Mimimum Degree and the Number of Chords*

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Abstract

We address the following problem: What minimum degree forces a graph on n vertices to have a cycle with at least c chords? We prove that any graph with minimum degree δ has a cycle with at least $\frac{(\delta+1)(\delta-2)}{2}$ chords. We investigate asymptotic behaviour for large n and c and we consider the special case where $n = c$.

1 Introduction

It is an easy exercise to prove that a graph with minimum degree at least three has a cycle with at least one chord (it must actually have a cycle with two chords). A natural question is: What minimum degree forces a graph on n vertices to have a cycle with c chords? We mean by "a cycle with c chords" "a cycle containing at least c chords". Peter Hamburger asked about the following special case of that problem: What minimum degree forces a graph on n vertices to have a cycle with n chords? The first results on this problem can be found in [2]:

Theorem 1 *Let G be a graph on n vertices with minimum degree at least $2\sqrt{n}$. Then G has a cycle with at least n chords.*

This result was improved in [1]:

Theorem 2 *Let G be a graph on n vertices with minimum degree at least $1 + \sqrt{2n+1}$. Then G has a cycle with at least n chords.*

Theorem 2 is a corollary of the following theorem, also proved in [1], which gives a partial answer to a more general problem "What minimum degree forces a graph to have a cycle with at least c chords?":

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Theorem 3 *Let G be a graph with minimum degree δ. Then G has a cycle with $\lceil \frac{\delta^2-2\delta}{2} \rceil$ chords.*

We improve this bound to $\frac{(\delta+1)(\delta-2)}{2}$ in Section 2 (Theorem 4); this value cannot be further improved without additional assumptions on the graph G. Thus we improve the bound of Theorem 2 to $1/2+\sqrt{2n+9/4}$. We address the original Hamburger's problem in Section 3 — we calculate lower and upper bounds on the minimum degree of graphs on n vertices from Hamburger's problem which differ by at most one (Theorem 5); our bounds meet for about half of n's.

We consider the generalization of Hamburger's problem in Section 4. We introduce the function $f(n,c)$ which is equal to the minimum degree which forces a graph on n vertices to have a cycle with c chords. Note that $f(n,c)$ is defined only for $0 \leq c \leq \frac{n^2-3n}{2}$. We prove that $f(n,c)$ is linear in $\sqrt{c}$ and it does not depend substantially on n (Theorem 6). We investigate the behaviour of $f(n,c)$ for n going to infinity for various choices of c as function of n in Theorem 7.

2 Tight Bound for the Number of Chords

We prove the lower bound on the number of chords in a cycle in a graph with minimum degree δ in this section. This improves Theorem 3. Our bound is sharp — it is achieved by infinitely many graphs (consult the proof of Theorem 7), e.g. by $K_{\delta+1}$.

Theorem 4 *Let G be a graph with minimum degree $\delta \geq 2$. Then G contains a cycle with at least $\frac{(\delta+1)(\delta-2)}{2}$ chords.*

Proof: The theorem trivially holds for $\delta = 2$; thus we assume further that δ is at least 3. We set for a path $P = v_1, \dots, v_n$ the number $k(P)$ to be the maximal i such that there is an edge v_1v_i in G and the number $l(P)$ to be the length of P (i.e. n). Let P be the path with maximal $l(P)$ (i.e. the longest path in G) and among all such paths the path with maximal $k(P)$. Note that all the neighbours of v_1 are due to the maximality of $l(P)$ and the definition of $k(P)$ among the vertices $v_2, \dots, v_{k(P)}$. Since $k(P) \geq \delta + 1$ from trivial reasons, $k(P)$ must be at least 4. Let $\overline{P}$ be the path $v_{k(P)-1}, \dots, v_1, v_{k(P)}, \dots, v_n$; we write $\overline{v_i} = v_{k(P)-i}$ for $i < k(P)$ and $\overline{v_i} = v_i$ for $i \geq k(P)$, i.e. $\overline{P} = \overline{v_1}, \dots, \overline{v_n}$. Note that $k(P) = k(\overline{P})$ and $l(P) = l(\overline{P})$, thus $\overline{P}$ is a path which could be chosen instead of P. Thus also all the neighbours of $\overline{v_1}$ are among the vertices $\overline{v_2}, \dots, \overline{v_{k(P)}}$. Let C be the cycle $v_1, \dots, v_{k(P)}$. Note that $v_1\overline{v_1}$ is not an edge of C. We prove that C has at least $\frac{(\delta+1)(\delta-2)}{2}$ chords.

We consider v_0 to be $v_{k(P)}$ and $\overline{v_0}$ to be $\overline{v_{k(P)}}$; this makes more clear some arguments in the proof, since we can say e.g. that the edges of C are the edges $v_{i-1}v_i$ for $1 \leq i \leq k(P)$. We write $N(v)$ for the set of all the neighbours of v and we write $V(C)$ for the set of the vertices of C. Let M^+ be $N(v_1)$; note that $M^+ \subseteq V(C)$. Let M be the set $\{v_i | v_{i+1} \in M^+\}$. Note that $|M| = |M^+| = |N(v_1)| \geq \delta$. We claim that $N(v_i) \subseteq V(C)$ for each $v_i \in M$. If this was not true, let $v_i w$ be an edge of G such that $w \notin V(C)$; i cannot be 1, since $N(v_1) \subseteq V(C)$. Consider the path $v_i, \ldots, v_1, v_{i+1}, \ldots, v_n$. If w is not any of v_j for $1 \leq j \leq n$, we can extend the path by w contradicting the maximality of $l(P)$. Otherwise, if $w = v_j$ it must hold $j > k(P)$ contradicting the maximality of $k(P)$. Thus no such v_i can exist and our claim is true. We define $\overline{M^+}$ to be $N(\overline{v_1})$ and $\overline{M}$ to be the set $\{\overline{v_i} | \overline{v_{i+1}} \in \overline{M^+}\}$; it also holds that $N(\overline{v_i}) \subseteq V(C)$ for all $\overline{v_i} \in \overline{M}$ and $|\overline{M}| \geq \delta$. Note also that the vertex $v_0 = \overline{v_0} = v_{k(P)} = \overline{v_{k(P)}}$ is neither in M nor in $\overline{M}$.

We distinguish two cases to finish the proof:

- $|M| \geq \delta + 1$ or $|\overline{M}| \geq \delta + 1$ or $M \neq \overline{M}$
  It holds that $|M \cup \overline{M}| \geq \delta + 1$ in this case and it also holds $N(w) \subseteq V(C)$ for each $w \in M \cup \overline{M}$. Thus each vertex of $M \cup \overline{M}$ is adjacent to at least $\delta - 2$ chords and the cycle contains at least $\frac{(\delta+1)(\delta-2)}{2}$ chords (each chord can be counted at most twice, once for each of its ends).

- $M = \overline{M}$ and $|M| = |\overline{M}| = \delta$
  We use discharging technique argument in this case. We assign each chord of C two units and to the edges v_0v_1 $\overline{v_0}\overline{v_1}$ one unit. We distribute these units to the vertices of M according to the following rules:

  - The chord connecting two vertices in M will give one unit to each of them.
  - The chord connecting a vertex in M different from v_1 and $\overline{v_1}$ to a vertex not in M will give two units to the vertex in M.
  - The edges v_1v_i for $3 \leq i \leq k(P)$ will give one unit to v_{i-1} and in case that $i < k(P)$ also one unit to v_1.
  - The edges $\overline{v_1}\overline{v_i}$ for $3 \leq i \leq k(P)$ will give one unit to $\overline{v_{i-1}}$ and in case that $i < k(P)$ also one unit to $\overline{v_1}$.

  Each vertex of M got at least $\delta - 1$ units:

  - Let $w \in M$ such that both its neighbours (in C) are also in M. Note that w is neither v_1 nor $\overline{v_1}$, since v_0 is not in M. There are at least $\delta - 2$ chords adjacent to w. Each chord connecting w

with another vertex of M gave one unit to w. There is at most $\delta - 3$ such chords, since w and both its neighbours are in M. All the other chords gave two units to w. Thus w got at least $\delta - 1$ units.

- Let $w \in M$ such that at least one of its neighbours (in C) is not in M.
  Each of $\delta - 2$ adjacent chords to w gave it one unit. Let u be that neighbour of w which is not in M. Assume w.l.o.g. that $w = v_i$ and $u = v_{i+1}$; otherwise the same will be used for $w = \overline{v_j} = v_i$ and $u = \overline{v_{j+1}} = v_{i-1}$. The vertex u is in M^+ and thus adjacent to v_1, since $w \in M$. The edge uv_1 gave one unit to w. Thus w got at least $\delta - 1$ units.

The vertices of M got together at least $\delta(\delta - 1)$ units. They got at least $\delta(\delta - 1) - 2 = (\delta + 1)(\delta - 2)$ units from chords of the cycle C and thus there are at least $\frac{(\delta+1)(\delta-2)}{2}$ chords.

□

The immediate corollary of this theorem related to Hamburger's problem is following:

Corollary 1 *Let G be a graph with minimum degree k on $n \leq \frac{k^2-k-2}{2}$ vertices. Then G contains a cycle with n chords.*

3 Hamburger's Problem

We address the original Hamburger's problem in this section. We say that G has minimum degree almost k iff there is at most one vertex of degree $k - 1$ in G and the degree of all the other vertices of G is at least k. We write L_n^k $(k < n)$ for a graph on n vertices with minimum degree almost k. The existence of L_n^k follows easily from the fact that K_n is 2-factorable if n is odd and 1-factorable if n is even. Note that the number of chords in any cycle of L_n^k is at most $\frac{n(k-2)}{2}$. We first develop some construction techniques for graphs without cycles with a lot of chords.

Lemma 1 *Let $G_1, \ldots, G_l$ $(l \geq 2)$ be graphs with minimum degree almost k which do not contain a cycle with c chords. Let n_i be the number of vertices of G_i. If $\sum_{i=1}^{l} n_i - l + 1 \leq n \leq \sum_{i=1}^{l} n_i$, then there exists a graph on n vertices and with minimum degree k which does not contain a cycle with c chords.*

Proof: Let v_i be a vertex of G_i of degree $k - 1$ if G_i contains such vertex or any of its vertices otherwise. Let $l' = n - \sum_{i=1}^{l} n_i + l$. Let us consider a graph consisting of vertex–disjoint copies of G_i. If $l' < l$, we identify vertices $v_{l'}, \dots, v_l$, i.e. we remove vertices $v_{l'+1}, \dots, v_l$ and we let the edges leading to them to lead to $v_{l'}$. If $l' > 1$, we place a cycle $v_1, \dots, v_{l'}$ to the graph. The just obtained graph contains $\sum_{i=1}^{l} n_i - (l - l') = n$ vertices and its minimum degree is k. Each its cycle different from $v_1, \dots, v_{l'}$ is entirely contained in a copy of G_i for some i. Thus the just obtained graph does not contain a cycle with c chords. □

Lemma 2 *There exists a graph on n vertices and with minimum degree k which does not contain a cycle with n chords if $\frac{k^2-k+2}{2} \leq n$ and k is odd.*

Proof: Let $n = \alpha k + \beta$ where $1 \leq \beta \leq k$; note that $\alpha \geq \frac{k-1}{2}$. We distinguish three cases:

- If $\beta \leq \alpha$, we set G_i to be L_{k+1}^k for $1 \leq i \leq \alpha$. The sum of the numbers of vertices of G_i is $\alpha(k+1)$: $\alpha(k+1) - \alpha + 1 = \alpha k + 1 \leq n = \alpha k + \beta \leq \alpha(k+1)$. Since the number of chords in any cycle of L_{k+1}^k is at most $\frac{(k+1)(k-2)}{2} = \frac{k^2-k-2}{2}$, the existence of the graph follows from Lemma 1.
- If $\alpha < \beta \leq 2\alpha$, we set G_i to be L_{k+2}^k for $1 \leq i \leq \alpha$. The sum of the numbers of vertices of G_i is $\alpha(k+2)$: $\alpha(k+2) - \alpha + 1 \leq \alpha k + \beta = n \leq \alpha(k+2)$. The number of chords in any cycle of L_{k+2}^k is at most $\frac{(k+2)(k-2)}{2} = \frac{k^2-4}{2}$. But it holds that $\alpha k + \alpha + 1 \leq n$ and $\frac{k-1}{2} \leq \alpha$, thus $\frac{k^2+1}{2} \leq n$. The existence of the graph follows from Lemma 1.
- The remaining case if $2\alpha < \beta$. Since $\beta \leq k$, it must be $\alpha = \frac{k-1}{2}$ and $\beta = k$ in this case. We set G_i to be L_{k+3}^k for $1 \leq i \leq \alpha$. The sum of the numbers of vertices of G_i is $\alpha(k+3)$: $\alpha(k+3) - \alpha + 1 \leq \alpha k + \beta = n \leq \alpha(k+3)$. The number of chords in any cycle of L_{k+3}^k is at most $\frac{(k+3)(k-2)}{2} = \frac{k^2+k-6}{2}$; since $n = \alpha k + \beta = \frac{k^2+k}{2}$, the number of chords is less than n. The existence of the graph follows from Lemma 1.

□

Lemma 3 *There exists a graph on n vertices and with minimum degree k which does not contain a cycle with n chords if $\frac{k^2+2}{2} \leq n$ and k is even.*

Proof: Let $n = \alpha k + \beta$ where $1 \leq \beta \leq k$; note that $\alpha \geq k/2$ and thus it holds $\beta \leq 2\alpha$ for all n. We proceed as in the proof of Lemma 2. If $\beta \leq \alpha$ then we use Lemma 1 for L_{k+1}^k, we use it for L_{k+2}^k otherwise. Since the number of chords in any cycle of L_{k+1}^k and L_{k+2}^k is at most

$\frac{(k+2)(k-2)}{2} = \frac{k^2-4}{2}$, Lemma 1 actually ensures the existence of the desired graph. □

If n is a bit smaller than in the just proven lemma and k is even, we can use another technique to construct the desired graph:

Lemma 4 *There exists a graph on n vertices and with minimum degree k which does not contain a cycle with n chords if $\frac{k^2-k/2}{2} \leq n \leq \frac{k^2-4}{2}$ and $k \geq 8$ is even.*

Proof: Let K be the complete graph on $k+1$ vertices and let L be a graph on $k+2$ vertices with the degree sequence equal to $(k-2l, k, \ldots, k)$ (l is going to be chosen later). The graph L exists, since it is enough to consider the complement of $K_{1,2l+1} \cup K_2 \cup \ldots \cup K_2$. Let $\beta = n - \frac{k^2-k-2}{2}$ and $\alpha = \frac{k-2}{2} - \beta$ ($\alpha \geq 0$, since $n \leq \frac{k^2-4}{2}$). We first establish that $\alpha(k+1) + \beta(k+2) = n$:

$$\alpha(k+1) + \beta(k+2) = (\frac{k-2}{2} - \beta)(k+1) + \beta(k+2) = \frac{k^2-k-2}{2} + \beta = n$$

Create the graph G from α vertex disjoint copies of K and β vertex disjoint copies of L. Choose vertices $v_1, \ldots, v_\alpha$ in each copy of K arbitrary and choose vertices $v_{\alpha+1}, \ldots, v_{\alpha+\beta}$ to be the vertices with the smallest degree in each copy of L; add the clique on vertices $v_1, \ldots, v_{\alpha+\beta}$ to G. We need that the minimum degree of G is at least k. Thus we want the following to hold:

$$\begin{aligned} k &\leq k - 2l + \left(\frac{k-2}{2} - 1\right) \\ 2l &\leq \frac{k-4}{2} \\ l &\leq \frac{k-4}{4} \end{aligned}$$

We choose l to be $\lfloor \frac{k-4}{4} \rfloor$. Any cycle of G is fully contained in either the clique on vertices $v_1, \ldots, v_{\alpha+\beta}$ or in a copy of K or in a copy of L. It has at most $\frac{(\alpha+\beta)(\alpha+\beta-3)}{2} = \frac{k^2-10k+16}{8}$ chords if it is in the clique, $\frac{k^2-k-2}{2}$ chords if it is in a copy of K and $\frac{(k+2)(k-2)-2l}{2} = \frac{k^2-4-2l}{2}$ chords if it is in a copy of L. We estimate the last fraction:

$$\frac{k^2-4-2l}{2} \leq \frac{k^2-4-2\left(\frac{k-4}{4} - 1/2\right)}{2} = \frac{k^2-4-k/2+3}{2} = \frac{k^2-k/2-1}{2}$$

Thus if $\frac{k^2-k/2}{2} \leq n$, then the graph G has all the desired properties. □

The immediate corollary of Lemma 1, Lemma 2, Lemma 3 and Lemma 4 is the following theorem:

n	5	6	7	8	9	10	11	12	13	14
$f(n,n)$	4	4	4	4	5	5	6	6	6	6
lower bound	4	4	4	4	5	5	6	6	6	6
upper bound	4	5	5	5	5	6	6	6	6	6

n	15	16	17	18	19	20	21	22	23	24
$f(n,n)$	6	6	6	6	7	7	?	8	8	8
lower bound	6	6	6	6	7	7	7	8	8	8
upper bound	7	7	7	7	7	7	8	8	8	8

n	25	26	27	28	29	30	31	32	33	34
$f(n,n)$	8	8	8	?	?	9	?	?	9	9
lower bound	8	8	8	8	8	9	8	8	9	9
upper bound	8	8	8	9	9	9	9	9	9	9

Table 1: The values of the function $f(n,n)$ from Hamburger's problem and lower and upper bounds from Theorem 5; $f(n,n)$ for $n = 6,7,8,10,15,16,17,18$ was calculated using methods not contained in this paper.

Theorem 5 *Let n be at least five. Then:*

- $f(n,n) = k$ *if* $\frac{k^2-3k+4}{2} \le n \le \frac{k^2-k-2}{2}$ *for even* k
- *either* $f(n,n) = k-1$ *or* $f(n,n) = k$ *if* $n = \frac{k^2-3k+2}{2}$ *for even* k
- $f(n,n) = k$ *if* $\frac{k^2-2.5k+1.5}{2} \le n \le \frac{k^2-2k-3}{2}$ *or* $\frac{k^2-2k+3}{2} \le n \le \frac{k^2-k-2}{2}$ *for odd* k
- *either* $f(n,n) = k-1$ *or* $f(n,n) = k$ *if* $\frac{k^2-3k+2}{2} \le n \le \frac{k^2-2.5k+1}{2}$ *or* $n = \frac{k^2-2k\pm1}{2}$ *for odd* k

Note that the cases in the theorem are disjoint and they cover all the possibilities ($\frac{(k-1)^2-(k-1)-2}{2} = \frac{k^2-3k}{2}$). The values of $f(n,n)$ for small n can be found in Table 1; we calculated some of the values $(6,7,8,10,15,16,17,18)$ in the table using some reasoning not contained in this paper. Theorem 5 gives us the following inequalities:

Corollary 2 *It holds that* $\lfloor 1+\sqrt{2n-2} \rfloor \le f(n,n) \le \lceil 1/2+\sqrt{2n+9/4} \rceil$ *for* $n \ge 5$.

4 Generalization of Hamburger's Problem

We want to address the generalization of Hamburger's problem in this section. We first develop some lower bounds on $f(n, c)$ for the general case:

Lemma 5 *Let $\kappa \geq 2$ be a fixed integer. Let c be at least $9/2(\kappa+1)^2$ and let n be at least $\kappa\sqrt{2c}$. There exists a graph G on n vertices and minimum degree at least $\sqrt{2c}\frac{\kappa}{\kappa+1}$ which does not contain a cycle with c chords.*

Proof: Let k be $\lfloor \frac{n}{\sqrt{2c}} \rfloor$; note that $k\sqrt{2c} \leq n \leq (k+1)\sqrt{2c}$ and $\kappa \leq k$. We split the n vertices of our graph G to k parts $V_1, \ldots, V_k$ of sizes $\lfloor n/k \rfloor$ and $\lceil n/k \rceil$. We claim that the following holds for all $1 \leq i \leq k$:

$$\sqrt{2c} - 1 \leq |V_i| \leq \sqrt{2c}(1 + 1/\kappa) + 1$$

The first inequality is clear from the definition of k. The average size of $|V_i|$ is at most $\frac{k+1}{k}\sqrt{2c} \leq \frac{\kappa+1}{\kappa}\sqrt{2c}$. The size of each V_i differs from the average size by at most one, thus the second inequality also holds. Let m be further $\sqrt{2c}(1 + 1/\kappa) + 1$, i.e. $|V_i| \leq m$ for all i.

Let δ be $\lceil \sqrt{2c}\frac{\kappa}{\kappa+1} \rceil$. We place a copy of $L^{\delta}_{|V_i|}$ on each vertex set V_i; we need to check that $|V_i| \geq \delta + 1$ for existence of $L^{\delta}_{|V_i|}$:

$$\begin{aligned} |V_i| \geq \sqrt{2c} - 1 &\geq \sqrt{2c}\frac{\kappa}{\kappa+1} + 2 \geq \delta + 1 \\ \sqrt{2c}\frac{1}{\kappa+1} &\geq 3 \\ c &\geq 9/2(\kappa+1)^2 \end{aligned}$$

Let w_i be any vertex of minimum degree in the subgraph placed on V_i. We add edges $w_i w_{i+1}$ for all $1 \leq i \leq k-1$. The minimum degree of all the vertices is δ, now. The just added edges do not introduce any new cycle in G and thus all the cycles are only in the subgraphs placed on V_i. We prove that $\delta < \frac{2c}{m} + 2$; then the number of chords of any cycle is bounded by $\frac{m(\delta-2)}{2} < \frac{m(2c/m)}{2} = c$ and G does not contain a cycle with c chords. We prove that bound on δ now and finish the proof of the lemma:

$$\frac{2c}{m} + 2 = \frac{2c}{\sqrt{2c}(1+1/\kappa)+1} + 2 + \frac{2c}{\sqrt{2c}(1+1/\kappa)} - \frac{2c}{\sqrt{2c}(1+1/\kappa)} =$$

$$\sqrt{2c}\frac{\kappa}{\kappa+1} + \frac{(2c)^{3/2}(1+1/\kappa) - (2c)^{3/2}(1+1/\kappa) - 2c}{2c(1+1/\kappa)^2 + \sqrt{2c}(1+1/\kappa)} + 2 =$$

$$\sqrt{2c}\frac{\kappa}{\kappa+1} - \frac{\sqrt{2c}}{\sqrt{2c}(1+1/\kappa)^2 + 1 + 1/\kappa} + 2 \geq \sqrt{2c}\frac{\kappa}{\kappa+1} + 1 > \delta$$

□

Lemma 6 *There is a graph G on n vertices and minimum degree at least $\frac{2c}{n}$ which does not contain a cycle with c chords.*

Proof: Let k be the largest even integer strictly smaller than $\frac{2c}{n}+2$. Note that k is at least $\frac{2c}{n}$. Let G be L_n^k and let C be any cycle of G. The number of its chords is at most $\frac{|C|(k-2)}{2} \leq \frac{n(k-2)}{2} < \frac{n}{2}\frac{2c}{n} = c$. □

We first state that $f(n,c)$ is linear in $\sqrt{c}$:

Theorem 6 *It holds that $\sqrt{c}/\sqrt{2} \leq f(n,c) \leq 3\sqrt{c}$ for $c \geq 1$.*

Proof:

It easily follows from Theorem 4 that $f(n,c) \leq 3/2 + \sqrt{2c+9/4}$. This gives the upper bound for $c \geq 2$. There is $f(n,c) = 3$ in the remaining case $c = 1$ and the upper bound holds also in this case. We distinguish several cases to prove $1/\sqrt{2}\sqrt{c} \leq f(n,c)$:

- $c \geq 41$ and $n \geq \sqrt{8c}$
  It follows $1/\sqrt{2}\sqrt{c} = 1/2\sqrt{2c} \leq f(n,c)$ from Lemma 5 used for $\kappa = 2$.

- $n \leq \sqrt{8c}$
  It holds that $2c/n \leq f(n,c)$ due to Lemma 6. But since $1/\sqrt{2}\sqrt{c} = 2c/\sqrt{8c} \leq 2c/n$, the lower bound follows.

- $c \leq 18$
  This case is trivial, since $1/\sqrt{2}\sqrt{c} \leq 3$ and $3 \leq f(n,c)$ from trivial reasons for $c > 0$. If $c = 0$, then $f(n,c) = 2$ and the lower bound also holds.

- $19 \leq c \leq 32$
  Since $1/\sqrt{2}\sqrt{c} \leq 4$, we need to establish that $4 \leq f(n,c)$. There is n at least 8 in this case in order $f(n,c)$ to be defined. We use Lemma 1 for G_i equal to L_4^3 for $1 \leq i \leq \lceil n/4 \rceil$ and $n = 8, 10, 11, \ldots$; we use Lemma 1 for $G_1 = L_4^3$ and $G_2 = L_5^3$ for $n = 9$. In both cases we have a graph on n vertices with minimum degree 3 which does not contain a cycle with 3 chords and thus $3 < f(n,c)$.

- $33 \leq c \leq 40$
  Since $1/\sqrt{2}\sqrt{c} \leq 5$, we need to establish that $5 \leq f(n,c)$. There is n at least 10 in this case in order $f(n,c)$ to be defined. We use Lemma 1 for G_i equal to L_5^4 for $1 \leq i \leq \lceil n/5 \rceil$ and $n = 13, 14, 15, 17, 18, \ldots$; we use Lemma 1 for $G_1 = G_2 = L_6^4$ for $n = 10, 11, 12$ and for $G_1 = G_2 = G_3 = L_6^4$ for $n = 16$. In all the cases we have a graph on n vertices with minimum degree 4 which does not contain a cycle with 7 chords and thus $4 < f(n,c)$.

□

We study asymptotic behaviour of $f(n,c)$ in the next theorem:

Theorem 7 *Let $c : \mathcal{N} \to \mathcal{N}$ be any function.*

- *If $\lim_{n\to\infty} c(n)$ exists, is finite and equal to c_0, then:*

$$\lim_{n\to\infty} f(n,c(n)) = \lceil 1/2 + \sqrt{2c_0 + 9/4} \rceil$$

- *If $\lim_{n\to\infty} c(n) = \infty$, then $\lim_{n\to\infty} f(n,c(n)) = \infty$.*
- *$\lim_{n\to\infty} f(n,c(n))$ is equal to $\lim_{n\to\infty} \lceil 1/2 + \sqrt{2c(n) + 9/4} \rceil$ if the latter limit exists and $\lim_{n\to\infty} f(n,c(n))$ does not exist otherwise.*
- *If $\lim_{n\to\infty} c(n) = \infty$ and $\lim_{n\to\infty} c(n)/n^2 = 0$, then:*

$$\lim_{n\to\infty} \frac{f(n,c(n))}{\sqrt{2c(n)}} = 1$$

Proof: In the first case, there exists n_0 such that $c(n) = c_0$ for $n \geq n_0$. Let $k = \lceil 1/2 + \sqrt{2c_0 + 9/4} \rceil$. The graph with minimum degree at least k contains a cycle with at least $\frac{(k+1)(k-2)}{2} \geq c_0$ chords due to Theorem 4. On the other hand, if we use Lemma 1 for K_k (for n sufficiently large), we immediately get that $\lceil 1/2 + \sqrt{2c_0 + 9/4} \rceil \leq f(n,c(n))$. If $\lim_{n\to\infty} c(n) = \infty$, the same construction gives that $\lim_{n\to\infty} f(n,c(n)) = \infty$. The third case easily follows from the first and second one.

Let us focus our attention to the fourth statement of the theorem. Theorem 5 assures the existence of a graph on n vertices without a cycle with c chords for $c \geq 9/2(\kappa+1)^2$ and $n \geq \kappa\sqrt{2c}$. Since $\lim_{n\to\infty} c(n) = \infty$ and $\lim_{n\to\infty} c(n)/n^2 = 0$, we can always choose large κ for n large enough to get the following:

$$\liminf_{n\to\infty} \frac{f(n,c(n))}{\sqrt{2c}} \geq 1$$

On the other hand, the opposite inequality easily follows from Theorem 4. □

Conclusions

We have proved lower and upper bounds on values of $f(n,n)$ which differs by at most one, calculating for about half of n's the exact value of $f(n,n)$. The interesting fact is that we managed to prove that $f(30,30) = f(33,33) = 9$, but we only proved that $8 \leq f(31,31), f(32,32) \leq 9$. We believe that $f(31,31) = f(32,32) = 8$ but we did not succeed in proving this. This would imply that $f(n,n)$ is not increasing in n. Our belief is supported by the fact that $f(15,15) = f(16,16) = f(17,17) = f(18,18) = 6$ which is an analogous smaller case of $f(31,31)$ and $f(32,32)$; we omit the proofs of these equalities in the paper, since their proofs are very boring and technical. This raises the following open problem: Prove that $f(31,31) = 8$ or $f(32,32) = 8$. Or even, calculate $f(n,n)$ for all n.

We addressed in Section 4 problems of more general interest. We calculated $\lim_{n\to\infty} \frac{f(n,c(n))}{\sqrt{2c}}$ for all functions $c(n)$ such that $\lim_{n\to\infty} c(n)/n^2 = 0$ (for those for which the limit exists). What is (if it exists) the limit for $c(n)$ such that $\lim_{n\to\infty} c(n)/n^2 = c_0$ where $0 < c_0 < 1/2$? The problem which is obligatory to mention, but it seems to be extremely hard, is the following: Calculate $f(n,c)$ for all n and c.

Note added in proof

Recently, our attention was pointed by Jeff Kahn to his joint paper [3]. This paper contains a slightly different proof of Theorem 4 which answers the problem posed in [1].

References

[1] A. A. Ali, W. Staton: The extremal question for cycles with chords, Ars Combinatoria Vol. 51, 1999, pp. 193–197.

[2] G. Chen, P. Erdös, W. Staton: Proof of a conjecture of Bollobás on nested cycles, J. Comb. Theory (B) Vol. 66, No. 1, 1996, pp. 38–43.

[3] R. P. Gupta, J. Kahn, N. Robertson: On the maximum number of diagonals of a circuit in a graph, Discrete Mathematics 32, 1980, pp. 37–43.