λ-Designs on $8p + 1$ Points

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Abstract

A λ-design on v points is a set of v distinct subsets (blocks) of a v-element set (points) such that any two different blocks meet in exactly λ points and not all of the blocks have the same size. Ryser's and Woodall's λ-design conjecture states that all λ-designs can be obtained from symmetric designs by a certain complementation procedure. The main result of the present paper is that the λ-design conjecture is true when $v = 8p + 1$, where $p \equiv 1$ or $7 \pmod 8$ is a prime number.

1 Introduction

Definition 1.1 *Given integers* λ *and* v *satisfying* $0 < \lambda < v$, *a* λ*-design* D *on* v *points is a pair* $(X, \mathcal{B})$, *where* X *is a set of cardinality* v *whose elements are called points and* $\mathcal{B}$ *is a set of* v *distinct subsets of* X *whose elements are called blocks, such that*

(i) For all blocks $A, B \in \mathcal{B}$, $A \neq B$, $|A \cap B| = \lambda$, *and*

(ii) There exist blocks $A, B \in \mathcal{B}$ *with* $|A| \neq |B|$.

λ-designs were first defined by Ryser [12], [13] and Woodall [22]. The only known examples of λ-designs are obtained from symmetric designs by the following complementation procedure. Let $(X, \mathcal{A})$ be a symmetric (v, k, μ)-design with $\mu \neq k/2$ and fix a block $A \in \mathcal{A}$. Put $\mathcal{B} = \{A\} \cup \{A \Delta B : B \in \mathcal{A}, B \neq A\}$, where Δ denotes the symmetric difference of sets (we refer to this procedure as complementing with respect to the block A). Then an elementary counting argument shows that $(X, \mathcal{B})$ is a λ-design on v points with $\lambda = k - \mu$. Any λ-design obtained in this manner is called a *type*-1 λ-design.

The λ-*design conjecture* of Ryser [12], [13] and Woodall [22] states that all λ-designs are type-1. The conjecture was proven for $\lambda = 1$ by deBruijn and Erdős [5], for $\lambda = 2$ by Ryser [12], for $3 \leq \lambda \leq 9$ by Bridges and Kramer [1], [10], [3], for $\lambda = 10$ by Seress [15], for $\lambda = 14$ by Tsaur [19], [4], and for all remaining $\lambda \leq 34$ by Weisz [20]. S. S. Shrikhande and Singhi [17] proved the conjecture for prime λ and Seress [16] proved it when λ is twice a prime.

Investigating the conjecture as a function of v rather than λ, Ionin and M. S. Shrikhande [8], [9] proved the conjecture for $v = p+1, 2p+1, 3p+1$, and $4p+1$, where p is any prime, and Hein [6], [7] proved it for $v = 5p+1$, where $p \not\equiv 2$ or $8 \pmod{15}$ is prime. The conjecture has also been verified by computer for all $v \leq 85$ [23]. Continuing along these lines, in the present paper we will prove the following result.

Theorem 1.2 *All λ-designs on $v = 8p+1$ points, $p \equiv 1$ or 7 (mod 8) a prime, are type-1.*

The method employed to prove Theorem 1.2 is a slight extension of the method of Ionin and M. S. Shrikhande developed in [8] and [9] and used in [6], [7]. However, whereas they were always able to reduce to the case of designs having at most two distinct block sizes, we examine a minimal counterexample and will have to deal with designs potentially possessing three different block sizes.

2 Preliminary results

Definition 2.1 *Given a λ-design $D = (X, \mathcal{B})$ and a point $x \in X$, the replication number of x is the number of blocks $A \in \mathcal{B}$ which contain x.*

Ryser [12] and Woodall [22] independently proved the following theorem concerning these replication numbers.

Theorem 2.2 *If $D = (X, \mathcal{B})$ is a λ-design on v points, then there exist integers $r > 1$ and $r^* > 1$, $r \neq r^*$, such that every point $x \in X$ has replication number r or r^* and $r + r^* = v + 1$. In addition, the integers r and r^* satisfy the equation*

$$\frac{1}{\lambda} + \sum_{A \in \mathcal{B}} \frac{1}{|A| - \lambda} = \frac{(v-1)^2}{(r-1)(r^*-1)}. \quad (1)$$

We will also need the following three theorems concerning the integers r and r^*. The first was stated without proof in [23]. For a proof see [14].

Theorem 2.3 *A λ-design on v points with replication numbers r and r^* is type-1 if and only if $r(r-1)/(v-1)$ or $r^*(r^*-1)/(v-1)$ is an integer.*

Theorem 2.4 [8], [9] *Let D be a λ-design with replication numbers r and r^* and put $g = \gcd(r-1, r^*-1)$. If $g = 1, 2$, or 4, then D is type-1.*

Theorem 2.5 [18], [16] *Let D be a λ-design with replication numbers r and r^*, $r > r^*$. Put $g = \gcd(r-1, r^*-1)$. If $\gcd(\lambda, (r-r^*)/g) = 1$ or 2, then D is type-1.*

Additionally, we will need the following three theorems concerning the validity of the λ-design conjecture for certain values of λ.

Theorem 2.6 [5], [12], [1], [10], [3], [15], [19], [4], [20] *The λ-design conjecture is true for $\lambda \leq 10$ and $\lambda = 12, 14, 15, 16$, and 18.*

Theorem 2.7 [17] *The λ-design conjecture is true for prime λ.*

Theorem 2.8 [16] *The λ-design conjecture is true when λ is twice a prime.*

3 The Ionin-Shrikhande method

Let $D = (X, \mathcal{B})$ be a λ-design on v points. Then Theorem 2.2 implies that every point of D has replication number r or r^* for some integers $r \neq r^*$. Therefore, the underlying set X of our λ-design is partitioned into two subsets, E and E^*, of points having replication numbers r and r^*, respectively. Let $|E| = e$ and $|E^*| = e^*$, so $e + e^* = v$. Also, for any block $A \in \mathcal{B}$, put $\tau_A = |A \cap E|$ and $\tau_A^* = |A \cap E^*|$, so $\tau_A + \tau_A^* = |A|$. We will frequently use the trivial inequalities $0 \leq \tau_A \leq e$ for all A.

The following simple relation among these parameters is the starting point of the Ionin-Shrikhande method developed in [8] and [9].

Lemma 3.1 *Let $D = (X, \mathcal{B})$ be a λ-design on v points with replication numbers r and r^*. Then the following relation holds for all blocks $A \in \mathcal{B}$:*

$$(r-1)(|A| - 2\tau_A) = (v-1)(|A| - \lambda - \tau_A). \tag{2}$$

Proof: Fixing a block $A \in \mathcal{B}$, we will count in two different ways all of the pairs $(x, B) \in X \times (\mathcal{B} \setminus A)$ such that $x \in A \cap B$. This gives us the equation $\tau_A(r-1) + \tau_A^*(r^*-1) = \lambda(v-1)$, which is easily transformed into equation (2).

□

Now, let $g = \gcd(r-1, r^*-1)$. Then, since $(r-1)+(r^*-1) = v-1$ by Theorem 2.2, we also have $g = \gcd(r-1, v-1) = \gcd(r^*-1, v-1)$. We put

$$q = \frac{v-1}{g}. \tag{3}$$

Then, since $\gcd((r-1)/g, q) = 1$, equation (2) implies that q divides $|A| - 2\tau_A$ for all blocks $A \in \mathcal{B}$. Therefore, for each block A we define an integer σ_A by

$$q\sigma_A = |A| - 2\tau_A. \tag{4}$$

Next, we define the quantity

$$s = \sum_{A \in \mathcal{B}} \sigma_A. \tag{5}$$

Also, equations (2) and (4) imply that

$$\tau_A = \lambda - \frac{r^*-1}{g}\sigma_A \tag{6}$$

and

$$\tau_A^* = \lambda + \frac{r-1}{g}\sigma_A \tag{7}$$

for all A. Adding equations (6) and (7) we obtain

$$|A| = 2\lambda + \frac{r-r^*}{g}\sigma_A \tag{8}$$

for all A.

Remark 3.2 *Note that equation (8) implies that for any two blocks $A, B \in \mathcal{B}$, $|A| = |B|$ if and only if $\sigma_A = \sigma_B$.*

The next three equations are easily verified:

$$\sum_{A \in \mathcal{B}} |A| = er + e^*r^*, \tag{9}$$

$$\sum_{A \in \mathcal{B}} \tau_A = er, \tag{10}$$

and

$$\sum_{A \in \mathcal{B}} \tau_A^* = e^* r^*. \tag{11}$$

Equations (4), (9), and (10) then imply that $sq = \sum_{A \in \mathcal{B}} (|A| - 2\tau_A) = e^* r^* - er = (v - e)(v - r + 1) - er$, which can be transformed into

$$sq = gq(gq - e - r + 3) - (2e + r - 2). \tag{12}$$

Equation (12) then implies that q divides $2e + r - 2$. Therefore, we define a positive integer m by

$$qm = 2e + r - 2. \tag{13}$$

Similarly, equations (4), (9), and (11) imply that q divides $2e^* + r^* - 2$. Thus, we define a positive integer m^* by

$$qm^* = 2e^* + r^* - 2. \tag{14}$$

Adding equations (13) and (14), we obtain

$$m + m^* = 3g. \tag{15}$$

Finally, equations (12), (13), and (15) imply that

$$s = g^2 q - g(e + r) + 3g - m. \tag{16}$$

Remark 3.3 *Upon further manipulation of the above equations, we eventually arrive at*

$$(r - r^*)(m^* - m) = g[v - (4\lambda - 1)]. \tag{17}$$

Note that equation (17) and the fact that $r \neq r^*$ *imply that* $v = 4\lambda - 1$ *if and only if* $m = m^*$.

The next lemma establishes formulae for e and r in terms of the parameters λ, g, q, and m. They follow easily from equations (13) and (17).

Lemma 3.4 [8] *If D is a λ-design on $v \neq 4\lambda - 1$ points, then*

$$e = \frac{g\lambda - (g-m)^2 q + g - m}{3g - 2m}$$

and

$$r = \frac{(2g-m)(gq+2) - 2g\lambda}{3g - 2m}.$$

The next result gives a way of constructing new λ-designs from old ones by complementing with respect to a fixed block. For a proof see [8].

Remark 3.5 *In what follows, if we complement with respect to the block A, the parameters of the new design will be denoted by $\lambda(A)$, $r(A)$, $m(A)$, etc.*

Lemma 3.6 *Let $D = (X, \mathcal{B})$ be a λ-design on v points with replication numbers r and r^* and let $A \in \mathcal{B}$. Put*

$$\mathcal{B}(A) = \{A\} \cup \{A \Delta B : B \in \mathcal{B}, B \neq A\}.$$

Denote by $D(A)$ the complemented set system $(X, \mathcal{B}(A))$. Then we have

(*i*) *If $A = E$ or E^*, then $D(A)$ is a symmetric $(v, |A|, |A| - \lambda)$-design,*

(*ii*) *If $A \neq E$ and $A \neq E^*$, then $D(A)$ is a $\lambda(A)$-design on v points with $r(A) = r$, $r^*(A) = r^*$, and $m(A) = m + 2\sigma_A$, where $\lambda(A) = |A| - \lambda$,*

(*iii*) *If $A \neq E$ and $A \neq E^*$ and D is type-1, then $D(A)$ is also type-1, and*

(*iv*) *$(D(A))(A) = D$.*

Finally, we will require the following simple lemma.

Lemma 3.7 *Let g be a fixed positive integer and suppose that there exists a non-type-1 λ-design with replication numbers r and r^* and $\gcd(r-1, r^*-1) = g$. Let D be such a design with minimal λ. Then all of the blocks of D have size at least 2λ.*

Proof: Suppose that D has a block A of size less than 2λ. Complementing with respect to A, we obtain the new design $D(A)$. Now, $D(A)$ is non-type-1 and has $g(A) = g$ by Lemma 3.6 (ii), (iii), and (iv). Also, $D(A)$ has new λ-value $\lambda(A) = |A| - \lambda$. However, $|A| - \lambda < \lambda$ since $|A| < 2\lambda$, contradicting the minimality of D's λ-value.

□

4 λ-designs with $g = 8$

We are now in a position to prove our main result. In what follows, the computer program Mathematica [21] was used extensively to carry out computations.

Theorem 4.1 *Let $D = (X, \mathcal{B})$ be a λ-design on v points with replication numbers r and r^*. If $g = \gcd(r-1, r^*-1) = 8$, then D is type-1.*

Proof: Suppose that Theorem 4.1 is false. Then there exists a non-type-1 λ-design with $g = 8$. Let $D = (X, \mathcal{B})$ be such a design with minimal λ. Then by Lemma 3.7, we know that $|A| \geq 2\lambda$ for all blocks $A \in \mathcal{B}$. We also know that $\lambda \geq 20$ by Theorems 2.6, 2.7, and 2.8.

By equation (3), we may write $v = 8q + 1$. For each integer i, let a_i denote the number of blocks $A \in \mathcal{B}$ with $\sigma_A = i$. We will frequently use the trivial fact that $a_i \geq 0$ for all i. Since the number of blocks is equal to the number of points, we clearly have

$$\sum_{i \in \mathbb{Z}} a_i = 8q + 1. \tag{18}$$

Also, equations (5), (16), and (17) and the formulae of Lemma 3.4 imply that

$$\sum_{i \in \mathbb{Z}} i a_i = \frac{(4q+1)(m^2 - 24m + 128) + 32\lambda}{12 - m} \tag{19}$$

if $m \neq 12$.

Next, equation (4) implies that for any block $A \in \mathcal{B}$, we have $|A| = 2\tau_A + q\sigma_A$. Using this and the formulae of Lemma 3.4, equation (1) is transformed into

$$\sum_{i \in \mathbb{Z}} \frac{(m-12)a_i}{\lambda(m-12) + i(2\lambda - 4q - 1)} =$$

$$\frac{4(m-12)^2 q^2}{[q(m-8) - 2\lambda + 1][q(m-16) + 2\lambda - 1]} - \frac{1}{\lambda} \tag{20}$$

if $m \neq 12$.

Now, since 8 divides $r - 1$, r is odd and equation (13) implies that m is odd as well. Also, equation (15) implies that $m + m^* = 24$. Without loss of generality, we may assume that $m \leq m^*$. Therefore, $m = 1, 3, 5, 7, 9$, or 11.

Case 1: $m = 1$.

In this case, the formulae of Lemma 3.4 imply that $e = (8\lambda - 49q + 7)/22$, $r = (60q - 8\lambda + 15)/11$, and $r^* = (28q + 8\lambda + 7)/11$. Also, equation (6) implies that $\tau_A = \lambda - [(7q + 2\lambda - 1)/22]\sigma_A$ for any block $A \in \mathcal{B}$. Then the inequalities $0 \leq \tau_A \leq e$ imply that $7 \leq \sigma_A \leq 22\lambda/(7q + 2\lambda - 1)$ for all A. Now, 8 divides $r - 1$ and $r > 1$, so $r \geq 9$. This gives us the inequality $q \geq (2\lambda + 21)/15$. Combining the last two inequalities, we obtain that $\sigma_A = 7$ for all A. Therefore, by Remark 3.2, all blocks have the same cardinality, contradicting the definition of a λ-design.

Case 2: $m = 3$.

In this case, Lemma 3.4 implies that $e = (8\lambda - 25q + 5)/18$, $r = (52q - 8\lambda + 13)/9$, and $r^* = (20q + 8\lambda + 5)/9$. Also, equation (6) implies that $\tau_A = \lambda - [(5q + 2\lambda - 1)/18]\sigma_A$ for any block A. Then the inequalities $0 \leq \tau_A \leq e$ imply that $5 \leq \sigma_A \leq 18\lambda/(5q + 2\lambda - 1)$ for all A. Next, $r \geq 9$ gives us the inequality $q \geq (2\lambda + 17)/13$. Combining the last two inequalities, we obtain that $\sigma_A = 5$ or 6 for all A. Hence, $a_i = 0$ for all i except possibly 5 and 6. Solving equations (18) and (19), we obtain $a_5 = (172q - 32\lambda - 11)/9$ and $a_6 = 4(8\lambda - 25q + 5)/9$.

Now, $\sigma_A > 0$ and $|A| \geq 2\lambda$ for all A, so we must have $r > r^*$ by equation (8). This implies that $q \geq \lambda/2$. However, this implies that $a_6 \leq -2(9\lambda - 10)/9 < 0$, a contradiction.

Case 3: $m = 5$.

In this case, Lemma 3.4 implies that $e = (8\lambda - 9q + 3)/14$, $r = (44q - 8\lambda + 11)/7$, and $r^* = (12q + 8\lambda + 3)/7$. Also, equation (6) implies that $\tau_A = \lambda - [(3q + 2\lambda - 1)/14]\sigma_A$ for any block A. Then the inequalities $0 \leq \tau_A \leq e$ imply that $3 \leq \sigma_A \leq 14\lambda/(3q + 2\lambda - 1)$ for all A. Also, $r \geq 9$ gives us the inequality $q \geq (2\lambda + 13)/11$. Combining the last two inequalities, we obtain that $\sigma_A = 3, 4$, or 5 for all A. Hence, $a_i = 0$ for all i except possibly 3, 4, and 5. Solving equations (18), (19), and (20) we obtain

$$a_3 = \frac{\alpha_{33}q^3 + \alpha_{32}q^2 + \alpha_{31}q + \alpha_{30}}{14\lambda(2\lambda - 11q - 1)(3q + 2\lambda - 1)},$$

where $\alpha_{33} = -10212\lambda + 7920$, $\alpha_{32} = 4909\lambda^2 - 6981\lambda + 60$, $\alpha_{31} = -288\lambda^3 + 1496\lambda^2 - 1588\lambda - 720$, and $\alpha_{30} = -64\lambda^4 - 152\lambda^3 + 107\lambda^2 - 59\lambda - 60$,

$$a_4 = \frac{\alpha_{43}q^3 + \alpha_{42}q^2 + \alpha_{41}q + \alpha_{40}}{7\lambda(2\lambda - 11q - 1)(3q + 2\lambda - 1)},$$

where $\alpha_{43} = 5328\lambda - 7920$, $\alpha_{42} = -6221\lambda^2 + 8099\lambda - 60$, $\alpha_{41} = 1392\lambda^3 - 2376\lambda^2 + 1752\lambda + 720$, and $\alpha_{40} = -64\lambda^4 + 288\lambda^3 - 147\lambda^2 + 61\lambda + 60$, and

$$a_5 = \frac{\alpha_{53}q^3 + \alpha_{52}q^2 + \alpha_{51}q + \alpha_{50}}{14\lambda(2\lambda - 11q - 1)(3q + 2\lambda - 1)},$$

where $\alpha_{53} = -4140\lambda + 7920$, $\alpha_{52} = 5741\lambda^2 - 8783\lambda + 60$, $\alpha_{51} = -2048\lambda^3 + 2584\lambda^2 - 1692\lambda - 720$, and $\alpha_{50} = 192\lambda^4 - 368\lambda^3 + 131\lambda^2 - 49\lambda - 60$.

Replacing q by a real variable x in the above expression for a_5, we obtain the function $a_5(x)$. Now, $\sigma_A > 0$ for all A, so $r > r^*$. This implies that $q \geq \lambda/2$. Also, the inequality $e \geq 1$ implies that $q \leq (8\lambda - 11)/9$. This implies $2\lambda - 11q - 1 < 0$. Therefore, $a_5(x)$ is a continuous functions of x on the interval $[\lambda/2, (8\lambda - 11)/9]$.

Now, the function $a_5(x)$ has zeros only at $(3\lambda - 5)/20$, $(8\lambda + 3)/9$, and $z_{51} = (8\lambda^2 - 5\lambda + 4)/(23\lambda - 44)$. Clearly, $(3\lambda - 5)/20, (8\lambda + 3)/9, z_{51} \notin [\lambda/2, (8\lambda - 11)/9]$, so $a_5(x)$ has no zeros on this interval. However,

$$a_5(\frac{3\lambda}{4}) = \frac{-(5\lambda + 12)(12\lambda + 5)(37\lambda^2 - 112\lambda - 16)}{14\lambda(17\lambda - 4)(25\lambda + 4)} < 0,$$

where $\lambda/2 < 3\lambda/4 < (8\lambda - 11)/9$. Therefore, $a_5(x)$ is negative on the interval $[\lambda/2, (8\lambda - 11)/9]$, a contradiction.

Case 4: $m = 7$.

In this case, Lemma 3.4 implies that $e = (8\lambda - q + 1)/10$, $r = (36q - 8\lambda + 9)/5$, and $r^* = (4q + 8\lambda + 1)/5$. Also, equation (6) implies that $\tau_A = \lambda - [(q + 2\lambda - 1)/10]\sigma_A$ for any block A. Then the inequalities $0 \leq \tau_A \leq e$ imply that $1 \leq \sigma_A \leq 10\lambda/(q + 2\lambda - 1)$ for all A. Also, $r \geq 9$ gives us the inequality $q \geq (2\lambda + 9)/9$. Combining the last two inequalities, we obtain that $\sigma_A = 1, 2, 3$, or 4 for all A. Therefore, since $\sigma_A > 0$ for all A, we must have $r > r^*$. From this we obtain the inequality $\lambda \leq 2q$. Now, suppose there exists a block A with $\sigma_A = 4$. Then the inequality $\tau_A \geq 0$ implies that $\lambda \geq 2q - 2$. Therefore, $\lambda = 2q - 2, 2q - 1$, or $2q$. If $\lambda = 2q - 2$, then $(r - r^*)/8 = 1$, a contradiction by Theorem 2.5. If $\lambda = 2q - 1$, then $r = (20q + 17)/5$ is not an integer, a contradiction. If $\lambda = 2q$, then $r = (20q + 9)/5$ is not an integer, a contradiction. Therefore, we must have $a_4 = 0$. Hence, $a_i = 0$ for all i except possibly 1, 2, and 3. Solving equations (18), (19), and (20) we obtain

$$a_1 = \frac{\alpha_{13}q^3 + \alpha_{12}q^2 + \alpha_{11}q + \alpha_{10}}{10\lambda(2\lambda - 9q - 1)(q + 2\lambda - 1)},$$

where $\alpha_{13} = -924\lambda + 216$, $\alpha_{12} = 331\lambda^2 - 997\lambda - 138$, $\alpha_{11} = 512\lambda^3 - 280\lambda^2 - 444\lambda - 72$, and $\alpha_{10} = -192\lambda^4 + 56\lambda^3 - 11\lambda^2 - 35\lambda - 6$,

$$a_2 = \frac{\alpha_{23}q^3 + \alpha_{22}q^2 + \alpha_{21}q + \alpha_{20}}{5\lambda(2\lambda - 9q - 1)(q + 2\lambda - 1)},$$

where $\alpha_{23} = 168\lambda - 216$, $\alpha_{22} = -1387\lambda^2 + 1615\lambda + 138$, $\alpha_{21} = 336\lambda^3 - 408\lambda^2 + 576\lambda + 72$, and $\alpha_{20} = 64\lambda^4 + 96\lambda^3 - 45\lambda^2 + 41\lambda + 6$, and

$$a_3 = \frac{\alpha_{33}q^3 + \alpha_{32}q^2 + \alpha_{31}q + \alpha_{30}}{10\lambda(2\lambda - 9q - 1)(q + 2\lambda - 1)},$$

where $\alpha_{33} = -132\lambda + 216$, $\alpha_{32} = 1163\lambda^2 - 1683\lambda - 138$, $\alpha_{31} = -864\lambda^3 + 616\lambda^2 - 548\lambda - 72$, and $\alpha_{30} = 64\lambda^4 - 208\lambda^3 + 61\lambda^2 - 37\lambda - 6$.

Replacing q by a real variable x in the above expressions for a_1 and a_3, we obtain two functions, $a_1(x)$ and $a_3(x)$. Now, we already know that $q \geq \lambda/2$. Also, the inequality $e \geq 1$ implies that $q \leq 8\lambda - 9$. This implies that $2\lambda - 9q - 1 < 0$. Therefore, $a_1(x)$ and $a_3(x)$ are continuous functions of x on the interval $[\lambda/2, 8\lambda - 9]$.

Now, the function $a_3(x)$ has zeros only at $(\lambda - 3)/12$, $8\lambda + 1$, and

$$z_{31} = \frac{8\lambda^2 - 3\lambda + 2}{11\lambda - 18}.$$

Clearly, $(\lambda - 3)/12, 8\lambda + 1 \notin [\lambda/2, 8\lambda - 9]$, so $a_3(x)$ has at most one zero on this interval. However,

$$a_3(\frac{\lambda}{2}) = \frac{(\lambda + 2)(5\lambda + 3)(15\lambda + 2)}{10\lambda(5\lambda - 2)} > 0$$

and

$$a_3(8\lambda - 9) = \frac{-4(\lambda - 2)(19\lambda - 21)}{\lambda(7\lambda - 8)} < 0.$$

Therefore, $a_3(x)$ has exactly one zero on the interval $[\lambda/2, 8\lambda - 9]$ at z_{31}. Then the above inequalities imply that $a_3(x)$ is negative on the interval $(z_{31}, 8\lambda - 9]$. Hence, we must have $q \in [\lambda/2, z_{31}]$.

Next, the function $a_1(x)$ has zeros only at $-(3\lambda + 1)/4$ and

$$z_{11}, z_{12} = \frac{128\lambda^2 - 116\lambda - 24 \mp 5\sqrt{64\lambda^4 - 680\lambda^3 + 49\lambda^2 + 204\lambda + 36}}{231\lambda - 54}.$$

Clearly, $-(3\lambda+1)/4, z_{11} \notin [\lambda/2, z_{31}]$, so $a_1(x)$ has at most one zero on this interval. However,

$$a_1(\frac{\lambda}{2}) = \frac{-(5\lambda+1)(5\lambda^2-52\lambda-12)}{10\lambda(5\lambda-2)} < 0$$

and

$$a_1(z_{31}) = \frac{7(13\lambda+2)}{11\lambda-18} > 0.$$

Therefore, $a_1(x)$ has exactly one zero on the interval $[\lambda/2, z_{31}]$ at z_{12}. Then the above inequalities imply that $a_1(x)$ is negative on the interval $[\lambda/2, z_{12})$. Hence, we must have $q \in [z_{12}, z_{31}]$.

Now, we easily obtain the inequality $z_{31} < (8\lambda+12)/11$. Next, the inequality $64\lambda^4 - 680\lambda^3 + 49\lambda^2 + 204\lambda + 36 > (8\lambda^2 - 43\lambda - 144)^2$ implies that $z_{12} > (168\lambda^2 - 331\lambda - 744)/(231\lambda - 54)$. This in turn implies that $z_{12} > (8\lambda - 16)/11$. Therefore, we have $q \in ((8\lambda-16)/11, (8\lambda+12)/11)$. Thus, since q is an integer, we must have $q \in \{(8\lambda+k)/11 : -15 \leq k \leq 11\}$.

Then $r = (200\lambda + 36k + 99)/55$ and $e = (80\lambda + 11 - k)/110$ for some integer $-15 \leq k \leq 11$. So, 5 must divide $36k + 99$ and 10 must divide $11 - k$. Therefore, we must have $k = -9, 1$, or 11. If $q = (8\lambda - 9)/11$, then $(r-1)/8 = (5\lambda - 7)/11$, so 11 divides $5\lambda - 7$. But, then $r(r-1)/(v-1) = 5(5\lambda-7)/11$ is an integer, a contradiction by Theorem 2.3. If $q = (8\lambda+1)/11$, then $(r^*-1)/8 = (3\lambda-1)/11$, so 11 divides $3\lambda - 1$. But, then $r^*(r^*-1)/(v-1) = 3(3\lambda-1)/11$ is an integer, a contradiction by Theorem 2.3. So, we must have $q = (8\lambda+11)/11$. However,

$$a_3(\frac{8\lambda+11}{11}) = \frac{-4(\lambda-22)(17\lambda+33)}{33\lambda(5\lambda+11)} < 0$$

for $\lambda \geq 23$. Therefore, we must have $\lambda = 20, 21$, or 22. But, the only such value of λ that makes q integral is $\lambda = 22$. Since 22 is twice a prime number, we obtain a contradiction by Theorem 2.8.

Case 5: $m = 11$.

If there exists a block A with $\sigma_A \leq -2$, then $m(A) \leq 7$ and $D(A)$ is non-type-1 with $g(A) = 8$ by Lemma 3.6 (ii), (iii), and (iv). Thus, we obtain a contradiction by cases 1, 2, 3, and 4. If there exists a block A with $\sigma_A \geq 3$, then $m(A) \geq 17$, so $m^*(A) \leq 7$ and once again we are done by previous cases. Therefore, we must have $\sigma_A = -1, 0, 1$, or 2 for all blocks A. Hence, $a_i = 0$ for all i except possibly $-1, 0, 1$, or 2. Now, if $r > r^*$, then by equation (8) we must have $\sigma_A \geq 0$ for all A since $|A| \geq 2\lambda$ for all

A. Similarly, if $r < r^*$, then we must have $\sigma_A \leq 0$ for all A. Thus, we have the following two subcases.
Subcase 5a: $\sigma_A \leq 0$ for all A.

In this subcase, $\sigma_A = -1$ or 0 for all A. Solving equations (18) and (19) we obtain $a_{-1} = 60q - 32\lambda + 15$ and $a_0 = 32\lambda - 52q - 14$. Now, $r < r^*$, which implies that $q \leq (\lambda - 1)/2$. However, this implies that $a_{-1} \leq -(2\lambda + 15) < 0$, a contradiction.
Subcase 5b: $\sigma_A \geq 0$ for all A.

In this subcase, $\sigma_A = 0, 1,$ or 2 for all A. Solving equations (18), (19), and (20), we obtain

$$a_0 = \frac{\alpha_{02}q^2 + \alpha_{01}q + \alpha_{00}}{2(2\lambda - 5q - 1)(2\lambda - 3q - 1)},$$

where $\alpha_{02} = -323\lambda - 30$, $\alpha_{01} = 288\lambda^2 - 136\lambda - 16$, and $\alpha_{00} = -64\lambda^3 + 64\lambda^2 - 13\lambda - 2$,

$$a_1 = \frac{\alpha_{13}q^3 + \alpha_{12}q^2 + \alpha_{11}q + \alpha_{10}}{(2\lambda - 5q - 1)(2\lambda - 3q - 1)},$$

where $\alpha_{13} = 1140$, $\alpha_{12} = -1373\lambda + 893$, $\alpha_{11} = 528\lambda^2 - 696\lambda + 228$, and $\alpha_{10} = -64\lambda^3 + 132\lambda^2 - 87\lambda + 19$, and

$$a_2 = \frac{\alpha_{23}q^3 + \alpha_{22}q^2 + \alpha_{21}q + \alpha_{20}}{2(2\lambda - 5q - 1)(2\lambda - 3q - 1)},$$

where $\alpha_{23} = -2040$, $\alpha_{22} = 2813\lambda - 1598$, $\alpha_{21} = -1280\lambda^2 + 1432\lambda - 408$, and $\alpha_{20} = 192\lambda^3 - 320\lambda^2 + 179\lambda - 34$.

Replacing q by a real variable x in the above expressions for a_1 and a_2, we obtain two functions, $a_1(x)$ and $a_2(x)$. Now, the inequalities $r > r^* \geq 9$ imply that $\lambda/2 \leq q \leq (2\lambda - 3)/3$. This implies $(2\lambda - 5q - 1)(2\lambda - 3q - 1) < 0$. Therefore, $a_1(x)$ and $a_2(x)$ are continuous functions of x on the interval $[\lambda/2, (2\lambda - 3)/3]$.

Now, the function $a_1(x)$ has zeros only at $(\lambda - 1)/4$ and

$$z_{11}, z_{12} = \frac{136\lambda - 76 \mp \sqrt{256\lambda^2 - 1292\lambda + 361}}{285}.$$

Clearly, $(\lambda - 1)/4, z_{11} \notin [\lambda/2, (2\lambda - 3)/3]$, so $a_1(x)$ has at most one zero on this interval. However,

$$a_1(\frac{\lambda}{2}) = \frac{(\lambda+1)(3\lambda-38)}{\lambda-2} > 0$$

and

$$a_1(\frac{2\lambda-3}{3}) = \frac{-(5\lambda-9)(7\lambda^2-56\lambda+114)}{6(\lambda-3)} < 0.$$

Therefore, $a_1(x)$ has exactly one zero on the interval $[\lambda/2, (2\lambda-3)/3]$ at z_{12}. The above inequalities then imply that $a_1(x)$ is negative on the interval $(z_{12}, (2\lambda-3)/3]$. Hence, we must have $q \in [\lambda/2, z_{12}]$.

Next, the function $a_2(x)$ has zeros only at $(3\lambda-2)/8$ and

$$z_{21}, z_{22} = \frac{128\lambda - 68 \mp \sqrt{64\lambda^2 - 1088\lambda + 289}}{255}.$$

Clearly, $(3\lambda-2)/8, z_{21} \notin [\lambda/2, z_{12}]$, so $a_2(x)$ has at most one zero on this interval. However,

$$a_2(\frac{\lambda}{2}) = \frac{-\lambda^2+16\lambda+68}{2(\lambda-2)} < 0$$

and

$$a_2(z_{12}) = \frac{64\lambda + 19 - 4\sqrt{256\lambda^2 - 1292\lambda + 361}}{38} > 0.$$

Therefore, $a_2(x)$ has exactly one zero on the interval $[\lambda/2, z_{12}]$ at z_{22}. The above inequalities then imply that $a_2(x)$ is negative on the interval $[\lambda/2, z_{22})$. Hence, we must have $q \in [z_{22}, z_{12}]$.

Now, the inequalities $64\lambda^2 - 1088\lambda + 289 > (8\lambda-96)^2$ and $256\lambda^2 - 1292\lambda + 361 < (16\lambda-40)^2$ imply that $z_{22} > (8\lambda-10)/15$ and $z_{12} < (8\lambda-6)/15$. Therefore, since q is an integer, we must have $q = (8\lambda-9)/15, (8\lambda-8)/15$ or $(8\lambda-7)/15$. If $q = (8\lambda-9)/15$, then

$$a_2(\frac{8\lambda-9}{15}) = \frac{-(2\lambda-51)(19\lambda-42)}{15(\lambda-3)(\lambda+2)} < 0$$

for $\lambda \geq 26$. Therefore, we must have $20 \leq \lambda \leq 25$. However, no such value of λ makes q integral, a contradiction. If $q = (8\lambda-8)/15$, then $e = (16\lambda+9)/10$ is not an integer, a contradiction. If $q = (8\lambda-7)/15$, then

$$5 < a_1(\frac{8\lambda - 7}{15}) = \frac{(7\lambda - 38)(17\lambda - 13)}{15(\lambda - 2)(\lambda + 1)} < 8.$$

Therefore, we must have $a_1((8\lambda - 7)/15) = 6$ or 7. If $a_1((8\lambda - 7)/15) = 6$, then λ is not an integer, a contradiction. If $a_1((8\lambda - 7)/15) = 7$, then $\lambda = 44$. But, then $(r - r^*)/8 = 5$ and we apply Theorem 2.5 to obtain a contradiction.

Case 6: $m = 9$.

If there exists a block A with $\sigma_A \leq -1$, then $m(A) \leq 7$ and we are done by previous cases. If there exists a block A with $\sigma_A \geq 4$, then $m^*(A) \leq 7$ and we are done. If there exists A with $\sigma_A = 1$, then $m(A) = 11$ and we are finished by case 5. If there exists A with $\sigma_A = 2$, then $m^*(A) = 11$ and we are finished. Therefore, $\sigma_A = 0$ or 3 for all A. Hence, $a_i = 0$ for all i except possibly 0 and 3. Solving equations (18) and (19), we obtain $a_0 = 4(25q - 8\lambda + 4)/9$ and $a_3 = (32\lambda - 28q - 7)/9$. Inserting these expressions for a_0 and a_3 into equation (20) and manipulating the result, we arrive at

$$(2\lambda - 4q - 1)^2[175q^2 - (256\lambda - 200)q + 64\lambda^2 - 64\lambda + 25] = 0.$$

Now, $2\lambda - 4q - 1 \neq 0$ since $v \neq 4\lambda - 1$ because $m \neq 12$. Therefore, we obtain that

$$175q^2 - (256\lambda - 200)q + 64\lambda^2 - 64\lambda + 25 = 0.$$

Now, the left-hand-side of the above equation is a quadratic polynomial in q, and therefore its discriminant must be a perfect square. This yields the equation

$$576\lambda^2 - 1600\lambda + 625 = N^2$$

for some integer N. This equation can then be transformed into

$$(31104\lambda - 6N - 43200)(31104\lambda + 6N - 43200) = 17500.$$

However, by considering all possible ways of factoring 17500 into the product of two integers, the above equation can be shown to have no integral solutions, a contradiction. This concludes the proof of Theorem 4.1.

□

Corollary 4.2 *All λ-designs on $v = 8p + 1$ points, $p \equiv 1$ or 7 (mod 8) a prime, are type-1.*

Proof: If p does not divide g, then $g = 1, 2, 4$, or 8 and D is type-1 by Theorems 2.4 and 4.1. If p does divide g, then p also divides $r - 1$ and $r^* - 1$. Without loss of generality, we may assume $r > r^*$. Therefore, either $r = 7p + 1$ and $r^* = p + 1$, $r = 6p + 1$ and $r^* = 2p + 1$, or $r = 5p + 1$ and $r^* = 3p + 1$.

If $r = 6p + 1$ and $r^* = 2p + 1$, then $(r - r^*)/g = 2$ and D is type-1 by Theorem 2.5. If $r = 5p + 1$ and $r^* = 3p + 1$, then $(r - r^*)/g = 2$ and D is type-1 by Theorem 2.5. Thus, we may assume that $r = 7p + 1$ and $r^* = p + 1$.

If $p \equiv 1$ (mod 8), then 8 divides $7p+1$, so $r(r-1)/(v-1) = 7(7p+1)/8$ is an integer and D is type-1 by Theorem 2.3. If $p \equiv 7$ (mod 8), then 8 divides $p+1$, so $r^*(r^* - 1)/(v-1) = (p+1)/8$ is an integer and D is type-1 by Theorem 2.3.

□

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