Cycles of length four through a given arc in almost regular multipartite tournaments.

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Abstract

If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x = y$). If $i_g(D) = 0$, then D is regular and if $i_g(D) \leq 1$, then D is almost regular.

A c-partite tournament is an orientation of a complete c-partite graph. It is easy to see that there exist regular c-partite tournaments with arbitrary large c which contain arcs that do not belong to a directed cycle of length 3. In this paper we show, however, that every arc of an almost regular c-partite tournament is contained in a directed cycle of length four, when $c \geq 8$. Examples show that the condition $c \geq 8$ is best possible

Keywords: Multipartite tournaments; Cycles; Cycles of length four

1. Terminology and introduction

In this paper all digraphs are finite without loops or multiple arcs. The vertex set of a digraph D is denoted by $V(D)$. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \rightarrow Y$. The number of arcs from X to Y is denoted by $d(X,Y)$. The *out-neighborhood* $N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *in-neighborhood* $N^-(x)$ is the set of vertices dominating x. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively. If we speak of a *cycle*, then we mean a directed cycle, and a cycle of length m is called an *m-cycle*. By $\alpha(D) = \alpha$ we denote the *independence number* of (the underlying graph of) the digraph D.

There are several measures of how much a digraph differs from being regular. In [6], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)} \{d^+(x), d^-(x)\} - \min_{x \in V(D)} \{d^+(x), d^-(x)\}.$$

If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leq 1$, then D is called *almost regular*.

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices.

It is very easy to see that every arc of a regular tournament belongs to a 3-cycle. Our first example shows that this is not valid for regular multipartite tournaments in general.

Example 1.1 Let C, C' and C'' be three induced cycles of length 4 such that $C \rightarrow C' \rightarrow C'' \rightarrow C$. The resulting 6-partite tournament D_1 is 5-regular, but no arc of the three cycles C, C', and C'' is is contained in a 3-cycle.

Let H, H_1, and H_2 be three copies of D_1 such that that $H \rightarrow H_1 \rightarrow H_2 \rightarrow H$. The resulting 18-partite partite tournament is 17-regular, but no arc of the cycles corresponding to the cycles C, C', and C'' is contained in a 3-cycle.

If we continue this process, we arrive at regular c-partite tournaments with arbitrary large c which contain arcs that do not belong to any 3-cycle.

However, recently the author [5] showed that every arc of a regular c-partite tournament belongs to a 4-cycle, when $c \geq 6$. We even proved the following more general result.

Theorem 1.2 (Volkmann [5]) Let $V_1, V_2, \ldots, V_c$ be the partite sets of an almost regular c-partite tournament D. If $c \geq 6$ and $|V_1| = |V_2| = \ldots = |V_c|$, then every arc of D is contained in a 4-cycle, and the condition $c \geq 6$ is best possible.

In this paper we present the following supplement and extension of Theorem 1.2. Let D be an arbitrary almost regular c-partite tournament. If $c \geq 8$ or if $c = 7$ and there are at least two vertices in every partite set, then every arc of D is contained in a 4-cycle. Examples show that these conditions are best possible.

Further results on cycles containing a given arc in regular multipartite tournaments can be found in papers by Guo [1] and Guo and Kwak [2].

2. Preliminary results

The following results play an important role in our investigations. The first three lemmas can be found in a recent article of Tewes, Volkmann, and Yeo [3].

Lemma 2.1 If $V_1, V_2, \ldots, V_c$ are the partite sets of an almost regular c-partite tournament, then $||V_i| - |V_j|| \leq 2$ for $1 \leq i \leq j \leq c$.

Lemma 2.2 If D is an almost regular multipartite tournament, then

$$d^+(x), d^-(x) \geq \frac{|V(D)| - \alpha(D) - 1}{2}$$

for every vertex x of D.

Lemma 2.3 If D is an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| = r + 2$, then $|V(D)| - \alpha(D) = |V(D)| - r - 2$ is even.

The next corollary is an immediate consequence of Lemma 2.2 and Lemma 2.3.

Corollary 2.4 If D is an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| = r + 2$, then

$$d^+(x), d^-(x) \geq \frac{|V(D)| - \alpha(D)}{2} = \frac{|V(D)| - r - 2}{2}$$

for every vertex x of D.

Lemma 2.5 (Volkmann [4] 1999) If X is a vertex set of an almost regular digraph D, then

$$|X| \geq |d(X, V(D) - X) - d(V(D) - X, X)|.$$

3. Main results

Theorem 3.1 Let D be an almost regular c-partite tournament.

If $c \geq 8$, then every arc of D is contained in a 4-cycle.

If $c = 7$ and there are at least two vertices in every partite set, then every arc of D is contained in a 4-cycle.

Proof. Let $V_1, V_2, \ldots, V_c$ be the partite sets of D, $|V(D)| = n$, and $\alpha(D) = \alpha$. Let now $e = uv$ be an arbitrary arc of D, and suppose to the contrary that $e = uv$ does not belong to any 4-cycle. In the following we denote by $V(u)$ and $V(v)$ the partite sets of D, containing u and v, respectively.

If there are two vertices $x, y \in S := N^+(v) \cap N^-(u)$ such that $x \to y$, then $uvxyu$ is a 4-cycle through $e = uv$, a contradiction. Consequently, $S = \emptyset$ or S is a subset of one partite set, say $S \subseteq V(S)$. Let $|S| = s$ and define $W = N^-(u) - S$. If $R := V(D) - (N^+(v) \cup W \cup \{u, v\})$, then

$$|R| \leq n - d^-(u) - d^+(v) + s - 2. \tag{1}$$

Since $e = uv$ does not belong to a 4-cycle, we observe that there is no arc from $N^+(v)$ to W. Hence, according to Lemma 2.5, we deduce that

$$\begin{aligned} |W| &\geq |d(W, V(D) - W) - d(V(D) - W, W)| \\ &\geq |W| + d(W, N^+(v) \cup \{v\}) - |R||W|. \end{aligned}$$

Let $W = V_1' \cup V_2' \cup \ldots \cup V_k'$ such that $V_i' \neq \emptyset$ and assume, without loss of generality, that $V_i' \subseteq V_i$ for $i = 1, 2, \ldots, k$. This yields together with the last inequality

$$|R||W| \geq d(W, N^+(v) \cup \{v\}) \geq \sum_{i=1}^{k} |V_i'|(d^+(v) + 1 + |V_i'| - |V_i|). \tag{2}$$

If $\min_{1 \leq i \leq c}\{|V_i|\} = r$, then, in view of Lemma 2.1, we have $\alpha \leq r + 2$. If $|V_i| \leq r + j$ for some $j \in \{0, 1, 2\}$ and for all $i = 1, 2, \ldots, k$, then, because of $|V_i'| \geq 1$ for $i = 1, 2, \ldots, k$, it follows from (2)

$$|R||W| \geq \sum_{i=1}^{k} |V_i'|(d^+(v) + 2 - r - j) = |W|(d^+(v) + 2 - r - j),$$

and thus

$$|R| \geq d^+(v) - r + 2 - j, \tag{3}$$

when $|W| \neq 0$. However, we will show at once that $|W| = 0$ is never possible. Suppose that $|W| = 0$. This implies $d^-(u) = |S| \leq \alpha$. If $\alpha \leq r+1$, then Lemma 2.2 leads to

$$r+1 \geq \alpha \geq |S| = d^-(u) \geq \frac{n-\alpha-1}{2} \geq \frac{cr-r-2}{2}.$$

Therefore, we obtain $3r+4 \geq cr$, a contradiction to $c \geq 8$ and $c=7$ and $r \geq 2$. In the case $\alpha = r+2$, Corollary 2.4 and the fact that $n \geq cr+2$ yield the same contradiction.

If in addition, $k \geq 2$ in inequality (2) and $|V_i| = r+2$ for exactly one $i \in \{1,2,\ldots,k\}$, say $|V_1| = r+2$, then it follows from (2) together with $|V_i| \leq r+1$ for $i = 2,3,\ldots,k$

$$\begin{aligned} |R||W| &\geq \sum_{i=1}^{k} |V_i'|(d^+(v)+1+|V_i'|-|V_i|) \\ &\geq |W|(d^+(v)+2) - |V_1'|(r+2) - \sum_{i=2}^{k} |V_i'|(r+1) \\ &= |W|(d^+(v)+1-r) - |V_1'|. \end{aligned}$$

Since $k \geq 2$, we have $|W| > |V_1'|$, and so the last inequality yields

$$|R| \geq d^+(v) - r + 1. \tag{4}$$

Now we start with the case that $\alpha = r$. Combining this condition with (1) and (3) for $j=0$, we find

$$n-d^-(u)-d^+(v)+r-2 \geq n-d^-(u)-d^+(v)+s-2 \geq |R| \geq d^+(v)-r+2$$

and hence, by Lemma 2.2

$$n+2r-4 \geq 2d^+(v)+d^-(u) \geq \frac{3}{2}(n-r-1).$$

This implies $7r-5 \geq n = cr$, a contradiction to $c \geq 7$.

Next we assume that $\alpha = r+1$. If $s \leq r$, then it follows from (1) and (3) with $j=1$

$$n-d^-(u)-d^+(v)+r-2 \geq n-d^-(u)-d^+(v)+s-2 \geq |R| \geq d^+(v)-r+1$$

and hence, by Lemma 2.2

$$n+2r-3 \geq 2d^+(v)+d^-(u) \geq \frac{3}{2}(n-r-2).$$

This implies $7r \geq n \geq cr+1$, a contradiction to $c \geq 7$.

If $s = r + 1$ and $|V_i| \le r$ for $i = 1, 2, \dots, k$, then (1) and (3) with $j = 0$ yield

$$n - d^-(u) - d^+(v) + r - 1 = n - d^-(u) - d^+(v) + s - 2 \ge |R| \ge d^+(v) - r + 2$$

and hence, by Lemma 2.2

$$n + 2r - 3 \ge 2d^+(v) + d^-(u) \ge \frac{3}{2}(n - r - 2).$$

This implies again the contradiction $7r \ge n \ge cr + 1$.

Let now $s = r + 1$ and $|V_i| = r + 1$ for at least one $i = 1, 2, \dots, k$, say $|V_1| = r + 1$. With respect to $|V(S)| = |S| = \alpha = r + 1$ and $|V_1| = r + 1$, we observe that $V(S) \cap V_1 = \emptyset$ and so we have $n \ge cr + 2$. We deduce from (1) and (3) with $j = 1$

$$n - d^-(u) - d^+(v) + r - 1 = n - d^-(u) - d^+(v) + s - 2 \ge |R| \ge d^+(v) - r + 1.$$

According to Lemma 2.2, we obtain

$$n + 2r - 2 \ge 2d^+(v) + d^-(u) \ge \frac{3}{2}(n - r - 2)$$

and thus $7r + 2 \ge n$, a contradiction for $n \ge cr + 3$ or $c \ge 8$. Therefore, it remains the case $c = 7$, $r \ge 2$, and $n = 7r + 2$. Since $V(S) \cap V(u) = \emptyset$ and $V_1 \cap V(u) = \emptyset$, we obtain $|V(u)| = r$. This leads to $d^+(u) + d^-(u) = 6r + 2$, and because of $i_g(D) \le 1$, we conclude that $d^+(u) = d^-(u) = 3r + 1$. Hence, (1), (3), and Lemma 2.2 imply the contradiction

$$n + 2r - 2 \ge 2d^+(v) + d^-(u) \ge n - r - 2 + 3r + 1.$$

It remains the case that $\alpha = r + 2$. This condition implies $n \ge cr + 2$. In the following we investigate four cases.

Case 1. Let $\alpha = r + 2$ and $s \le r - 1$. It follows from (1) and (3) with $j = 2$

$$n - d^-(u) - d^+(v) + r - 3 \ge n - d^-(u) - d^+(v) + s - 2 \ge |R| \ge d^+(v) - r.$$

Because of Corollary 2.4, we obtain

$$n + 2r - 3 \ge 2d^+(v) + d^-(u) \ge \frac{3}{2}(n - r - 2),$$

and this leads to the contradiction $7r \ge n \ge cr + 2$.

Case 2. Let $\alpha = r + 2$ and $s = r$. Analogously to Case 1, it follows

$$n + 2r - 2 \ge 2d^+(v) + d^-(u) \ge \frac{3}{2}(n - r - 2)$$

and thus $7r+2 \geq n$. This is a contradiction, when $n \geq cr+3$ or $c \geq 8$. In the remaining case $c=7$, $r \geq 2$, and $n = 7r+2$, we observe that $|V(u)| = r$ or $|V(v)| = r$ and therefore $d^-(u) \geq 3r+1$ or $d^+(v) \geq 3r+1$. Hence, we deduce from (1), (3), and Corollary 2.4 the contradiction $n+2r-2 \geq 2d^+(v)+d^-(u) \geq n+2r-1$.

Case 3. Let $\alpha = r+2$ and $s = r+1$. If $|V_i| \leq r$ for all $i = 1,2,\ldots,k$, then (1), (3) with $j=0$, and Corollary 2.4 lead to the contradiction $7r \geq n \geq cr+2$. In the case $|V_i| \leq r+1$ for all $i=1,2,\ldots,k$ and $|V_i| = r+1$ for at least one $i \in \{1,2,\ldots,k\}$, we obtain similarly the contradiction $7r+2 \geq n \geq cr+3$.

It remains the case that $|V_i| = r+2$ for at least one $i \in \{1,2,\ldots,k\}$, say $|V_1| = r+2$. It follows from (1) and (3) with $j=2$

$$n+2r-1 \geq 2d^+(v)+d^-(u). \tag{5}$$

In view of Corollary 2.4, we find $7r+4 \geq n$. This is a contradiction for $n \geq cr+5$, or $c \geq 10$, or $c=9$ and $n \geq 9r+3$, or $c=9$ and $r \geq 2$, or $c=8$ and $n \geq 8r+4$, or $c=8$ and $r \geq 3$. Consequently, there remain four subcases.

Subcase 3.1. Let $c=9$, $r=1$, and $n=cr+2=11$. Since $|V_1| = r+2 = 3$, we have $|V(u)| = r = 1$. This implies $d^-(u) \geq 5$, and so (5) and Corollary 2.4 yield the contradiction $12 = n+2r-1 \geq 2d^+(v)+d^-(u) \geq 8+5 = 13$.

Subcase 3.2. Let $c=8$, $r=1$, and $10 = 8r+2 \leq n \leq 8r+3 = 11$. Because of Lemma 2.3, the digraph D cannot consist of 10 vertices. In the remaining case $n=11$, we note that $V(u) \cap V(v) = \emptyset$, $V(u) \cap V(S) = \emptyset$, $V(v) \cap V(S) = \emptyset$, and $|V(S)| \geq |S| = r+1 = 2$. This implies $|V(u)| = r = 1$ or $|V(v)| = r = 1$ and thus $d^+(u) \geq 5$ or $d^-(u) \geq 5$. Combining this with (5) and Corollary 2.4, we obtain the contradiction $12 = n+2r-1 \geq 2d^+(v)+d^-(u) \geq 13$.

Subcase 3.3. Let $c=8$, $r=2$, and $18 = 8r+2 \leq n \leq 8r+3 = 19$. Because of Lemma 2.3, the case $n=19$ is not possible. If $n=18$, then $|V_1| = r+2 = 4$ implies $|V(u)| = r = 2$ and hence $d^-(u) \geq 8$. We deduce from (5) the contradiction $21 = n+2r-1 \geq 2d^+(v)+d^-(u) \geq 22$.

Subcase 3.4. Let $c=7$, $r \geq 2$, and $7r+2 \leq n \leq 7r+4$. Because of Lemma 2.3, the case $n = 7r+3$ is impossible.

Let $n = 7r+2$. Then, $|V(S)| = r+2$ and thus $|V(u)| = |V(v)| = r$, and so $d^+(v), d^-(u) \geq 3r+1$. Using (5), we obtain the contradiction $9r+1 = n+2r-1 \geq 2d^+(v)+d^-(u) \geq 9r+3$.

Now let $n = 7r+4$. If $|V(u)| = r$ or $|V(v)| = r$, then $d^-(u) \geq 3r+2$ or $d^+(v) \geq 3r+2$, and (5) together with Corollary 2.4 lead to the contradiction $9r+3 \geq 2d^+(v)+d^-(u) \geq 9r+4$. Therefore, we assume now that $|V(u)| \geq r+1$ and $|V(v)| \geq r+1$. Since $|S| \geq r+1$ and $|V_1| = r+2$, we deduce that $V_1 = V(S)$ or $V_1 = V(v)$.

If $V_1 = V(S)$, then $|V_1'| = 1$ and $|V(u)| = |V(v)| = r+1$. In the case $W = V_1'$, Corollary 2.4 implies the contradiction $r+2 = d^-(u) \geq 3r+1$. Otherwise, there are vertices of at least two partite sets in W. Then, (1), (4), and Corollary 4 yield the contradiction $9r+2 = n+2r-2 \geq 2d^+(v) + d^-(u) \geq 9r+3$.

If $V_1 = V(v)$, then necessarily $|V(S)| = |V(u)| = r+1$ and, since $v \in V_1$ but $v \notin V_1'$, we see that $|V_1'| \leq r+1$. In the case $W = V_1'$, we conclude $d^-(u) \leq 2r+2$ and therefore $d^+(u) \geq n-(2r+2)-(r+1) = 4r+1$. This implies $d^+(u) - d^-(u) \geq 2r-1 \geq 2$ for $r \geq 2$, a contradiction to the hypothesis $i_g(D) \leq 1$. Otherwise, there are vertices of at least two partite sets in W, and we obtain a contradiction analogously to the case $V_1 = V(S)$.

Case 4. Let $\alpha = r+2$ and $s = r+2$. If $|V_i| \leq r$ for all $i = 1, 2, \ldots, k$, then (1), (3) with $j = 0$, and Corollary 2.4 lead to $7r+2 \geq n$, a contradiction, when $n \geq cr+3$. In the remaining case $n = 7r+2$, we observe that $|V(u)| = r$ and so $d^-(u) \geq 3r+1$, which leads to the contradiction $n + 2r - 2 \geq 2d^+(v) + d^-(u) \geq n - r - 2 + 3r + 1$.

In the case $|V_i| \leq r+1$ for all $i = 1, 2, \ldots, k$ and $|V_i| = r+1$ for at least one $i \in \{1, 2, \ldots, k\}$, say $|V_1| = r+1$, we have $n \geq cr+3$. It follows from (1) and (3) with $j = 1$

$$n + 2r - 1 \geq 2d^+(v) + d^-(u). \tag{6}$$

According to Corollary 2.4, we deduce that $7r+4 \geq n$. This is a contradiction for $n \geq cr+5$, or $c \geq 9$, or $c = 8$ and $n \geq 8r+4$, or $c = 8$ and $r \geq 2$. Consequently, there remain two subcases.

Subcase 4.1. Let $c = 8$, $r = 1$, and $11 = 8r+3 = n$. The condition $|V(S)| = |S| = r+2 = 3$ yields $V(S) \cap V_1 = \emptyset$. Therefore, $|V_1| = r+1 = 2$ leads to $|V(u)| = r = 1$ and hence $d^-(u) \geq 5$. Now we obtain from (6) the contradiction $12 \geq 2d^+(v) + d^-(u) \geq 13$.

Subcase 4.2. Let $c = 7$, $r \geq 2$, and $7r+3 \leq n \leq 7r+4$. Because of Lemma 2.3, the case $n = 7r+3$ is not possible. If $n = 7r+4$, then $|V(S)| = |S| = r+2$ and $|V_1| = r+1$ show that $|V(u)| \leq r+1$. In the case $|V(u)| = r$ or $|V(v)| = r$, we see that $d^-(u) \geq 3r+2$ or $d^+(v) \geq 3r+2$, and this yields together with (6) and Corollary 2.4 the contradiction $9r+3 \geq 2d^+(v) + d^-(u) \geq 9r+4$. Therefore, we assume now that $|V(u)| = r+1$ and $|V(v)| \geq r+1$. But this implies $V_1 = V(v)$.

Let $W = V_1' \subseteq V_1 = V(v)$. Since $v \notin V_1'$, we conclude $d^-(u) \leq 2r+2$ and thus $d^+(u) \geq n-(2r+2)-(r+1) = 4r+1$. This leads to $d^+(u) - d^-(u) \geq 2r-1 \geq 2$ for $r \geq 2$, a contradiction to $i_g(D) \leq 1$.

Otherwise, there are vertices of at least two partite sets in W. Since $|V_i'| \leq r$ for all $i = 2, 3, \ldots, k$, we obtain, analogously to (4), the estimate $|R| \geq d^+(v) - r + 2$. Using (1) and Corollary 2.4, we deduce $9r+2 = n+2r-2 \geq 2d^+(v) + d^-(u) \geq 9r+3$, a contradiction.

Finally, we investigate the case that $|V_i| = r + 2$ for at least one $i \in \{1, 2, \ldots, k\}$, say $|V_1| = r+2$. Then we have $n \geq cr+4$, and it follows from (1) and (3) with $j = 2$

$$n + 2r \geq 2d^+(v) + d^-(u). \tag{7}$$

With the help of Corollary 2.4, we deduce that $7r + 6 \geq n$. This yields a contradiction for $n \geq cr + 7$, or $c \geq 10$, or $c = 9$ and $n \geq 9r + 5$, or $c = 9$ and $r \geq 2$, or $c = 8$ and $n \geq 8r+6$, or $c = 8$ and $r \geq 3$. Thus, there remain four subcases.

Subcase 4.3. Let $c = 9$, $r = 1$, and $13 = 9r + 4 = n$. Since $|V(S)| = r + 2 = 3$ and $|V_1| = r + 2 = 3$, we see that $|V(u)| = r = 1$ and hence $d^-(u) \geq 6$. Thus, inequality (7) leads to the contradiction $15 = n + 2r \geq 2d^+(v) + d^-(u) \geq 16$.

Subcase 4.4. Let $c = 8$, $r = 1$, and $12 = 8r + 4 \leq n \leq 8r + 5 = 13$. Because of Lemma 2.3, the case $n = 12$ is not possible. Let now $n = 13$. If $|V(u)| = r = 1$ or $|V(v)| = r = 1$, then $d^-(u) = 6$ or $d^+(v) = 6$, and we obtain by (7) the contradiction $15 = n + 2r \geq 2d^+(v) + d^-(u) \geq 16$. Therefore, it remains the case $|V(u)| = r + 1 = 2$ and $V_1 = V(v)$. If $W = V_1' \subset V_1 = V(v)$, then because of $d^+(v) \geq 5$ and $|V(u)| = 2$, there exists a vertex $z \in N^+(v)$ such that $z \notin (V(S) \cup V(u) \cup V(v))$. By the assumption that $e = uv$ is not contained in a 4-cycle, we deduce that $N^-(z) \supseteq (N^-(u) \cup \{u, v\})$, a contradiction to $i_g(D) \leq 1$. If there are vertices of at least two partite sets in W, then it follows from (1), (4), and Corollary 2.4 that $14 = n + 2r - 1 \geq 2d^+(v) + d^-(u) \geq 15$, a contradiction.

Subcase 4.5. Let $c = 8$, $r = 2$, and $20 = 8r + 4 \leq n \leq 8r + 5 = 21$. Because of Lemma 2.3, we see that $n \neq 21$, and analogously to Subcase 4.3, we obtain a contradiction, when $n = 20$.

Subcase 4.6. Let $c = 7$, $r \geq 2$, and $7r + 4 \leq n \leq 7r + 6$. Because of Lemma 2.3, the case $n = 7r + 5$ is not possible.

Firstly, we discuss the case $n = 7r + 4$. Since $|V(S)| = |V_1| = r + 2$, we conclude that $|V(u)| = r$ and so $d^-(u) = 3r + 2$. If $|V(v)| = r$, then $d^+(v) = 3r + 2$, and consequently, (7) leads to the contradiction $9r + 4 = n + 2r \geq 2d^+(v) + d^-(u) = 9r + 6$. Therefore, it remains the case that $|V(v)| \geq r + 1$ and thus $V(v) = V_1$. If $W = V_1' \subset V_1$, then $d^-(u) \leq 2r + 3$ and thus $d^+(u) \geq 7r+4-(2r+3)-r = 4r+1$. This implies $d^+(u)-d^-(u) \geq 2r - 2 \geq 2$ for $r \geq 2$, a contradiction to $i_g(D) \leq 1$. Now we assume that there are vertices of at least two partite sets in W. Combining (1), (4), the fact that $d^-(u) = 3r + 2$, and Corollary 2.4, we deduce the contradiction $9r + 3 \geq 2d^+(v) + d^-(u) \geq 9r + 4$.

Finally, let $n = 7r + 6$. If $|V(u)| = r$ or $|V(v)| = r$, then (7) and Corollary 2.4 give the contradiction $9r + 6 \geq 2d^+(v) + d^-(u) \geq 9r + 7$. Therefore, it remains the case that $|V(u)| \geq r + 1$ and $|V(v)| \geq r + 1$. We distinguish again two subcases.

Let $V(v) \cap V_1 = \emptyset$. This implies $|V(v)| = |V(u)| = r+1$. If $W \subseteq V_1$ and $|W| < |V_1|$, then $d^-(u) \leq 2r+3$ and thus $d^+(u) \geq n-(2r+3)-(r+1) = 4r+2$, and hence $d^+(u) - d^-(u) \geq 2r-1 \geq 2$ for $r \geq 2$, a contradiction to $i_g(D) \leq 1$. If $W = V_1$, then let $z \in W$. By the assumption that $e = uv$ is not contained in a 4-cycle, we deduce that $N^+(z) \supseteq (N^+(v) \cup \{u, v\})$, a contradiction to $i_g(D) \leq 1$. If there are vertices of at least two partite sets in W, then, because of $|V(u)| = r+1$, it follows from (1), (4), and Corollary 2.4 that $9r+5 \geq 2d^+(v) + d^-(u) \geq 9r+6$, a contradiction.

Let $V(v) = V_1$. If $W \subset V_1$, then $d^-(u) \leq 2r+3$ and thus $d^+(u) \geq n-(2r+3)-(r+2) = 4r+1$. Consequently, $d^+(u) - d^-(u) \geq 2r-2 \geq 2$ for $r \geq 2$, a contradiction to $i_g(D) \leq 1$. If there are vertices of at least two partite sets in W, then, because of $|V(u)| \geq r+1$, it follows from (1), (4), and Corollary 2.4 that $9r+5 \geq 2d^+(v) + d^-(u) \geq 9r+6$, a contradiction.

Since we have discussed all possible cases, the proof is complete. □

Example 3.2 Let $V_1 = \{u, u_2\}$, $V_2 = \{v, v_2\}$, $V_3 = \{w_1, w_2, w_3\}$, $V_4 = \{x\}$, $V_5 = \{y\}$, $V_6 = \{z\}$, and $V_7 = \{a\}$ be the partite sets of a 7-partite tournament such that $u \to v \to u_2 \to \{a, x, y, z\} \to v_2 \to u \to \{a, x, y, z\} \to v \to V_3 \to u$, $v_2 \to u_2$, $v_2 \to V_3 \to u_2$, $w_1 \to a \to x \to y \to z \to a \to y \to w_1 \to z \to x \to w_1$, $w_2 \to z \to w_3 \to a \to w_2 \to x \to w_3 \to y \to w_2$ (see Figure 1). The resulting 7-partite tournament is almost regular, however, the arc uv is not contained in a 4-cycle. Consequently, the condition $c \geq 8$ in Theorem 3.1 is best possible.

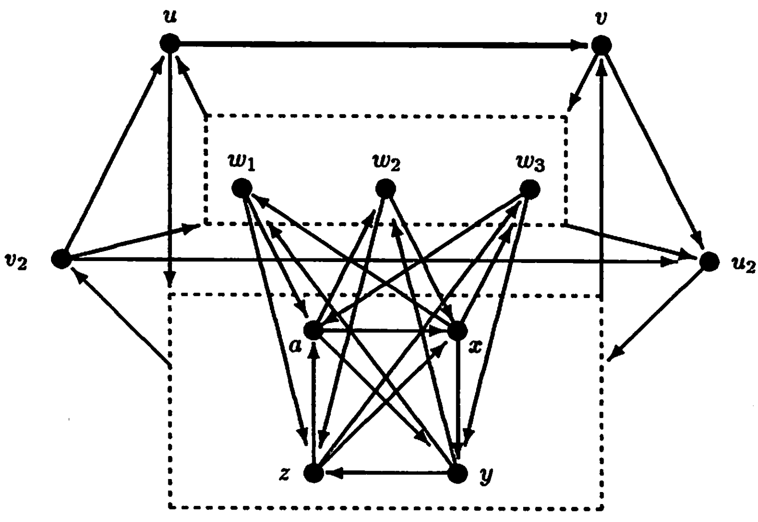

Figure 1: An almost regular 7-partite tournament with the property that the arc uv is not contained in a 4-cycle

Since there are only a finite number of almost regular 7-partite tournaments with the partite sets $V_1, V_2, \ldots, V_7$ such that $\min_{1\le i\le 7}\{|V_i|\} = 1$, Theorem 3.1 leads immediately to the following corollary.

Corollary 3.3 Every arc of an almost regular 7-partite tournament is contained in a 4-cycle, except for a finite number of such multipartite tournaments. In addition, Example 3.2 shows that such exceptions really exist.

Example 3.4 Let $V_1 = \{u\} \cup V_1'$ with $|V_1'| = 2$, Let $V_2 = \{v\} \cup V_2'$ with $|V_2'| = 2$, $V_3 = V_3' \cup V_3''$ with $|V_3'| = |V_3''| = 2$, and V_4, V_5, V_6 with $|V_4| = |V_5| = |V_6| = 2$ with $V_4 = \{x, y\}$ be the partite sets of a 6-partite tournament such that $u \to v \to V_1' \to (V_4 \cup V_5 \cup V_6) \to V_2' \to u \to (V_4 \cup V_5 \cup V_6) \to v$, $V_2' \to V_3 \to u$, $v \to V_3 \to V_1'$, $V_2' \to V_1'$ $V_4 \to V_5 \to V_6 \to V_4$, and $V_3' \to (V_6 \cup \{x\}) \to V_3'' \to (V_5 \cup \{y\}) \to V_3'$ (see Figure 2). The resulting 6-partite tournament is almost regular with at least two vertices in every partite set, however, the arc uv is not contained in a 4-cycle. Therefore, the condition $c \ge 7$ in the second part of Theorem 3.1 is also best possible.

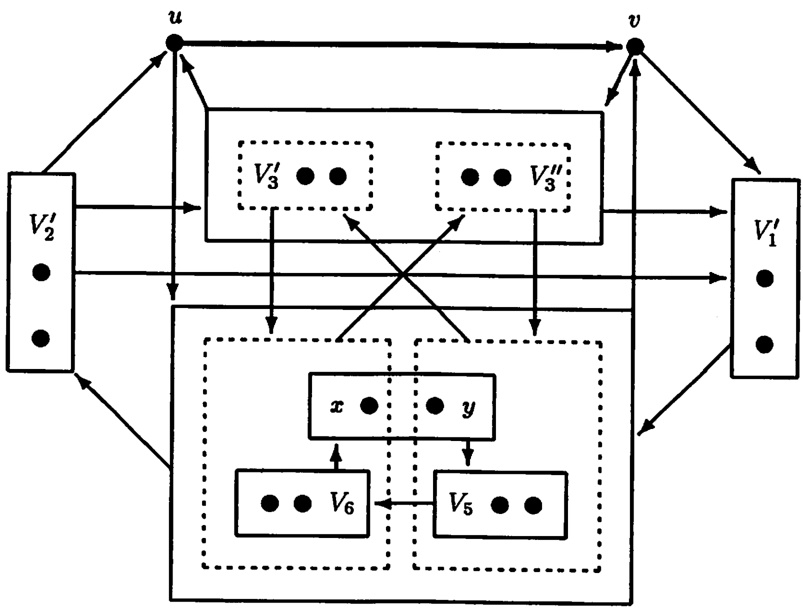

Figure 2: An almost regular 6-partite tournament with the property that the arc uv is not contained in a 4-cycle

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