Upper Bounds of Dynamic Chromatic Number

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Abstract

A proper vertex k-coloring of a graph G is dynamic if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors. The smallest integer k such that G has a dynamic k-coloring is the dynamic chromatic number $\chi_d(G)$. We prove in this paper the following best possible upper bounds as an analogue to Brook's Theorem, together with the determination of chromatic numbers for complete k-partite graphs.

(1) If $\Delta \leq 3$, then $\chi_d(G) \leq 4$, with the only exception that $G = C_5$, in which case $\chi_d(C_5) = 5$.

(2) If $\Delta \geq 4$, then $\chi_d(G) \leq \Delta + 1$.

(3) $\chi_d(K_{1,1}) = 2$, $\chi_d(K_{1,m}) = 3$ and $\chi_d(K_{m,n}) = 4$ for $m, n \geq 2$; $\chi_d(K_{n_1,n_2,\cdots,n_k}) = k$ for $k \geq 3$.

1. Introduction

Graphs in this note are simple and finite. For a graph G and $v \in V(G)$, $d_G(v)$ and $N_G(v)$ denote the degree of v in G and the set of vertices adjacent to v in G, respectively. δ_G and Δ_G denote the smallest degree and the largest degree in G, respectively. The subscript G may be dropped if G is clear from the context. The cycle of k vertices will be denoted by C_k, while $C(k)$ denotes a set of k colors.

A **dynamic vertex k-coloring** of a graph G is a map $c : V(G) \mapsto C(k)$ such that

(C1) If $uv \in E(G)$, then $c(u) \neq c(v)$, and

(C2) for each vertex $v \in V(G)$, $|c(N(v))| \geq \min\{2, d_G(v)\}$.

The smallest integer k such that G has a dynamic k-coloring is the dynamic chromatic number $\chi_d(G)$.

This coloring number has been initiated and studied in [2]. In this paper, we prove the following three theorems. Theorem 1 and Theorem 2 are analogous to Brook's Theorem and provide best possible upper bounds for the dynamic chromatic number. Theorem 3 completely determines the dynamic chromatic numbers for complete k-partite graphs.

Theorem 1 If $\Delta \leq 3$, then $\chi_d(G) \leq 4$, with the only exception that $G = C_5$, in which case $\chi_d(C_5) = 5$.

Theorem 2 If $\Delta \geq 4$, then $\chi_d(G) \leq \Delta + 1$.

Theorem 3 $\chi_d(K_{1,1}) = 2$, $\chi_d(K_{1,m}) = 3$, for $m \geq 2$, and $\chi_d(K_{m,n}) = 4$ for $m, n \geq 2$;

$\chi_d(K_{n_1,n_2,\cdots,n_k}) = k$ for $k \geq 3$.

2. Proof of Theorem 1

We start with a lemma. An **arc** of G is an (x, y)-path P from a vertex x to a vertex y (or a cycle if $x = y$), where $x, y \in V(G)$, such that all the internal vertices of P have degree 2 in G, while x and y have degree at least 3.

Lemma 1 Let G be a connected graph with $\delta = 2$. Then there exists an arc of length ≥ 2 or G is a cycle.

Proof of Lemma 1 Let $v \in V(G)$ be such that $d(v) = 2$ and let $P = a \cdots v \cdots b$ be the longest path through v such that any internal vertex is of degree 2. Since $\delta \geq 2$, then $d(a), d(b) \geq 2$. But any internal vertex is of degree 2 and this path is the longest, thus either $ab \in E$, or $d(a), d(b) \geq 3$.

If $d(a) = d(b) = 2$, then G is a cycle since it is connected.

If $d(a), d(b) \geq 3$, then P is an arc of length ≥ 2.

Otherwise we may assume that $d(a) = 2, d(b) \geq 3$. Then $ab \in E$ and so bP is an arc of length ≥ 3. This completes the proof of Lemma 1.

Proof of Theorem 1 We may assume that G is connected. We argue by induction on $|V(G)|$. The conclusion holds trivially if $|V(G)| \leq 4$. So we assume that $|V(G)| \geq 5$.

Case 1 G has a cut vertex v.

Then G has two connected subgraphs G_1 and G_2, each having at least 2 vertices, such that $V(G_1) \cap V(G_2) = \{v\}$. By induction, either $G_i = C_5$

or $\chi_d(G_i) \leq 4$.

Suppose $G_1 \ncong C_5$ and $G_2 \ncong C_5$. Then by induction, there are dynamic 4-colorings $c_i : V(G_i) \mapsto C(4)$, for each $i = 1, 2$. We may assume that $c_1(v) = c_2(v)$, and since each G_i is connected with at least two vertices, we may also assume that one neighbor of v in G_1 receives a different color from a neighbor of v in G_2. Therefore, a dynamic 4-coloring of G can be obtained by combining c_1 and c_2.

Suppose $G_1 = vv_2v_3v_4v_5v$ is a C_5 and $G_2 \ncong C_5$. Let $c_2 : V(G_2) \mapsto C(4)$ be a dynamic 4-coloring of G_2. We may assume that $c_2(v) = 1$ and for some $u \in V(G_2)$ with $uv \in E(G_2)$, $c_2(u) = 4$. Then define $c(z) = c_2(z)$ if $z \in V(G_2)$ and $c(v_2) = c(v_5) = 2$, $c(v_3) = 3$ and $c(v_4) = 4$. Then coloring c is a dynamic 4-coloring of G.

Finally, we notice that since $\Delta \leq 3$, we can NOT have $G_1 \cong G_2 \cong C_5$ for in this case we will have $d(v) = 4$. This completes Case 1.

Below we assume that G is 2-connected. $|V| \geq 5$, hence $\delta \geq 2$. Yet $\Delta \leq 3$, so $\delta \leq 3$.

Case 2 G is 2-connected and $\delta = 2$.

Thus Lemma 1 holds.

Case 2A $G \cong C_n$.

One can easily verify that for $n \geq 3$,

$$\chi_d(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3 \\ 4 & \text{if } n \not\equiv 0 \pmod 3 \text{ and } n \neq 5 \\ 5 & \text{if } n = 5. \end{cases}$$

Case 2B G has an arc $P = v_1v_2\cdots v_m$ for some $m \geq 4$. Let $G' = G - \{v_2, \cdots, v_{m-1}\}$. By induction, either $G' \cong C_5$, or $\chi_d(G) \leq 4$. Since G is 2-connected, then $v_1 \neq v_m$, for otherwise $v_1 = v_m$ is a cut vertex.

Case 2B1 $G' \cong C_5$.

If $v_1v_m \in E(G')$, then let $G' = v_1v_mu_4u_3u_2v_1$ and $c : V(G') \mapsto C(4)$ is given by $c(u_i) = i$, for $2 \leq i \leq 4$, $c(v_1) = 1$ and $c(v_m) = 2$; if $v_1v_m \notin E(G')$, then let $G' = v_1u_3u_4v_mu_2v_1$ and $c : V(G') \mapsto C(4)$ is given by $c(u_i) = i$, for $3 \leq i \leq 4$, $c(u_2) = 3$, $c(v_1) = 1$ and $c(v_m) = 2$.

For $2 \leq i \leq m - 1$, define

$$c(v_i) = \begin{cases} 4 & \text{if } i \equiv 2 \pmod 3 \\ 3 & \text{if } i \equiv 0 \pmod 3 \\ 1 & \text{if } i \equiv 1 \pmod 3). \end{cases}$$

Then c is a dynamic 4-coloring of G.

Case 2B2 $G' \ncong C_5$.

Then by induction, $\chi_d(G') \leq 4$. Let $c : V(G') \mapsto C(4)$ be a dynamic 4-coloring of G'. Now we extend c to G. Note that both v_1 and v_m have degree at least 2 in G', and so each of v_1 and v_m is adjacent to vertices in G' with at least two colors.

If $c(v_1) = c(v_m)$, then we assume that $c(v_1) = c(v_m) = 2$; if $c(v_1) \neq c(v_m)$, then we assume that $c(v_1) = 1$ and $c(v_m) = 2$. Define, for $2 \leq i \leq m-1$,

$$c(v_i) = \begin{cases} 4 & \text{if } i \equiv 2 \pmod 3 \\ 3 & \text{if } i \equiv 0 \pmod 3 \\ 1 & \text{if } i \equiv 1 \pmod 3). \end{cases}$$

Then c is a dynamic 4-coloring of G. This completes the proof for Case 2B.

Case 2C Every arc in G is of length ≤ 2 and there is at least one arc of length 2.

Note that in this case there could not exist an edge xy in G such that $d(x) = d(y) = 2$ for otherwise a reasoning similar as in the proof of Lemma 1 will lead to the existence of an arc of length ≥ 3 or G is a cycle, either of which is a contradiction with the assumption of this case.

Let $d(v) = 2$ and assume that $N(v) = \{x, y\}$. Then by the assumption of this case, $d(x), d(y) \geq 3$; thus, $d(x) = d(y) = 3$. Denote $N(x) - \{v\} = \{a, b\}, N(y) - \{v\} = \{c, d\}$. Since G is simple, $x \neq y$.

Case 2C1 $xy \in E$. Let $G' = G - v$. Then either $G' \cong C_5$, or $\chi_d(G') \leq 4$.

We have $G' \ncong C_5$, for otherwise we can find an arc of length 4. Hence, by induction we may assume that $c : V(G') \mapsto C(4)$ is a dynamic 4-coloring for G'. Since G is simple, then $d_{G'}(x), d_{G'}(y) \geq 2$. Therefore each of x and y is adjacent to vertices in G' with at least two different colors. Since $xy \in E(G')$, then$c(x) \neq c(y)$. Pick k from $C(4) \setminus c(x, y)$ and define $c(v) = k$. Then c is a dynamic 4-coloring for G.

Now we assume that $xy \notin E$. Then $\{a, b, c, d\} \cap \{x, y\} = \emptyset$.

Case 2C2 $xy \notin E$ and $\{a, b\} \cap \{c, d\} \neq \emptyset$.

Assume that $a \in N(x) \cap N(y) \setminus \{v\}$. Let $G' = G - v + xy$. Then G is still connected and $d_{G'} = d(G) = 3$; hence, $G' \ncong C_5$. By induction, let $c : V(G') \mapsto C(4)$ be a dynamic 4-coloring for G', then $c(x) \neq c(y)$. Pick k from $C(4) \setminus c(x, y, a)$ and define $c(v) = k$; then c is a dynamic 4-coloring for G.

Case 2C3 $xy \notin E$ and $\{a,b\} \cap \{c,d\} = \emptyset$.

In this case we have: a,b,c,d are distinct and $a,b \notin N(y)$, $c,d \notin N(x)$. $\delta = 2, \Delta = 3$.

We may assume, by symmetry, that $d(a) \le d(b)$.

Case 2C3.1 $d(a) = 2 = d(b)$.

Note that in this case a,b are not adjacent as we have mentioned at the beginning of Case 2C.

If $N(a) = N(b) = \{x,z\}$. Let $G' = G - \{a,b\} + xz$. Then G' is still connected and since $a,b \notin N(y)$, $d_{G'}(y) = d_G(y) = 3$. Hence, $G' \ncong C_5$. By induction, we may assume that $c: V(G') \mapsto C(4)$ is a dynamic 4-coloring for G'. $xz \in E(G')$, thus $c(x) \neq c(z)$. We may assume that $c(x) = 1, c(z) = 2$. Define $c(a) = 3, c(b) = 4$. Then c is a dynamic 4-coloring for G.

Now we assume that $N(a) = \{e,x\}$ and $N(b) = \{f,x\}$, where $e \neq f$. By assumption of Case 2C, $d(e) = d(f) = 3$.

Let $G' = G - \{a,b,x,v\}$, then $\delta(G') = 2$. What's more, G' is connected for otherwise x will be a cut-vertex for G, contrary to G being 2-connected. By induction, either $G' \cong C_5$ or $\chi_d(G') \le 4$.

If $G' \cong C_5$, then the three vertices e,f,y are symmetric in term of x. Among them there is at least one pair of adjacent vertices since they are vertices of C_5. Assume that e and f are adjacent. Note that the two vertices in G' other than e,f,y are not adjacent, for otherwise we have an arc of length 3. So $G' = efv_1yv_2$. Thus, $c: V(G) \mapsto C(4)$ can be given by $c(f) = c(v_2) = 1, c(e) = c(v) = c(v_1) = 2, c(a) = c(b) = c(y) = 3, c(x) = 4$.

Now we assume that $G' \ncong C_5$. By induction, we may assume that $c: V(G') \mapsto C(4)$ is a dynamic 4-coloring for G'. We may assume that $c(e) = 1, c(f) \in \{1,2\}, c(y) \in \{1,2,3\}$. Define $c(a) = 2, c(x) = 4, c(b) = 3$, and pick $c(v)$ from $C(4) - \{4, c(y)\}$; then c is a dynamic 4-coloring for G. This completes the proof for Case 2C3.1.

Case 2C3.2 $d(a) = 2, d(b) = 3$.

Suppose a,b are adjacent. Let $G' = G - v + xy$. Then G' is connected and since G' has a triangle abx, thus $G' \ncong C_5$. By induction, we may assume $c: V(G') \mapsto C(4)$ is a dynamic 4-coloring for G'. Since $ab, xy \in E$, so $c(x) \neq c(y)$, $c(a) \neq c(b)$. We may assume that $c(x) = 1, c(y) = 2$. $d(y) = 3$, hence y has a neighbor colored differently from 1 or 2. We may assume that the neighbor receives color 3. Define $c(v) = 4$, then c is a dynamic 4-coloring for G.

Otherwise, a, b are not adjacent. Denote e as the neighbor of a other than x. Let $G' = G - \{a, x, v\}$. G' must be connected for otherwise x will be a cut-vertex in G. If $G' \ncong C_5$. By induction, we may assume that $c : V(G') \mapsto C(4)$ is a dynamic 4-coloring for G'. Since $d_{G'}(e) = d_{G'}(b) = d_{G'}(y) = 2$, then each of e, b, y is adjacent to vertices of at least 2 different colors. We may assume that $c(e) = 1, c(y) \in \{1, 2\}, c(b) \in \{1, 2, 3\}$. Define $c(a) = 2, c(x) = 4, c(v) = 3$, then c is a dynamic 4-coloring for G.

Now assume that $G' \cong C_5$. Note that e, y are symmetric with respect to x. If $eb \notin E, yb \notin E$, then $ey \in E$; otherwise, we may assume that $eb \in E$. Denote the two vertices other than e, b, y as w, z. By the assumption of Case 2C, w and z are not adjacent. Therefore, it suffices to deal with the following two cases:

(i). $G' = eywbz$. Then $c : V(G) \mapsto C(4)$ is given by $c(e) = c(w) = 1, c(a) = c(y) = 2, c(x) = c(z) = 3, c(b) = c(v) = 4$.

(ii). $G' = ewyzb$. Then $c : V(G) \mapsto C(4)$ is given by $c(b) = c(v) = 1, c(x) = c(w) = 2, c(a) = c(y) = 3, c(e) = c(z) = 4$.

This completes the proof for Case 2C3.2.

Case 2C3.3 $d(a) = 3 = d(b)$. By the symmetry of x and y, and the previous two cases, we may assume also that $d(c) = 3 = d(d)$.

Let $G' = G - \{x, v\}$. G' is simple and G' is still connected, for otherwise v will be a cut-vertex in G. $d_{G'}(c) = 3$ implies that $G' \ncong C_5$. Thus, by induction we may assume that $c : V(G') \mapsto C(4)$ is a dynamic 4-coloring for G'.

We may assume that $c(y) = 1, c(a) \in \{1, 2\}, c(b) \in \{1, 2, 3\}$. Define $c(x) = 4, c(v) = 3$; then c can be extended to a dynamic 4-coloring for G.

This completes the proof for Case 2C3.3 as well as the proof for Case 2C.

Case 3 G is 2-connected and $\delta = 3$.

Since $\Delta \leq 3$, then $\Delta = \delta = 3$ and G is cubic. Let x, y be an adjacent pair of vertices in G and assume that $N(x) = \{a, b, y\}, N(y) = \{c, d, y\}$. Since G is simple, then $a \neq b, c \neq d$ and $\{a, b, c, d\} \cap \{x, y\} = \emptyset$.

Case 3A $\{a, b\} \cap \{c, d\} \neq \emptyset$.

Assume that $a \in N(x) \cap N(y)$. Let $G' = G - x$. Then G' is still connected for G is 2-connected. Since $d_{G'}(w) = 3$ for any $w \notin \{a, b, x, y\}$, hence $G' \ncong C_5$. By induction, we may assume that $c : V(G') \mapsto C(4)$ is a dynamic 4-coloring for G'.

Since a and y are adjacent, then $c(a) \neq c(y)$. Since $d_{G'}(a) = d_{G'}(b) = d_{G'}(y) = 2$, then each of a, b, y is adjacent to vertices of 2 different colors. Pick k from $C(4) \setminus c(a, b, y)$ and extend c to G by defining $c(x) = k$. Then c is a dynamic 4-coloring for G.

Case 3B $\{a, b\} \cap \{c, d\} = \emptyset$.

Let $G' = G - x - y$. G is cubic and so $d_{G'}(w) = d_G(w) = 3$ for every w in G' expect for a, b, x, y. Thus G' can not have a component isomorphic to C_5. By induction we may assume that $c : V(G') \mapsto C(4)$ is a dynamic 4-coloring for G'.

Suppose $c(a) \neq c(b), c(c) \neq c(d)$. Pick m from $C(4) \setminus \{a, b\}$, pick n from $C(4) \setminus c(c, d) \setminus \{m\}$ and extend c to G so that $c(x) = m, c(y) = n$. Then c is a dynamic 4-coloring for G.

Otherwise, without loss of generality, we may assume that $c(a) = c(b)$. Pick n from $C(4) \setminus c(a, c, d)$, pick m from $C(4) \setminus c(a, c) \setminus \{n\}$ and extend c to G by assigning $c(x) = m, c(y) = n$. Then c is a dynamic 4-coloring for G.

This proves Case 3 and thus completes the proof of Theorem 1.

3. Proof of Theorem 2

We use induction on $|V(G)|$. Note that $\Delta \geq 4$ implies that $\Delta + 1 \geq 5$. The conclusion holds trivially if $|V| \leq 5$. Assume that $|V| \geq 6$.

Let H be a subgraph of G with a fewer number of vertices. Then $\Delta(H) \leq \Delta(G)$. If $\Delta(H) \leq 3$, then by Theorem 1, $\chi_d(H) \leq 5 \leq \Delta(G) + 1$; otherwise, $\Delta(H) \geq 4$ and by induction, $\chi_d(H) \leq \Delta(H) + 1 \leq \Delta(G) + 1$. So, in either case we have $\chi_d(H) \leq \Delta(G) + 1$.

Case 1 $\delta = 1$.

Let v be a vertex in G with $d(v) = 1$ and consider $G' = G - v$. By induction, $\chi_d(G') \leq \Delta(G) + 1$. Let c be a dynamic $(\Delta(G) + 1)$-coloring for G'. Denote the only neighbor of v as w. We may assume that $d(w) \geq 2$. Let u be a neighbor of w other than v. Pick k from $C(\Delta(G) + 1) \setminus c(w, u)$. Then we can extend c to G by assigning $c(v) = k$.

Case 2 $\delta = 2$.

Let $d(v) = 2$ and denote $N(v) = \{x, y\}$. Consider $G' = G - v + xy$. Then by induction, $\chi_d(G') \leq \Delta(G) + 1$. Let c be a dynamic $(\Delta(G) + 1)$-coloring for G'. Since $d(x), d(y) \geq 2$, then we can choose x' and y' from $N(x) -$

$\{v\}$ and $N(y)-\{v\}$, respectively. Pick k from $C(\Delta(G)+1)\setminus c(x,y,x',y')$. Then we can extend c to G by assigning $c(v)=k$.

Case 3 $\delta \geq 3$.

Denote x, y as a pair of adjacent vertices in G.

Case 3A $N(x)\cap N(y)\neq\emptyset$.

Let $z\in N(x)\cap N(y)$ and denote $G'=G-x$. By induction, $\chi_d(G')\leq\Delta(G)+1$. Let c be a dynamic $(\Delta(G)+1)$-coloring for G'. Then $c(y)\neq c(z)$. Pick k from $C(\Delta(G)+1)\setminus c(N(x))$. Then we can extend c to G by assigning $c(v)=k$.

Case 3B $N(x)\cap N(y)=\emptyset$.

Let $G'=G-x-y$. By induction, $\chi_d(G')\leq\Delta(G)+1$. Let c be a dynamic $(\Delta(G)+1)$-coloring for G'.

Denote $N_x=N(x)\setminus\{y\}, N_y=N(y)\setminus\{x\}$. Then $|N_x|\leq\Delta(G)-1, |N_y|\leq\Delta(G)-1$.

If $|c(N_x)|\geq 2, |c(N_y)|\geq 2$. Pick $m\in C(\Delta(G)+1)\setminus c(N_x)$ and pick $n\in C(\Delta(G)+1)\setminus c(N_y)\setminus\{m\}$. Then we can extend c to G by assigning $c(x)=m, c(y)=n$.

Otherwise we may assume that $|c(N_y)|=1$. Let x' be a neighbor of x other than y. Pick $m\in C(\Delta(G)+1)\setminus c(N_x\cup N_y)$ and pick $n\in C(\Delta(G)+1)\setminus c(N_y)\setminus\{m\}\setminus c(x')$. Then we can extend c to G by assigning $c(x)=m, c(y)=n$. This proves Case 3 and thus finishes the proof of Theorem 2.

The following example shows that the result is best possible. Denote SK_n as a graph obtained from K_n by subdividing every edge in the complete graph. It is easy to verify that $\Delta(SK_n)=n-1$ and every pair of vertices in the original complete graph must be colored differently. Therefore, we need at least n colors, that is, $\Delta+1$ colors in a dynamic coloring for SK_n. Also, by the Theorem, we know that we need exactly n colors for $n\geq 5$.

4. Proof of Theorem 3

It is easy to verify the first equality. For $K_{1,m}$ with $m\geq 2$, denote the two bipartitions as $\{a\}$ and $\{b_1,b_2,\cdots,b_m\}$. Then a dynamic 3-coloring $c: K_{1,m}\mapsto C(3)$ can be given by defining $c(a)=1, c(b_1)=2$ and $c(b_i)=3$ for $i\geq 2$. On the other hand, a has at least two neighbors and so its neighbors must receive at least two distinct colors different from the color of a. Hence, it is necessary to use 3 colors. So, $\chi_d(K_{1,m}) = 3$.

Now we show that $\chi_d(K_{m,n}) = 4$ for $m, n \geq 2$. Denote the two bipartitions of $K_{m,n}$ as X and Y. Let $X = \{x_1, x_2, \cdots, x_m\}$, $Y = \{y_1, y_2, \cdots, y_n\}$. Then a dynamic 4-coloring $c : X \cup Y \mapsto C(4)$ can be given as follows: $c(x_1) = 1, c(x_i) = 2$, for $i \geq 2$ and $c(y_1) = 3, c(y_j) = 4$, for $j \geq 2$. This shows that $\chi_d(K_{m,n}) \leq 4$. On the other hand, let c' be a dynamic coloring for $K_{m,n}$. Since $m, n \geq 2$, then each vertex has degree at least 2, so its neighbors must receive at least two different colors. Hence $|c'(X)|, |c'(Y)| \geq 2$. Also, by the definition of proper coloring, $c'(X) \cap c'(Y) = \emptyset$, as any vertex in X is adjacent to any vertex in Y. Thus, $|c'(K_{m,n})| \geq 4$ and so $\chi_d(K_{m,n}) \geq 4$. Therefore, $\chi_d(K_{m,n}) = 4$.

If $k \geq 3$, then it suffices to use k colors in a dynamic coloring of $K_{n_1,n_2,\cdots,n_k}$, for we can assign to vertices in each partition a distinct color and it is easy to verify that this is a dynamic k-coloring of the graph. It is also necessary to use k colors since the complete k-partite graph obviously contains a clique K_k. Hence, the dynamic chromatic number for complete k-partite graphs is k.

References

[1] J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications", American Elsevier, New York, 1976.

[2] B. Montgomery, "Dynamic Coloring", Ph.D. Dissertation, West Virginia University, 2001.