Finite sets in $\mathbf{R}^n$ given up to translation have reconstruction number three.

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Abstract: We prove that a finite set A of points in the n-dimensional Euclidean space $\mathbf{R}^n$ is uniquely determined up to translation by three of its subsets of cardinality $|A|-1$ given up to translation, i.e. the *Reconstruction Number* of such objects is three. This result is best-possible.

Keywords: reconstruction; reconstruction number; translation

1 Introduction

In [5] Harary and Plantholt defined the *Graph Reconstruction Number* of a finite graph of order n as the minimum number of its induced subgraphs of order $n-1$ given up to isomorphism that uniquely determine the graph up to isomorphism.

This parameter is related to the famous *Graph Reconstruction Conjecture* due to Kelly [6] and Ulam [14] which states that every finite graph of order n is uniquely determined up to isomorphism by the multiset of its induced subgraphs of order $n-1$ given up to isomorphism (see e.g. [3] for a good survey on this conjecture). The observation that many graphs of order n are already uniquely determined up to isomorphism by very few of their induced subgraphs of order $n-1$ given up to isomorphism led to the definition of the Graph Reconstruction Number. There are several results about this parameter for special cases [4], [8], [9] and almost all graphs have Reconstruction Number three [2].

The Graph Reconstruction Number and the Graph Reconstruction Conjecture have been generalized to various different combinatorial objects and reconstruction problems for finite sets of points in $\mathbf{R}^n$ given up to isometry have been considered by Alon, Caro, Krasikov and Roditty in [1] and by Krasikov and Roditty in [7]. In [11] Radcliffe and Scott considered reconstruction problems for infinite sets of real numbers given up to translation. They observed that every finite set $A \subseteq \mathbf{R}$ is uniquely determined up to translation by the multiset of its subsets of cardinality three given up to

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translation. This observation is easily extended to an arbitrary dimension. Similar problems for subsets of $\mathbf{Z}_n$ have been studied in [10]. In general only few things are known about this kind of reconstruction problems for infinite objects (cf. [11], [12] and [13]).

In the present paper we will prove that finite sets of points in $\mathbf{R}^n$ have reconstruction number three, i.e. every finite set $A \subseteq \mathbf{R}^n$ is uniquely determined up to translation by (at most) three of its subsets of cardinality $|A|-1$ given up to translation. This result is best-possible.

2 Results

We will first prove our result for dimension $n = 1$ and then extend it to an arbitrary dimension. Two sets $A, B \subseteq \mathbf{R}^n$ will be called *isomorphic*, if A is a translate of B, i.e. $A = B + x := \{b + x | b \in B\}$ for some $x \in \mathbf{R}^n$. If $A, B \subseteq \mathbf{R}^n$ are isomorphic, we will write $A \cong B$.

Theorem 1 *Let $A, B \subseteq \mathbf{R}$ with $|A| = |B| \geq 4$ be two finite sets that are not isomorphic and let $a'_1, a'_2, ..., a'_r \in A$ and $b'_1, b'_2, ..., b'_r \in B$ be $r \in \mathbf{N}$ different elements of A and B, respectively. For $1 \leq i \leq r$ let the sets $A \setminus \{a'_i\}$ and $B \setminus \{b'_i\}$ be isomorphic.*

Then $r \leq 2$.

Proof: Let $A' = \{a'_1, a'_2, ..., a'_r\}$ and $B' = \{b'_1, b'_2, ..., b'_r\}$. We assume that $r \geq 3$.

If we have $\max(A) - \min(A) > \max(B) - \min(B)$, then clearly $A' \subseteq \{\min(A), \max(A)\}$ and therefore $r \leq 2$. By symmetry, we assume that $\max(A) - \min(A) = \max(B) - \min(B)$. If $A' \cap (A \setminus \{\max(A), \min(A)\}) = \emptyset$, then again $r \leq 2$ and hence we assume that $a'_1 \in (A \setminus \{\max(A), \min(A)\})$. This implies that $b'_1 \in (B \setminus \{\max(B), \min(B)\})$. Let $a' = a'_1$ and $b' = b'_1$. Possibly translating B, we may assume that

$$A \setminus \{a'\} = B \setminus \{b'\} = \{a_1, ..., a_{|A|-1}\}$$

with

$$a_1 < a_2 < a_3 < ... < a_{|A|-1} \text{ and } a' \neq b'.$$

There are indices $1 \leq i, j \leq |A| - 2$ such that $a_i < a' < a_{i+1}$ and $a_j < b' < a_{j+1}$. Since $a' \in A \setminus B$ and $b' \in B \setminus A$, we have that $A' \setminus \{a'\} \subseteq \{a_1, a_{|A|-1}\}$.

If either $a_1 \notin A'$ or $a_{|A|-1} \notin A'$, then $r \leq 2$. Hence we assume that $A' = \{a_1, a', a_{|A|-1}\}$ and, by symmetry, $B' = \{a_1, b', a_{|A|-1}\}$. Since $a' \in A \setminus B$, we obtain that

$$A \setminus \{a_1\} \cong B \setminus \{a_{|A|-1}\} \text{ and } A \setminus \{a_{|A|-1}\} \cong B \setminus \{a_1\}.$$

If $i < j$, then, since $a_1 \in A'$, we obtain that $a_{i+1} - a_i = a_{i+1} - a'$ which is a contradiction. Hence, by symmetry, we assume that $i = j$. Since $|A| \geq 4$, either $i \geq 2$ or $|A| - i \geq 2$. We assume without loss of generality that $i \geq 2$ (the proof for the case $|A| - i \geq 2$ works analogously.)

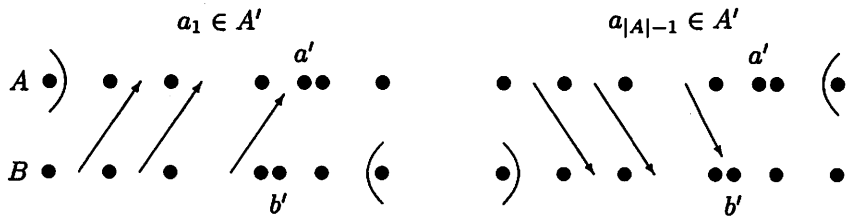

Figure 1

Since $a_1 \in A'$, we obtain $a_i - a_{i-1} = a' - a_i$ and since $a_{|A|-1} \in A'$, we obtain $a_i - a_{i-1} = b' - a_i$ (see Figure 1). This implies that $a' = b'$ and hence $A = B$ which is a contradiction and the proof is complete. □

The two sets $\{1, 2, 4\}$ and $\{1, 3, 4\}$ are not isomorphic but their decks share all three elements. Hence the assumption $|A| = |B| \geq 4$ is essential for Theorem 1. The following example gives an infinite class with $r = 2$. For $\nu, \mu \in \mathbf{N}$ with $\nu, \mu \geq 2$ let

$$A_{\nu,\mu} = \{1, 2, ..., \nu\} \cup \{\nu + 2, \nu + 3, ..., \nu + \mu + 1\}$$

and

$$B_{\nu,\mu} = \{1, 2, ..., \nu - 1\} \cup \{\nu + 1, \nu + 2, ..., \nu + \mu + 1\}.$$

Clearly,

$$A_{\nu,\mu} \ncong B_{\nu,\mu},\ A_{\nu,\mu} \setminus \{\nu\} \cong B_{\nu,\mu} \setminus \{\nu + 1\} \text{ and } A_{\nu,\mu} \setminus \{1\} \cong B_{\nu,\mu} \setminus \{\nu + \mu + 1\}.$$

The deletion of every other pair of elements of $A_{\nu,\mu}$ and $B_{\nu,\mu}$ does not create two isomorphic sets. We now extend Theorem 1 to an arbitrary dimension $n \in \mathbf{N}$.

Theorem 2 *Let $A, B \subseteq \mathbf{R}^n$ with $|A| = |B| \geq 4$ be two finite sets that are not isomorphic and let $a_1, a_2, ..., a_r \in A$ and $b_1, b_2, ..., b_r \in B$ be $r \in \mathbf{N}$ different elements of A and B, respectively. For $1 \leq i \leq r$ let the sets $A \setminus \{a_i\}$ and $B \setminus \{b_i\}$ be isomorphic.*
Then $r \leq 2$.

Proof: If the elements of A are collinear, then either the elements of B are collinear or obviously $r \leq 1$. If the elements of both, A and B, are collinear, then the result easily follows from Theorem 1. Hence we assume that neither the elements of A nor the elements of B are collinear. Furthermore, we may assume that for some (unique) $x \in \mathbf{R}^n$

$$A \setminus \{a_1\} = B \setminus \{b_1\} \text{ and that } A \setminus \{a_2\} = (B \setminus \{b_2\}) + x.$$

Since $a_1 \in (A \setminus \{a_2\}) \setminus (B \setminus \{b_2\})$, we have that $x \neq (0, 0, ..., 0)$.

We will often use the trivial fact that $S \neq S + x$ for some finite, non-empty set S. We consider two cases.

Case 1: $a_1 \in \{b_1 + i \cdot x | i \in \mathbf{Z}\}$.

Since $|A \cap \{b_1 + i \cdot x | i \in \mathbf{Z}\}| = |B \cap \{b_1 + i \cdot x | i \in \mathbf{Z}\}|$, we have that either $a_2, b_2 \in \{b_1 + i \cdot x | i \in \mathbf{Z}\}$ or $a_2, b_2 \notin \{b_1 + i \cdot x | i \in \mathbf{Z}\}$. Note that $A, B \not\subseteq \{b_1 + i \cdot x | i \in \mathbf{Z}\}$, since neither A nor B are collinear.

If $a_2, b_2 \in \{b_1 + i \cdot x | i \in \mathbf{Z}\}$, then

$$A \setminus \{b_1 + i \cdot x | i \in \mathbf{Z}\} = (B \setminus \{b_1 + i \cdot x | i \in \mathbf{Z}\}) + x = (A \setminus \{b_1 + i \cdot x | i \in \mathbf{Z}\}) + x$$

and $A \setminus \{b_1 + i \cdot x | i \in \mathbf{Z}\}, B \setminus \{b_1 + i \cdot x | i \in \mathbf{Z}\} \neq \emptyset$ which is a contradiction to the finiteness of A and B. Hence $a_2, b_2 \notin \{b_1 + i \cdot x | i \in \mathbf{Z}\}$.

If for some $a \in A$ we have $a \in \{b_1 + i \cdot x | i \in \mathbf{Z}\}$ and $a \neq a_1$, then $a \in B$ and $a + x \in A$. This and the finiteness of A easily imply that $a_1 = b_1 + j \cdot x$ for some $j \in \mathbf{N}$ and

$$A \cap \{b_1 + i \cdot x | i \in \mathbf{Z}\} = \{b_1 + x, b_1 + 2 \cdot x, ..., b_1 + j \cdot x\}$$

and

$$B \cap \{b_1 + i \cdot x | i \in \mathbf{Z}\} = \{b_1, b_1 + x, b_1 + 2 \cdot x, ..., b_1 + (j - 1) \cdot x\}.$$

If for some $a \in A$ we have $a \in \{a_2 + i \cdot x | i \in \mathbf{Z}\}$ and $a \neq b_2$, then $a \in B$ and $a + x \in A$. This and the finiteness of A easily imply that $b_2 = a_2 + k \cdot x$ for some $k \in \mathbf{N}_0 = \{0, 1, 2, ...\}$ and

$$A \cap \{a_2 + i \cdot x | i \in \mathbf{Z}\} = B \cap \{a_2 + i \cdot x | i \in \mathbf{Z}\} = \{a_2, a_2 + x, ..., b_2 = a_2 + k \cdot x\}.$$

Again using the finiteness of A and B and similar arguments as above we obtain that

$$A, B \subseteq \{b_1 + i \cdot x | i \in \mathbf{Z}\} \cup \{a_2 + i \cdot x | i \in \mathbf{Z}\}$$

and hence (see Figure 2)

$$A = \{b_1 + x, b_1 + 2 \cdot x, ..., b_1 + j \cdot x\} \cup \{a_2, a_2 + 1 \cdot x, ..., a_2 + k \cdot x\}$$

and

$$B = \{b_1, b_1 + x, ..., b_1 + (j-1) \cdot x\} \cup \{a_2, a_2 + x, ..., a_2 + k \cdot x\}.$$

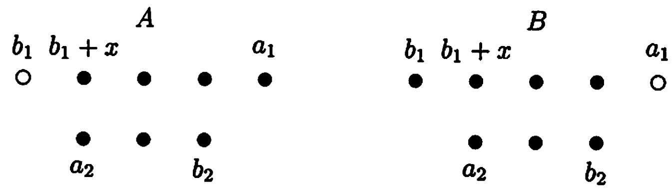

Figure 2

Now it follows easily from the assumption $|A| = |B| \geq 4$ that $r \leq 2$.

Case 2: $a_1 \notin \{b_1 + i \cdot x | i \in \mathbf{Z}\}$.

If for some $a \in A$ we have $a \notin \{b_1 + i \cdot x | i \in \mathbf{Z}\}$ and $a \neq a_2$ (note that a_1 satisfies these assumptions), then $a - x \in B$ and $a - x \in A$. This and the finiteness of A easily imply that $a_2 = a_1 - j \cdot x$ for some $j \in \mathbf{N}$ and

$$A \cap \{a_1 + i \cdot x | i \in \mathbf{Z}\} = \{a_2, a_2 + x, ..., a_1 = a_2 + j \cdot x\}$$

and

$$B \cap \{a_1 + i \cdot x | i \in \mathbf{Z}\} = \{a_2, a_2 + x, ..., a_2 + (j-1) \cdot x\}.$$

If for some $b \in B$ we have $b \in \{b_1 + i \cdot x | i \in \mathbf{Z}\}$ and $b \neq b_2$ (note that b_1 satisfies these assumptions), then $b + x \in A$ and $b + x \in B$. This and the finiteness of A easily imply that $b_2 = b_1 + k \cdot x$ for some $k \in \mathbf{N}$ and

$$A \cap \{b_1 + i \cdot x | i \in \mathbf{Z}\} = \{b_1 + x, b_1 + 2 \cdot x, ..., b_2 = b_1 + k \cdot x\}$$

and

$$B \cap \{b_1 + i \cdot x | i \in \mathbf{Z}\} = \{b_1, b_1 + x, b_1 + 2 \cdot x, ..., b_2 = b_1 + k \cdot x\}.$$

Again using the finiteness of A and B and similar arguments as above we obtain that (see Figure 3)

$$A = \{a_2, a_2 + x, ..., a_1 = a_2 + j \cdot x\} \cup \{b_1 + x, b_1 + 2 \cdot x, ..., b_2 = b_1 + k \cdot x\}$$

and

$$B = \{a_2, a_2 + x, ..., a_2 + (j-1) \cdot x\} \cup \{b_1, b_1 + x, b_1 + 2 \cdot x, ..., b_2 = b_1 + k \cdot x\}.$$

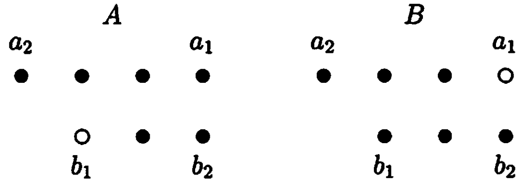

Figure 3

Now it follows easily from the assumption $|A| = |B| \geq 4$ that $r \leq 2$. □

With some more effort the arguments used in the proof of Theorem 2 will yield a characterization of all pairs of finite sets A and B in $\mathbf{R}^n$ for which the decks share two elements.

As an open problem it is possible to consider the analogous question for subsets of smaller cardinality $|A| - k$ for some $k \geq 2$.

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