(v+1, 3, 6).*

**Proof:** We construct blocks for Y by using one copy of the blocks for X, four copies of a cyclic Z = BIBD(v, 3, 1), and expand new point x with a difference set as in the proof of Theorem 2.3. Since x will appear six times in blocks with each other element, the index is six.

**Theorem 3.2** *Suppose $v \equiv 3 \pmod 6$ and X = BIBD(v, 3, 2). Then there exists a minimal enclosing Y = BIBD(v+1, 3, 4).*
**Proof:** We construct blocks for Y by using the blocks from X and from two copies of a Kirkman triple system Z = BIBD(v, 3, 1). Thus the index is 4 for pairs of elements of V. Now just expand two resolution classes with x.

**Example 3.3:** For the family $v \equiv 6 \pmod{12}$, we have an enclosing for the smallest v, an enclosing for a triple system (6, 3, 2) into a triple system (7, 3, 3). Take the blocks of the (6, 3, 2) together with the following blocks:
{7, i, 4}, {7, i, 5}, {7, i, 6}, for i= 1, 2, 3, and {1, 2, 3}, {4, 5, 6}.

**Theorem 3.4** *We can minimally enclose a triple system BIBD(v, 3, 2) into a triple system BIBD(v+1, 3, 3) when $v \equiv 0, 4 \pmod{12}$.*
**Proof:** If $v \equiv 0, 4 \pmod{12}$, then a group divisible design of block size three, group size four, index 1, exists [12]. The result follows from the Theorem 1.1.

Note that if v is even, then for a triple system (v, 3, 2) to exist, $v \equiv 0, 4$ (mod 6); so the construction for Theorem 3.7 gives only half the necessary enclosings for Case (c) and Case (d).

**Example 3.5.a:** The following case is of interest. Suppose we wish to enclose a triple system X = BIBD(10, 3, 2) into a triple system Y = BIBD(11, 3, 3). We know [3, p.236] that a PBIBD(10, 10, 3, 3, 6, 3, 1, 0) exists. The blocks of Y are the blocks of X together with the blocks of the PBIBD

{1, 2, 3}, {1, 5, 6}, {1, 4, 7}, {2, 5, 8}, {2, 4, 9},
{3, 6, 8}, {3, 4, 10}, {5, 7, 9}, {6, 7, 10}, {8, 9, 10}.

Then construct new blocks with element 11 based on the blocks of the PBIBD as follows:

{11, 1, 8}, {11, 1, 9}, {11, 1, 10}, {11, 2, 6}, {11, 2, 7},
{11, 2, 10}, {11, 3, 5}, {11, 3, 7}, {11, 3, 9}, {11, 4, 5},
{11, 4, 6}, {11, 4, 8}, {11, 5, 10}, {11, 6, 9}, {11, 7, 8}.

Note that if element a has three second associates, say b, c, and d, we will construct new blocks {x, a, b}, {x, a, c}, {x, a, d}. Having $n_2$, the number of second associates, equal 3 is critical to the construction.

**Example 3.5.b:** We next illustrate another enclosing with the same parameters of Example 3.5.a, but the examples are non-isomorphic. We use

{1, 2, 5} as a partial difference set to generate 10 blocks. We observe that the non-zero differences generated by {1, 2, 5}, namely 1, 3, 4, 6, 7, and 9, omit the differences 2, 5, and 8. This means that the element 1 is not in a block with 1+2, 1+5, or 1+8. Also, 2 is not in a block with 2+2, 2+5, or 2+8; and so on. Thus, each element has three second associates. It follows by Theorem 1.1 that a BIBD(10, 3, 2) may be enclosed in a BIBD(11, 3, 3).

It is convenient to restate Theorem 1.1 with parameters needed for this section.

**Lemma 3.6** Suppose $v = 6t$ for some $t \geq 1$ and suppose there exists a partially balanced design with parameters ($v = 6t$, $b = v(v-4)/6$, $r = (v-4)/2$, $k = 3$, $n_1 = v - 4$, $n_2 = 3$, $\lambda_1 = 1$, $\lambda_2 = 0$). *Then we can minimally enclose a triple system (v, 3, 2) into a triple system (v+1, 3, 3).*

**Lemma 3.7** *If a GDD($6^t$) exists (i.e., with group size 6, index 1, block size 3), then there exists a PBIBD with parameters as in Lemma 3.6.*
**Proof:** Suppose G is the GDD. For each group, $G_1, G_2, \ldots, G_t$ we partition the group arbitrarily into two subsets of size 3, $G_{ij}$ for $i = 1, 2, \ldots, t$ and $j = 1, 2$. The blocks of the PBIBD are the blocks of the GDD and the blocks $G_{ij}$ for all i, j. Now each v in $G_{i1}$ is a second associate for the 3 elements in $G_{i2}$, and vice-versa.

**Theorem 3.8** *If $v \equiv 0 \pmod 6$, then there is a minimal enclosing for any BIBD(v, 3, 2) into a BIBD(v+1, 3, 3).*
**Proof:** To enclose BIBD(6, 3, 2) into BIBD(7, 3, 3), see Example 3.3. To enclose BIBD(12, 3, 2) into BIBD(13, 3, 3), add Z = GDD($4^3$). Next, for any group G = {a, b, c, d} in Z, put each pair of points of G in a block with 13. For v > 12, the GDD required for Lemma 3.7 exists [12]. Now the theorem follows by Lemma 3.6.

This theorem completes Case d, above. For $\lambda = 2$, it only remains to deal with $v \equiv 10 \pmod{12}$. The next example is suggestive of a method which was the key element in our proof of this last case.

**Example 3.9** We give an example for v = 22 similar to Example 3.5.b. We create blocks by developing the following difference sets: {1, 2, 5}, {1, 3, 9}, and {1, 8, 18}, mod 22. These 66 blocks give a PBIBD(22, 66, 9, 3, $n_1 = 18$, $n_2 = 3$, $\lambda_1 = 1$, $\lambda_2 = 0$) design. It may be checked that the differences not created by the three difference sets are 9, 11, and 13. Then, for any u in the design, {u+9, u+11, u+13}, calculated mod 22, are second associates for u. This shows we can enclose a BIBD(22, 3, 2) into a BIBD(23, 3, 3) by Theorem 1.1.

**Theorem 3.10** *Suppose $v \equiv 4 \pmod 6$. Then there is a minimal enclosing of $X = BIBD(v, 3, 2)$ into $Y = BIBD(v+1, 3, 3)$.*
**Proof:** To inclose BIBD(4, 3, 2) into BIBD(5, 3, 3), put each pair of points of X in a block with new point x. The rest of case v = 12t + 4 is done in Theorem 3.4. The case v = 10 is in Example 3.5. Now suppose v = 12t + 10 for t ≥ 1. We provide a (partial) difference family for a PBIBD(v, b, r, 3, $n_1 = v-4$, $n_2 = 3$, $\lambda_1 = 1$, $\lambda_2 = 0$):

$\{0, 1, 3t+3\}, \{0, 3, 3t+4\}, \{0, 5, 3t+5\}, \ldots, \{0, 2t+1, 4t+3\}$, and
$\{0, 2, 5t+5\}, \{0, 4, 5t+6\}, \{0, 6, 5t+7\}, \ldots, \{0, 2t, 6t+4\}$.

It is easily checked that three differences are missing: 5t + 4, 6t + 5 (the center point), and 7t + 6. For any u in the design, these determine the second associates of u, as in Example 3.9. Now the result follows by Theorem 1.1.

## 4. λ = 3.

In this case, BIBD(v, 3, 3) exist for v ≡ 1, 3, 5 (mod 6), and the respective minimum index values for an enclosing are 6, 4, and 4, from Table 1.

**Theorem 4.1** *For every $X = BIBD(v, 3, 3)$ with $v \equiv 3 \pmod 6$, there is a minimal enclosing $Y = BIBD(v+1, 3, 4)$.*
**Proof:** Let Z be a Kirkman triple system BIBD(v, 3, 1). The blocks of Y are the blocks of X, the blocks of Z, and we expand new point x with two resolution classes of Z. Clearly Λ = 4.

**Theorem 4.2** *For every $X = BIBD(v, 3, 3)$ with $v \equiv 1 \pmod 6$, there is a minimal enclosing $Y = BIBD(v+1, 3, 6)$.*
**Proof:** The proof uses the same construction as the proof of Theorem 2.3.

**Theorem 4.3** *For every $X = BIBD(v, 3, 3)$ with $v \equiv 5 \pmod 6$, there is a minimal enclosing $Y = BIBD(v+1, 3, 4)$.*
**Proof:** First suppose v = 12t + 11. Then the same difference family from the proof of Theorem 3.10 generates a cyclic PBIBD(12t+11, 3, $n_1 = 12t+6$, $n_2 = 4$, $\lambda_1 = 1$, $\lambda_2 = 0$) – just develop the difference family mod 12t+11 instead of 12t+10. Now, however, there are 4 differences missing, 5t+4, 6t+5 and their complements (mod 12t+11), namely 7t+7 and 6t+6. This case now follows by Theorem 1.1. (For v = 11, use only the first difference set.) Next, to enclose BIBD(5, 3, 3) into BIBD(6, 3, 4), just put new point x in blocks with each pair of points of X. Suppose t > 0 and v = 12t + 5. Then the following difference family,

$\{0, 1, 3t+1\}, \{0, 3, 3t+2\}, \{0, 5, 3t+3\}, \ldots, \{0, 2t-1, 4t\}$,
$\{0, 2, 5t+3\}, \{0, 4, 5t+4\}, \{0, 6, 5t+5\}, \ldots, \{0, 2t, 6t+2\}$,

when developed, produces a cyclic PBIBD(12t+5, 3, $n_1$ = 12t, $n_2$ = 4, $\lambda_1$ = 1, $\lambda_2$ = 0). The four differences missing are 4t + 1, 8t + 4, 5t + 2, and 7t + 3. Again, this case now follows by Theorem 1.1.

## 5. $\lambda = 4$

In this case, v ≡ 0, 1 (mod 3). Set $\lambda$+m = $\Lambda$. The subcases to consider are similar to the $\lambda$ = 2 case:

a. v ≡ 1 (mod 6) and $\Lambda$ = 6;
b. v ≡ 3 (mod 6) and $\Lambda$ = 6;
c. v ≡ 4 (mod 6) and $\Lambda$ = 6; and
d. v ≡ 0 (mod 6) and $\Lambda$ = 5.

The values for the smallest possible $\Lambda$ just above depend on v+1 (mod 6) and Table 1. For example, consider Case b just above. Since v + 1 ≡ 4 ≡ 1 (mod 3), the smallest suitable index greater than 4, from Table 1 again, is $\Lambda$ = 6.

**Theorem 5.1.** *Suppose there exists X = BIBD(v, 3, 4) for v ≡ 1 (mod 6). Then there exists a minimal enclosing Y = BIBD(v+1, 3, 6).*
**Proof:** We construct blocks for Y by using one copy of the blocks for X and two copies of a cyclic Z = BIBD(v, 3, 1). Thus the new index is 6 for any pair of elements of V. As in the proof of Theorem 2.3, we expand new point x with one of the difference sets for one of the copies of Z.

**Theorem 5.2** *Suppose v ≡ 3 (mod 6) and X = BIBD(v, 3, 4). Then there exists a minimal enclosing Y = BIBD(v+1, 3, 6).*
**Proof:** The proof uses the same construction as Theorem 2.2 but with two Kirkman triple systems and 3 resolution classes.

**Theorem 5.3** *Suppose v ≡ 4 (mod 6). Then for any X = BIBD(v, 3, 4) there is a minimal enclosing of X into Y = BIBD(v+1, 3, 6).*
**Proof:** To enclose BIBD(4, 3, 4) into BIBD(5, 3, 6), form two copies of each pair of points of X and put them in blocks with new point x. Suppose v = 12t + 4 for t > 0. In this case a cyclic BIBD(v, 3, 2) exists. Use the construction from Theorem 2.3. Suppose v = 12t + 10 for t ≥ 0. Let Z be the PBIBD(v, 3, 2) from Theorem 3.10. Then the blocks of Y are the blocks of X and the blocks of Z supplemented as follows. For each u in X, with second associates (b, c, d), we add two copies of blocks {x, u, b}, {x, u, c}, and {x, u, d}, without further repeating blocks. Then the new index is 6.

**Proposition 5.4** Suppose X = BIBD(6t, 3, 4). Then there is a minimal enclosing of X into Y = BIBD(6t+1, 3, 5).
**Proof:** To enclose BIBD(6, 3, 4) into BIBD(7, 3, 5), put each pair of points of X in a block with new point x. To enclose BIBD(12, 3, 4) into BIBD(13,

3, 6), add the blocks of Z, a resolvable GDD($4^3$). Next, for each group {a, b, c, d} of Z, add blocks

{a, b, 13}, {a, c, 13}, { a, d, 13}, {b, c, 13}, {b, d, 13}, {c, d, 13}.

Next expand a resolution class of Z with new point 13. For t ≥ 3, there exists a GDD($6^t$). The group-mates of any element u are second associates of u so Theorem 1.1 may be applied for the remaining cases.

## 6. λ = 5

In this case, v ≡ 1, 3 (mod 6).

**Theorem 6.1** *If v ≡ 1 (mod 6), then for every BIBD(v, 3, 5) there is a minimal enclosing BIBD(v+1, 3, 6).*

**Proof:** The construction is just like that in the proof of Theorem 2.3, with one copy of Z.

**Theorem 6.2** *Suppose v ≡ 3 (mod 6) and X = BIBD(v, 3, 5). Then there is a minimal enclosing Y = BIBD(v+1, 3, 6).*

**Proof:** The blocks of Y are those of X and, as in the proof of Theorem 2.2, those of Z, a Kirkman triple system. Note Λ = 6 for elements of V. Since the number of resolution classes is (v-1)/2, there are at least 4 resolution classes for any Z we are considering since v ≥ 9. Therefore, expand new point x with any 3 resolution classes of Z.

## 7. λ = 6

In this case v > 3. Here are the cases with λ+m = Λ:

a. v ≡ 1(mod 6), Λ = 12.
b. v ≡ 2(mod 6), Λ = 7.
c. v ≡ 3 (mod 6), Λ = 8.
d. v ≡ 4 (mod 6), Λ = 9.
e. v ≡ 5 (mod 6), Λ = 8.
f. v ≡ 0 (mod 6), Λ = 7.

**Theorem 7.1** *Suppose X = BIBD(6t+1, 3, 6). Then there exists a minimal enclosing Y = BIBD(6t+2, 3, 12).*

**Proof:** Suppose Z = (6t+1, 3, 1) is a cyclic Steiner triple system. Then the blocks of Y are the blocks of X, the blocks of six copies of Z, and expand new point x with two difference sets.

**Theorem 7.2** *There exists a cyclic PBIBD(6t+2, 3, $n_1$ = 6t − 6, $n_2$ = 7, $\lambda_1$ = 1, $\lambda_2$ = 0) for t > 1.*

**Proof:** We first construct a nearly full set of difference triples {a, b, a+b} where a+b ≤ v/2 = 3t+1. A cyclic PBIBD will occur by replacing each difference triple with a corresponding difference set {0, a, a+b} and expanding the difference set. Each difference set determines the differences $\pm$ a, $\pm$ b, $\pm$ (a+b) evaluated mod v. First suppose t is even and v/2 = 6n+1 for some n. The triples are

{1, 3n, 3n+1}, {3, 3n-1, 3n+2}, ... {2n-1, 2n+1, 4n}, and

{2, 5n, 5n+2}, {4, 5n-1, 5n+3}, ... {2n, 4n+1, 6n+1}.

Note the point 5n+1 is not among the triples. We remove the last triple, {2n, 4n+1, 6n+1}, from our list. The remaining triples are used to form a cyclic PBIBD. The differences not determined by the difference sets are $\pm$ 2n, $\pm$ (4n+1), $\pm$ (6n+1), $\pm$ (5n+1). Since 6n+1 = -(6n+1) evaluated mod v, there are 7 differences missing from the difference sets, and we have $n_2 = 7$ in the PBIBD. This proves the theorem for even t using Theorem 1.1.

Now suppose t is odd, say t = 2j-1 for some j, and v/2 = 6j-2. The difference triples are

{1, 3j-1, 3j}, {3, 3j-2, 3j+1}, ... {2j-1, 2j, 4j-1}, and

{2, 5j-2, 5j}, {4, 5j-3, 5j+1}, ... {2j-2, 4j, 6j-2}.

The point not listed is 5j-1. As in the previous case, we delete the last triple containing v/2 = 6j-2, and the result follows as before.

**Theorem 7.3** *Suppose X = BIBD(6t + 2, 3, 6). Then there exists a minimal enclosing into Y = BIBD(6t+3, 3, 7).*
**Proof:** To enclose BIBD(8, 3, 6) into BIBD(9, 3, 7) just put each pair of points of X with new point x. For t > 1, use Theorem 7.2 and Theorem 1.1.

**Theorem 7.4** *Let v = 6t + 3. There is a minimal enclosing of X = BIBD(v, 3, v-3) into Y = BIBD(v+1, 3, v–1).*
**Proof:** Let R be the replication number for Y. Then R = v(v-1)/2 which is the number of pairs of elements for V. In order to construct Y from X, we just need to add the blocks of Z = BIBD(v, 3, 1) and add block {x, a, b} to X for each pair a,b in V, with no repetition of blocks. Note the index for x is v-1, the number of pairs of V containing each element u of X.

**Theorem 7.5** *Suppose X = BIBD(6t+3, 3, 6). Then X may be enclosed in a minimal Y = BIBD(6t + 4, 3, 8).*
**Proof:** Let $Z_1$ = BIBD(6t + 3, 3, 1) be a cyclic Steiner triple system (we may assume v > 9 since Theorem 7.4 covers the v = 9 case) and let $Z_2$ = BIBD(6t+3, 3, 1) be a Kirkman triple system on the same elements as X. Form blocks for Y using X, $Z_1$, and $Z_2$. This makes the index 8 for all elements of Y except x. For one difference set in $Z_1$, and the blocks developed from it, we expand with new point x. This brings the index up to 6 for x (and leaves the index for elements of X unchanged). In one resolution class of $Z_2$, expand with new point x. Then the index is 8 for all elements of Y.

**Theorem 7.6** *(a) If X = BIBD(6t+4, 3, 6) then there is a minimal enclosing of X into Y = BIBD(6t+5, 3, 9). (b) If X = BIBD(6t+4, 3, 6) then there is an enclosing of X into Y = BIBD(6t+5, 3, 12).*
**Proof:** For part (a), first enclose BIBD(4, 3, 6) into BIBD(5, 3, 9) by taking 3 copies of each pair from (1, 2, 3, 4} and make blocks with 5. To enclose BIBD(10, 3, 6) into BIBD(11, 3, 9), take each pair formed from{1, 2, ... , 10} and make blocks with 11; then add the blocks of any BIBD(10, 3, 2). Now let Z = GDD($4^{3t+1}$) the group design on 12t+4 points, block size 3, index 1, and group size 4. Add the blocks of 3 copies of Z to those of X, and, for each group of 4 elements, take 3 copies of each of the pairs from those elements to make blocks with 6t+5. For v = 12t+10, let Z be the PBIBD from Theorem 3.10. Add the blocks of 3 copies of Z to those of X. Then, for any point u of X, use 3 copies of the pairs {u, 5t+4+u}, {u, 6t+5+u}, and {u, 7t+6+u} and make blocks with 12t + 11.

For part (b), enclose BIBD(4, 3, 6) into BIBD(5, 3, 12) using 4 copies of each pair of points of X in a block with new point x. Next add the blocks of one copy of BIBD(4, 3, 2). To enclose BIBD(10, 3, 6) into BIBD(11, 3, 12), add 3 copies of a cyclic BIBD(4, 3, 2) and expand two difference sets with new point x. Now let t > 0. Suppose v = 12t + 10. Let $Z_1$ denote any BIBD(12t+10, 3, 2), and let $Z_2$ denote the PBIBD on 12t+10 points constructed in the proof of Theorem 3.10. Then the blocks of Y are the blocks of X and the blocks from two copies of $Z_1$ and two copies of $Z_2$. Now the index is 12 for all elements that are first associates in $Z_2$. For element u in X there are three second associates say {b, c, d} in $Z_2$. For each u, add two copies of blocks {x, u, b}, {x, u, c}, and {x, u, d} without further repeating blocks. Now the index is 12 for all elements of X and is six for the element x. Since $Z_2$ is cyclic, use one difference set to augment the set of blocks as in the proof of Theorem 2.3. Now suppose v = 12t + 4. In this case a cyclic BIBD(v, 3, 2), say Z, exists. Then the blocks for Y are the blocks from X and 3 copies of Z. These are supplemented using two of the difference sets for Z as in Theorem 2.3.

**Theorem 7.7** *Suppose X = BIBD(6t+5, 3, 6). Then there is an minimal enclosing of X into Y = BIBD(6t+6, 3, 8).*
**Proof:** To enclose BIBD(5, 3, 6) into BIBD(6, 3, 8), put new point x into 2 blocks with each pair of points of X. The two general cases v = 12t + 5 and v = 12t + 11 are both handled using the PBIBD's developed in the proof of Theorem 4.3. In each case for v, we use the blocks from X, the blocks from two copies of the PBIBD, and two copies each of the 4 blocks indicated as in the proof of Theorem 4.3.

For the last case of this section, v = 6t, for some t > 0. We first complete the question posed by Example 1.2., i.e., what BIBD gives a minimal embedding for a (6, 3, 6) triple system?

**Example 7.8** We recall the v = 6 = λ case is the first in which the minimal parameters do not give the enclosing (Example 1.2). However, there exists a minimal enclosing of A = BIBD(6, 3, 6) into B = BIBD(7, 3, 8). The necessary blocks to create B from A are the following: use the blocks of A and

{7, 1, 2}, {7, 1, 2}, {7, 1, 3}, {7, 1, 3}, {7, 1, 4}, {7, 1, 4}, {7, 1, 5}, {7, 1, 6}, {7, 2, 5}, {7, 2, 5}, {7, 2, 6}, {7, 2, 6}, {7, 2, 3}, {7, 2, 4}, {7, 3, 4}, {7, 3, 5}, {7, 3, 5}, {7, 3, 6}, {7, 3, 6}, {7, 4, 5}, {7, 4, 5}, {7, 4, 6}, {7, 4, 6}, {7, 5, 6}, {1, 5, 6}, {2, 3, 4}.

**Theorem 7.9** Suppose t > 1. Then X = BIBD(6t, 3, 6) can be enclosed into Y = BIBD(6t+1, 3, 7).

**Proof:** In view of Example 1.2 we may assume t > 2. In this case there exists a resolvable Z = GDD($2^{3t}$). Use the blocks of X, of Z, expand point x with 3 resolution classes, and for each group {a, b} add block {x, a, b}.

## 8. A new Necessary General Condition

The purpose of this section is to establish the next theorem, a new general condition for enclosings for arbitrary k. Its consequences are quite far reaching.

**Theorem 8.1** A necessary condition for enclosing X = BIBD(v, k, λ) into Y = BIBD(v+s, k, λ+m) is that

$$s \leq \frac{k-1+2v}{2(k-1)} - \sqrt{\frac{(k-1+2v)^2(\lambda+m)(k-2)-8(k-1)mv(v-1)}{4(\lambda+m)(k-2)(k-1)^2}} \quad \text{or}$$

$$s \geq \frac{k-1+2v}{2(k-1)} + \sqrt{\frac{(k-1+2v)^2(\lambda+m)(k-2)-8(k-1)mv(v-1)}{4(\lambda+m)(k-2)(k-1)^2}}.$$

**Proof:** Suppose that R is the replication number for Y and that {$a_1$, $a_2$, ..., $a_s$} are the points of Y which are not in X. We count the "new" blocks for Y which are not in X. The point $a_1$ will appear in R distinct new blocks. Point $a_2$ must appear in R-(λ+m) new blocks without $a_1$. Point $a_3$ must appear in R-2(λ+m) new blocks (that is, without either $a_1$ or $a_2$; more new blocks are needed if $a_1$, $a_2$, and $a_3$ appear together in the same block), and so on. By adding these it follows that at least sR - (λ+m)s(s-1)/2 new blocks are needed. Now let B and b be the total number of blocks in Y and X, respectively. Since for any BIBD, vr = bk and λ(v-1) = r(k-1), we get:

$$B-b=\frac{(v+s)(\lambda+m)(v+s-1)}{k(k-1)}-\frac{\lambda v(v-1)}{k(k-1)}\geq sR-(\lambda+m)s(s-1)/2.$$

Substitute (λ+m)(v+s-1)/(k-1) for R. If the resulting inequality is solved for s, the theorem follows.

**Corollary 8.2** A necessary condition for enclosing BIBD(v, 3, λ) into BIBD(v+1, 3, λ+m) is that v - 2 ≥ λ/m.
**Proof:** This follows directly from the inequality developed in the proof of Theorem 8.1.

Alternate versions of Corollary 8.2 can be further easily specialized for selected values of s and m and k. The argument in Example 1.2 shows that when s = m = 1, the index λ must be less than the number v of points. The corollary shows that in fact v - 2 ≥ λ is necessary.

A further point may be illustrated by a new example. We have shown in Theorem 7.3 that any B = BIBD(14, 3, 6) may be minimally enclosed into C = BIBD(15, 3, 7). However, it follows from the previous theorem that the next point-enclosing for B, with m = 1 and for a smallest v+s, may possibly occur only for BIBD(27, 3, 7). For k = 3, v = 14, m = 1, we get

$$(s-13)(s-2)\geq 0.$$

The entire range of s strictly between 2 and 13 is inadmissible and s = 2 is not possible by Table 1. This type of gap will occur for any v, and it suggests that the spectrum of enclosings has to be studied at each end. In particular, the importance of considering small values of s is pronounced.

This curious gap was noticed in [1], but to explain fully we specialize the formula in Theorem 8.1 for k = 3:

$$s\leq\frac{1+v}{2}-\frac{\sqrt{(1+v)^2(\lambda+m)^2-4mv(v-1)(\lambda+m)}}{2(\lambda+m)}\quad\text{or}$$

$$s\geq\frac{1+v}{2}+\frac{\sqrt{(1+v)^2(\lambda+m)^2-4mv(v-1)(\lambda+m)}}{2(\lambda+m)}.$$

These can be seen to be exactly equivalent to the formulas in Lemma 1.2 of [1]. The derivation in [1] was for *faithful* enclosings, that is, for enclosings in which each new block has at least one new point. Our derivation here is much simpler, and it is for arbitrary k, not just k = 3. Furthermore, it shows that the condition is a necessary one for all enclosings not just the faithful type. The formula in [1] for k = 3 is also developed again in [8] but before mentioning faithfulness.

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