A census of critical sets in the Latin squares of order at most six

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ABSTRACT: A critical set in a Latin square of order n is a set of entries from the square which can be embedded in precisely one Latin square of order n, such that if any element of the critical set is deleted, the remaining set can be embedded in more than one Latin square of order n. In this paper we find all the critical sets of different sizes in the Latin squares of order at most six. We count the number of main and isotopy classes of these critical sets and classify critical sets from the main classes into various "strengths". Some observations are made about the relationship between the numbers of classes, particularly in the 6×6 case. Finally some examples are given of each type of critical set.

1 Introduction

A *Latin square* L of order n is an $n \times n$ array with entries chosen from a set N, of size n, such that each element of N occurs precisely once in each row and column. For convenience, a Latin square will sometimes be represented as a set of ordered triples $(i, j; k)$. This means that element k occurs in cell (i, j) of the Latin square L. For a Latin square of order n we use $1, 2, 3, \ldots, n$ as the entries, and rows and columns will also be labelled

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from $1, \ldots, n$. A *partial Latin square* P of order n is an $n \times n$ array with entries chosen from a set N, of size n, such that each element of N occurs at most once in each row and column. Thus P may contain a number of empty cells. We sometimes denote P by the set $\{(i, j; k) | i, j, k \in N\}$. If a Latin square L contains an $s \times s$ subarray S and if S is a Latin square of order s, then we say that S is a *Latin subsquare* of L. A *critical set* in a Latin square L of order n, is a set $C = \{(i, j; k) \mid i, j, k \in N\}$, such that

(1) L is the only Latin square of order n which has element k in cell (i, j) for each $(i, j; k) \in C$; and

(2) no proper subset of C satisfies (1).

A partial Latin square U of L which satisfies condition (1) is called a *uniquely completable* (UC) set in L. A *smallest critical set* in a Latin square L is a critical set of minimum cardinality.

Let L be a Latin square of order n and let $\{a, b, c\} = \{1, 2, 3\}$. The (a, b, c)-*conjugate* of L, written $L_{(a,b,c)}$, is defined as follows:

$$L_{(a,b,c)} = \{(x_a, x_b; x_c) \mid (x_1, x_2; x_3) \in L\}.$$

Two Latin squares L and L' of order n are *isotopic* if there are three bijections from the rows, columns, and symbols of L to the rows, columns, and symbols, respectively, of L', that map L to L'. Two Latin squares L and L' of order n are *main class isotopic* if L is isotopic to any conjugate of L'. Table 1 shows the number of main and isotopy classes for Latin squares of order $1 \leq n \leq 7$ (see Denes and Keedwell [7]).

$n =$	1	2	3	4	5	6	7
Main classes	1	1	1	2	2	12	147
Isotopic classes	1	1	1	2	2	22	564

Table 1

In the process of completing the critical set C to the Latin square L of order n which it characterizes, we say that adjunction of a triple $t = (r, c; s)$ is *forced* (see [11]) in the process of completion of a set T of triples ($|T| < n^2$, $C \subseteq T \subset L$) to the complete set of triples which represents L, if either

(i) $\forall r' \neq r$, $\exists z \neq c$ such that $(r', z; s) \in T$ or $\exists z \neq s$ such that $(r', c; z) \in T$, or

(ii) $\forall c' \neq c$, $\exists z \neq r$ such that $(z, c'; s) \in T$ or $\exists z \neq s$ such that $(r, c'; z) \in T$, or

(iii) $\forall s' \neq s$, $\exists z \neq r$ such that $(z, c; s') \in T$ or $\exists z \neq c$ such that $(r, z; s') \in T$.

The critical set C is called *strong* if we can define a sequence of sets of triples $C = F_1 \subset F_2 \subset F_3 \subset \ldots \subset F_r = L$ such that each triple $t \in F_{i+1} \setminus F_i$ is forced in F_i for $1 \leq i \leq r-1$. A critical set which is not strong is called *weak*. A critical set is called *totally weak* if no cell is forced. A *smallest weak critical set* in a Latin square L is a weak critical set of minimum cardinality. (Note that the authors of [3] have used the term *semi-strong* in place of strong as defined above.)

The following extension of the concept of the strong critical set is taken from Bedford and Whitehouse [4].

Let P be a partial Latin square of order n defined on an element set N. Then A_P is an *array of alternatives* for P if

1. A_P is an $n \times n$ array;

2. whenever the cell at (i, j) in P is filled, the cell at (i, j) of A_P is empty; and

3. whenever the cell at (i, j) in P is empty, the cell at (i, j) of A_P contains all the elements of N which do not appear in the i^{th} row or j^{th} column of P.

We denote the set of elements in cell (i, j) of A_P by $A_P(i, j)$.

Let P be a partial Latin square. We say that the symbol $k' \in A_P(i, j)$ is *forced out* of A_P if either:

(1) there exists $r > 0$ and $i_1, i_2, \ldots, i_r$ (all $\neq i$) with $k' \in A_P(i_1, j) \cup \ldots \cup A_P(i_r, j)$ and $|A_P(i_1, j) \cup \ldots \cup A_P(i_r, j)| = r$; or

(2) $\theta(i, j, k')$ satisfies 1 in $A_{P_{\theta(1,2,3)}}$ for some $\theta \in S_3$.

The reduced array of alternatives, RA_P, is the array obtained from A_P by successively removing symbols which are forced out until no more symbols can be forced out. Then the addition of an entry $(i, j; k)$ to P is said to be *semi-forced* if either:

1. k is the only symbol in $RA_P(i, j)$; or

2. k occurs exactly once in either the i^{th} row or j^{th} column of RA_P.

Then a UC set U is *near-strong* to the Latin square L if we can find a sequence of sets $U = S_1 \subset \ldots \subset S_f = L$ such that each triple $t \in S_{v+1} \setminus S_v$ is semi-forced in S_v. We call a UC set *Bedford-Whitehouse totally weak*

(BW totally weak) if no cell is semi-forced. If a UC set is BW totally weak, this implies that it is also totally weak.

We denote the number of critical sets of size x in a main class $y.z$ by $CS(y, z, x)$. (The notation $y.z$ denotes main class z in a Latin square of order y, as in the CRC Handbook of Combinatorial Designs, [5].) The number of isotopy classes of critical sets of size x in a main class $y.z$ shall be denoted $IC(y, z, x)$, and the number of main classes of these critical sets shall be denoted $MC(y, z, x)$. The greatest common divisor of the number of critical sets of all sizes in a particular main class $y.z$ will be referred to as GCDCS(y, z).

The sizes of smallest critical sets for the Latin squares of orders four and five were determined in [8, 6]. Howse in [10] finds smallest critical sets for all the Latin squares of order six. Adams and Khodkar in [1] give smallest critical sets for all the Latin squares of order at most seven. They also find [2] smallest weak and smallest totally weak critical sets for the Latin squares of order at most seven. The size of smallest strong critical sets in a Latin square has also been considered in the past (see [3]).

This paper deals with critical sets of different sizes in the Latin squares of order at most six. First, for each main class of Latin square of order at most six, we calculate every possible critical set. (These will be of various sizes.) Then, again for each main class and possible size, the main classes and isotopy classes for the relevant set of critical sets are determined. Next, we determine which of the main classes are strong, near-strong, totally weak, and BW totally weak. Some interesting properties concerning the greatest common divisors of numbers of critical sets across main classes in the 6×6 Latin squares and ratios of various kinds are discussed.

In the appendix, we list examples for each possible size of critical set in each main class of 6×6 Latin square.

2 Algorithms

There were two basic algorithms used to independently calculate all critical sets of a given size m for a given main class of $n \times n$ Latin squares. The first examined all $\binom{n^2}{m}$ size m subsets of the Latin square. If any subset U had unique completion, all the proper subsets of U of size $\mid U \mid -1$ were tested for unique completion. If no such subset had unique completion, U was output as a critical set.

The second algorithm used Latin interchanges (see [9]) to speed up this process. Interchanges are subsets of a Latin square which have the property that any critical set must intersect every Latin interchange in at least one cell. This program divided the Latin square up into disjoint Latin interchanges, ensuring that each candidate for a critical set had at least one element in each of the Latin interchanges.

For the 6 × 6 case, for critical sets of size greater than 18, the process was further speeded up by ensuring that in each subset examined, no row or column was full and no symbol occurred six times. Such subsets cannot be critical sets as any cell may be removed from the relevant row, column or symbol set while maintaining the unique completion property.

3 Table of results

3.1 Explanation of headings

The first column in the tables of results (Tables 2, 3, 4 and 5) is the main class number $y.z$ (LS), followed by the size(s) of the critical set(s) for the main class (Size), the number of critical sets of that size in the main class (#CS); this is then followed by the number of isotopy classes (#Iso) and the number of main classes of those critical sets (#Main). (The notation $y.z$ denotes main class z in a Latin square of order y, as in the CRC Handbook of Combinatorial Designs, [5].)

For the next four columns, we consider representatives of each main class of critical sets, and list the number of critical sets of various "strengths" within the main classes of critical sets. That is, we calculate how many of the representatives of each main class of critical set have the various "strengths". We need only consider representatives of each main class of critical set, since, for example, a strong critical set remains a strong critical set when the rows, columns and symbols are interchanged or swapped. Similarly, a near-strong critical set remains near-strong under permutations or interchanges of rows, columns or symbols. These last four columns are, in order, the number of near-strong critical sets (#NS), the number of strong critical sets (#Strong), the number of totally weak critical sets (#TW), and the number of BW totally weak critical sets (#BWTW).

3.2 Latin squares of order 3

There is only one main class, denoted 3.1, for Latin squares of order three [5]:

1	2	3
2	3	1
3	1	2

3.1

For this Latin square, we have the results presented in Table 2 concerning the number of critical sets of every possible size.

LS	Size	#CS	#Iso	#Main	#NS	#Strong	#TW	#BWTW
3.1	2	9	1	1	1	1	0	0
	3	18	1	1	1	1	0	0

Table 2

3.3 Latin squares of order 4

There are two main classes, denoted 4.1 and 4.2, for Latin squares of order four [5]:

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

4.1

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

4.2

For these Latin squares, we have the results presented in Table 3 concerning the number of critical sets of every possible size.

LS	Size	#CS	#Iso	#Main	#NS	#Strong	#TW	#BWTW
4.1	4	32	1	1	1	1	0	0
	5	576	18	4	4	4	0	0
	6	128	4	2	2	2	0	0
4.2	5	96	1	1	1	1	0	0
	6	432	7	3	3	3	0	0
	7	48	1	1	1	1	0	0

Table 3

3.4 Latin squares of order 5

There are two main classes, denoted 5.1 and 5.2, for Latin squares of order five [5]:

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

5.1

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

5.2

For this Latin square, we have the results presented in Table 4 concerning the number of critical sets of every possible size.

LS	Size	#CS	#Iso	#Main	#NS	#Strong	#TW	#BWTW
5.1	6	50	1	1	1	1	0	0
	7	1000	10	4	4	4	0	0
	8	30900	312	57	57	57	0	0
	9	18800	188	37	37	37	0	0
	10	2500	25	6	6	6	0	0
5.2	7	600	50	11	10	10	1	1
	8	21588	1802	322	311	311	1	1
	9	23718	1981	348	348	348	0	0
	10	2340	198	39	38	36	2	0
	11	216	18	4	4	4	0	0

Table 4

3.5 Latin squares of order 6

There are twelve main classes, denoted 6.1, ..., 6.12, for Latin squares of order six [5]:

1	2	3	4	5	6
2	3	4	5	6	1
3	4	5	6	1	2
4	5	6	1	2	3
5	6	1	2	3	4
6	1	2	3	4	5

6.1

1	2	3	4	5	6
2	1	4	3	6	5
3	4	5	6	1	2
4	3	6	5	2	1
5	6	1	2	4	3
6	5	2	1	3	4

6.2

1	2	3	4	5	6
2	1	4	5	6	3
3	4	1	6	2	5
4	5	6	1	3	2
5	6	2	3	1	4
6	3	5	2	4	1

6.3

1	2	3	4	5	6
2	1	4	5	6	3
3	4	1	6	2	5
4	5	6	1	3	2
5	6	2	3	4	1
6	3	5	2	1	4

6.4

1	2	3	4	5	6
2	1	4	5	6	3
3	4	2	6	1	5
4	5	6	2	3	1
5	6	1	3	4	2
6	3	5	1	2	4

6.5

1	2	3	4	5	6
2	1	4	5	6	3
3	4	5	6	1	2
4	5	6	3	2	1
5	6	1	2	3	4
6	3	2	1	4	5

6.6

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	2	4
4	6	2	5	1	3
5	3	6	1	4	2
6	4	5	2	3	1

6.7

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	2	4
4	6	2	5	1	3
5	3	6	2	4	1
6	4	5	1	3	2

6.8

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	2	4
4	6	2	5	3	1
5	4	6	2	1	3
6	3	5	1	4	2

6.9

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	4	2
4	6	5	1	2	3
5	3	6	2	1	4
6	4	2	5	3	1

6.10

1	2	3	4	5	6
2	1	4	5	6	3
3	4	2	6	1	5
4	6	5	2	3	1
5	3	6	1	2	4
6	5	1	3	4	2

6.11

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
4	6	5	2	1	3
5	4	6	3	2	1
6	5	4	1	3	2

6.12

For these Latin squares, we have the results presented in Table 5 concerning the number of critical sets of every possible size.

LS	Size	#CS	#Iso	#Main	#NS	#Strong	#TW	#BWTW
6.1	9	72	1	1	1	1	0	0
	11	39384	547	97	97	95	0	0
	12	1161036	16149	2740	2541	2513	11	8
	13	3634344	50492	8481	7815	7792	19	14
	14	886428	12346	2090	1931	1920	10	9
	15	80064	1118	202	182	168	8	0
	16	3240	45	8	8	8	0	0
	17	108	3	1	0	0	0	0
6.2	11	7848	327	60	50	48	3	3
	12	658908	27477	4633	4370	4325	35	27
	13	3328908	138708	23267	22226	22187	52	36
	14	1800228	75035	12617	12267	12263	11	8
	15	192480	8022	1362	1354	1351	3	1
	16	15840	660	115	113	113	0	0
	17	240	10	3	3	3	0	0

Table 5

LS	Size	#CS	#Iso	#Main	#NS	#Strong	#TW	#BWTW
6.3	11	1200	10	7	2	2	0	0
	12	192360	1603	836	749	748	14	14
	13	1837440	15315	7757	7445	7440	33	29
	14	1727880	14400	7279	7252	7252	2	2
	15	378928	3162	1610	1610	1610	0	0
	16	20280	169	90	90	90	0	0
	17	840	7	4	4	4	0	0
6.4	10	56	7	5	5	5	0	0
	11	34000	4250	2149	2001	1980	16	10
	12	1590608	198826	99654	94485	94024	197	136
	13	5498076	687262	344044	328754	327801	331	232
	14	1931424	241428	120895	116390	115691	58	34
	15	168752	21095	10586	10102	9704	137	10
	16	13736	1717	871	821	780	24	1
	17	148	19	11	9	9	1	1
6.5	10	60	15	9	9	9	0	0
	11	42992	10748	5406	5132	5078	30	24
	12	1878236	469559	235063	224705	223776	401	292
	13	6475142	1618806	809952	778258	776251	648	473
	14	2182652	545663	273120	264790	263229	119	76
	15	192416	48104	24108	23304	22340	281	28
	16	16908	4227	2135	2041	1961	43	3
	17	112	28	16	15	15	0	0
6.6	11	12888	358	187	177	175	0	0
	12	856908	23803	12005	11191	11155	64	55
	13	4097790	113839	57151	54038	53898	162	120
	14	1476864	41024	20664	19770	19697	27	23
	15	155196	4311	2201	2139	2117	8	4
	16	12744	354	186	175	166	3	0
	17	216	6	4	4	4	0	0
6.7	12	4752	22	5	3	3	0	0
	13	212328	985	183	165	165	5	5
	14	893700	4151	736	706	705	3	2
	15	545508	2536	465	465	465	0	0
	16	125766	583	109	109	109	0	0
	17	8208	38	13	13	13	0	0
	18	648	3	1	1	1	0	0

Table 5 (continued)

LS	Size	#CS	#Iso	#Main	#NS	#Strong	#TW	#BWTW
6.8	11	3264	408	75	67	66	3	2
	12	324608	40576	6817	6023	5986	37	33
	13	1826592	228335	38265	35161	35063	123	80
	14	1093796	136729	22909	21804	21764	10	8
	15	106296	13290	2260	2178	2155	10	1
	16	8464	1058	185	175	167	6	1
	17	216	27	5	5	5	0	0
6.9	10	24	2	2	2	2	0	0
	11	13980	1165	596	546	535	7	7
	12	716352	59714	29939	27999	27723	127	84
	13	2784264	232027	116246	109378	108885	243	173
	14	1065876	88856	44575	42345	42068	24	19
	15	85884	7159	3607	3462	3314	72	4
	16	6960	580	302	283	259	15	1
	17	24	2	2	2	2	0	0
6.10	10	4	1	1	1	1	0	0
	11	13748	3437	587	555	547	6	3
	12	858348	214587	35899	33814	33644	169	123
	13	3894038	973520	162538	154803	154404	279	195
	14	1715492	428873	71685	69560	69375	38	29
	15	155000	38753	6513	6355	6232	34	4
	16	10540	2635	461	443	423	13	1
	17	120	30	6	6	6	0	0
6.11	10	40	10	3	3	3	0	0
	11	63540	15885	2673	2617	2590	9	5
	12	2292266	573254	95781	92453	92029	96	59
	13	7075888	1768972	295196	284917	284027	145	96
	14	2203696	550993	91977	88209	87499	39	22
	15	188344	47086	7890	7584	7175	161	8
	16	17172	4293	729	685	645	35	4
	17	36	9	2	1	1	0	0
6.12	11	143208	1326	232	229	228	0	0
	12	3518478	32664	5510	5384	5358	7	6
	13	9025344	83568	14037	13636	13584	17	14
	14	2104704	19506	3315	3146	3107	6	4
	15	200340	1855	326	316	297	8	1
	16	17820	165	32	29	28	1	0

Table 5 (continued)

Dénes and Keedwell [7] point out that, for a given order n, each isotopy class of $n \times n$ Latin squares has a number of Latin squares associated with it, and similarly each main class of $n \times n$ Latin squares has a number of isotopy classes associated with it. Similarly, for any given main class $n.z$ of $n \times n$ Latin squares and given size of critical set m, if we consider the main classes of critical sets of size m within the main class $n.z$, we have several associated isotopy classes of critical sets. In the same way, if we consider the isotopy classes of critical sets of size m within the main class $n.z$, we have several associated critical sets of size m in the main class $n.z$.

In Tables 6 to 9, the head line refers to the twelve main classes of 6×6 Latin squares, and the side line refers to the possible sizes of critical sets. For 6×6 Latin squares, we consider results related to these observations.

Each isotopy class of critical sets in 6×6 Latin squares has between 2 and 216 associated critical sets. This result is given in Table 6.

	6.1	6.2	6.3	6.4	6.5	6.6
9	72	-	-	-	-	-
10	-	-	-	8	4	-
11	72	24	120	8	4	36
12	12,36,72	8,12,24	8	8	4	36
13	36,72	12,24	60,120	4,8	2,4	18,36
14	36,72	12,24	60,120	8	4	36
15	36,72	12,24	24,40,120	4,8	4	36
16	72	24	120	8	4	36
17	36	24	120	4,8	4	36
18	-	-	-	-	-	-

Table 6

	6.7	6.8	6.9	6.10	6.11	6.12
9	-	-	-	-	-	-
10	-	-	12	4	4	-
11	-	8	12	4	4	108
12	216	8	6,12	4	2,4	54,108
13	108,216	4,8	6,12	2,4	4	108
14	108,216	4,8	6,12	4	2,4	54,108
15	108,216	4,8	6,12	2,4	4	108
16	54,216	8	12	4	4	108
17	216	8	12	4	4	-
18	216	-	-	-	-	-

Table 6 (continued)

Each main class of critical sets in 6×6 Latin squares has either 1, 2, 3 or 6 associated isotopy classes of critical sets. This result is given in Table 7.

	6.1	6.2	6.3	6.4	6.5	6.6
9	1	-	-	-	-	-
10	-	-	-	1,2	1,2	-
11	1,3,6	3,6	1,2	1,2	1,2	1,2
12	1,2,3,6	1,2,3,6	1,2	1,2	1,2	1,2
13	2,3,6	1,2,3,6	1,2	1,2	1,2	1,2
14	1,2,3,6	1,2,3,6	1,2	1,2	1,2	1,2
15	1,2,3,6	1,2,3,6	1,2	1,2	1,2	1,2
16	3,6	3,6	1,2	1,2	1,2	1,2
17	3	1,3,6	1,2	1,2	1,2	1,2
18	-	-	-	-	-	-

Table 7

	6.7	6.8	6.9	6.10	6.11	6.12
9	-	-	-	-	-	-
10	-	-	1	1	1,3,6	-
11	-	3,6	1,2	2,3,6	1,2,3,6	3,6
12	1,3,6	1,2,3,6	1,2	1,2,3,6	1,2,3,6	1,2,3,6
13	1,3,6	1,2,3,6	1,2	1,2,3,6	1,2,3,6	3,6
14	1,2,3,6	1,2,3,6	1,2	2,3,6	2,3,6	3,6
15	1,3,6	3,6	1,2	1,2,3,6	1,3,6	1,2,3,6
16	1,3,6	2,3,6	1,2	1,2,3,6	3,6	3,6
17	1,3,6	3,6	1	3,6	3,6	-
18	3	-	-	-	-	-

Table 7 (continued)

4 Some observations

We will concentrate on observations concerning the 6×6 Latin squares.

We find that when the main class z is fixed and x takes all possible values, $CS(6, z, x)$ / $IC(6, z, x)$ is in most cases close to an integer constant. There is one exception: the critical sets of size 17 in main class 6.1, where all other values of $CS(6, 1, x)/IC(6, 1, x)$ are approximately 72, but $CS(6, 1, 17)/IC(6, 1, 17) = 36$. We also find that this integer constant is a multiple of GCDCS$(6, z)$. These ratios are given in Table 8, truncated at two decimal places. The bottom line tabulates the values of GCDCS$(6, z)$.

	6.1	6.2	6.3	6.4	6.5	6.6
9	72.00	-	-	-	-	-
10	-	-	-	8.00	4.00	-
11	72.00	24.00	120.00	8.00	4.00	36.00
12	71.89	23.98	120.00	8.00	4.00	36.00
13	71.97	23.99	119.97	7.99	3.99	35.99
14	71.79	23.99	119.99	8.00	4.00	36.00
15	71.61	23.99	119.83	7.99	4.00	36.00
16	72.00	24.00	120.00	8.00	4.00	36.00
17	36.00	24.00	120.00	7.78	4.00	36.00
18	-	-	-	-	-	-
gcd	36	12	2	4	2	18

Table 8

	6.7	6.8	6.9	6.10	6.11	6.12
9	-	-	-	-	-	-
10	-	-	12.00	4.00	4.00	-
11	-	8.00	12.00	4.00	4.00	108.00
12	216.00	8.00	11.99	4.00	3.99	107.71
13	215.56	7.99	11.99	3.99	4.00	108.00
14	215.29	7.99	11.99	4.00	3.99	107.90
15	215.10	7.99	11.99	3.99	4.00	108.00
16	215.72	8.00	12.00	4.00	4.00	108.00
17	216.00	8.00	12.00	4.00	4.00	-
18	216.00	-	-	-	-	-
gcd	54	4	12	2	2	54

Table 8 (continued)

In seven of the twelve 6×6 main classes (6.1, 6.2, 6.7, 6.8, 6.10, 6.11, and 6.12), the ratio $MC(6, z, x)/IC(6, z, x)$ is close to 6 with a few exceptions. In the other five main classes (6.3, 6.4, 6.5, 6.6, and 6.9) this ratio is close to 2 with a few exceptions. These ratios are given in Table 9, truncated at two decimal places.

	6.1	6.2	6.3	6.4	6.5	6.6
9	1.00	-	-	-	-	-
10	-	-	-	1.40	1.66	-
11	5.63	5.45	1.42	1.97	1.98	1.91
12	5.89	5.93	1.91	1.99	1.99	1.98
13	5.95	5.96	1.97	1.99	1.99	1.99
14	5.90	5.94	1.97	1.99	1.99	1.98
15	5.53	5.88	1.96	1.99	1.99	1.95
16	5.62	5.73	1.87	1.97	1.97	1.90
17	3.00	3.33	1.75	1.72	1.75	1.50
18	-	-	-	-	-	-

Table 9

	6.7	6.8	6.9	6.10	6.11	6.12
9	-	-	-	-	-	-
10	-	-	1.00	1.00	3.33	-
11	-	5.44	1.95	5.85	5.94	5.71
12	4.40	5.95	1.99	5.97	5.98	5.92
13	5.38	5.96	1.99	5.98	5.99	5.95
14	5.63	5.96	1.99	5.98	5.99	5.88
15	5.45	5.88	1.98	5.95	5.96	5.69
16	5.34	5.71	1.92	5.71	5.88	5.15
17	2.92	5.40	1.00	5.00	4.50	-
18	3.00	-	-	-	-	-

Table 9 (continued)

In each 6×6 main class $6.z$, GCDCS$(6, z)$ is a multiple of 2, and for those main classes with 3×3 subsquares (6.1, 6.6, 6.7 and 6.12) this number is a multiple of 18.

We also note that in the 4×4 and 6×6 main classes of Latin squares, the smallest and largest possible critical sets (4 and 7 for the 4×4 case and 9 and 18 for the 6×6 case) each have only one isotopy and main class.

5 Appendix

In this appendix, we give examples of critical sets of each possible size in each of the twelve main classes of 6×6 Latin squares.

1	2	3			
2	3				
3					
					4
				4	5

6.1, size 9

					1
				1	2
			1	2	3
		1	2	3	4
6					

6.1, size 11

					1
				1	2
			1	2	3
	6	1	2	3	
6					5

6.1, size 12

					1
				1	2
		6	1	2	
	6	1	2	3	
			3	4	5

6.1, size 13

		1			4
	1	2		4	5
					1
		4	3	1	2
	4	5		2	

6.1, size 14

					1
				1	2
			1	2	3
		1	2	3	4
	1	2	3	4	5

6.1, size 15

					1
		5		1	2
	5	6	1	2	
	6	1	2		4
	1	2	3		5

6.1, size 16

			5	6	1
		5		1	2
	5	6	1	2	3
	6	1		3	
	1	2	3		

6.1, size 17

	1		3		5
			6	1	
	3	6		2	
5					
		2			4

6.2, size 11

	1		3		5
				1	2
	3	6			
5			2		
6		2			4

6.2, size 12

	1		3		5
				1	2
	3		5	2	
	6	1		4	
		2			4

6.2, size 13

	1		3		5
				1	2
	3		5	2	
		1	2	4	
	5	2			4

6.2, size 14

	1		3		5
				1	2
	3		5	2	
		1	2	4	
6		2	1	3	

6.2, size 15

	1		3		5
				1	2
	3	6	5	2	1
	6	1			
	5	2	1	3	

6.2, size 16

	1		3		5
				1	2
	3		5	2	1
		1	2		3
	5	2	1	3	4

6.2, size 17

				5	6
	1		3		
		1			
4					2
	4			1	
6		2			

6.3, size 11

					6
	1		3		
		1		2	
		5	1		
5			2		
6		2		4	

6.3, size 12

					6
	1		3		
		1		2	
			1	3	2
5	4				
	3		5	4	

6.3, size 13

					6
	1		3		
		1		2	
			1	3	2
	4			1	3
6		2		4	

6.3, size 14

	1		3		5
		1		2	4
			1	3	2
		6	2	1	
6		2		4	

6.3, size 15

	1		3		5
		1		2	4
			1	3	2
		6	2	1	
	3		5	4	1

6.3, size 16

	1		3		5
		1		2	4
			1	3	2
	4			1	3
	3	2	5	4	1

6.3, size 17

			4		6
	1				
	5	1			
4					3
6		2			4

6.4, size 10

	1		3		5
		1			2
			1		3
	4	6			
6		2			

6.4, size 11

	1		3		5
		1			2
			1		3
	4	6			
6	3		5		

6.4, size 12

	1		3		5
		1			2
			1		3
		6	2	3	
6	3			1	

6.4, size 13

	1		3		5
		1			2
			1		3
		6	2	3	
	3		5	1	4

6.4, size 14

	1		3		5
		1			2
			1		3
	4	6	2	3	
	3	2		1	4

6.4, size 15

	1		3		5
		1			2
			1		3
	4		2	3	1
	3	2	5	1	4

6.4, size 16

	1		3		5
3		1			2
4		5	1		3
5	4		2		
	3	2	5		4

6.4, size 17

					6
	1		3		5
		5			
	5				1
		2		4	
				2	

6.5, size 10

	1		3		5
				1	2
	5				
		2		4	
		1		2	4

6.5, size 11

	1		3		5
				1	2
			2	3	1
5	6				
6				2	

6.5, size 12

	1		3		5
				1	2
			2	3	1
		2	1	4	3
6					

6.5, size 13

	1		3		5
				1	2
			2	3	1
5				4	3
6	3			2	

6.5, size 14

	1		3		5
				1	2
			2	3	1
	6	2		4	
		1	5	2	4

6.5, size 15

	1		3		5
				1	2
			2	3	1
	6	2	1	4	
	3	1		2	4

6.5, size 16

	1		3		5
				1	2
			2	3	1
		2	1	4	3
	3	1	5	2	4

6.5, size 17

					6
	1		3		
				1	
4	5				
5	6				4
		2		4	

6.6, size 11

					6
	1		3		
				1	
			1	2	3
5			2		
6		2		4	

6.6, size 12

	1		3		5
				1	2
			1	2	3
5	6				
6		2		4	

6.6, size 13

	1		3		5
				1	2
			1	2	3
	6	1		3	
6		2		4	

6.6, size 14

	1		3		5
				1	2
			1	2	3
		1	2	3	4
6		2		4	

6.6, size 15

	1		3		5
				1	2
		6	1	2	
		1	2	3	4
	3		5	4	1

6.6, size 16

	1		3		5
				1	2
			1	2	3
		1	2	3	4
	3	2	5	4	1

6.6, size 17

					6
		1			
	1	2	5		
4	5				
5			3		
			2	3	1

6.7, size 12

		1			5
3				6	
			1	2	3
	6	4		1	
6	4	5			

6.7, size 13

		1			5
	1	2		6	
			1	2	3
	6		3	1	
6	4			3	

6.7, size 14

		1			5
	1	2		6	
			1	2	3
		4	3	1	2
6			2	3	

6.7, size 15

		1			5
	1	2		6	
			1	2	3
		4	3	1	2
	4	5	2		1

6.7, size 16

		1			5
	1	2		6	
	5	6	1	2	3
			3	1	2
		5	2	3	1

6.7, size 17

		1			5
	1	2	5		4
			1	2	3
		4	3	1	2
	4	5	2	3	1

6.7, size 18

	1		3	6	
	5	1			4
4					3
		6			
				3	2

6.8, size 11

	1		3		5
		1		2	4
4	6				
	3	6			
				3	2

6.8, size 12

	1		3		5
		1		2	4
		2	5		
	3			4	
	4	5			2

6.8, size 13

	1		3		5
		1		2	4
		2	5		
	3	6		4	
	4			3	2

6.8, size 14

	1		3		5
		1		2	4
		2	5		
5		6		4	1
6		5			2

6.8, size 15

	1		3		5
		1	6	2	
	6	2		1	3
	3	6		4	1
			1	3	

6.8, size 16

	1		3		5
3	5				4
4			5	1	3
5	3			4	1
	4			3	2

6.8, size 17

					6
	1	4	3		
	5	1			
	4			1	
6					2

6.9, size 10

	1		3		5
		1	6	2	
		2			
5	4				3
	3				

6.9, size 11

	1		3		5
		1		2	4
		2	5		
		6	2	1	
	3				

6.9, size 12

	1		3		5
		1		2	4
		2	5		
		6	2	1	
6				4	

6.9, size 13

	1		3		5
		1		2	4
		2	5		
			2	1	3
	3	5		4	

6.9, size 14

	1		3		5
		1		2	4
4	6				
5	4	6			3
6	3	5			

6.9, size 15

	1		3		5
3				2	4
4			5	3	1
5	4		2	1	3
					2

6.9, size 16

	2	3	4	5	6
		4	3		5
	5				4
	4	6			3
	3	5		4	2

6.9, size 17

1				5	6
			3		
		1			
4					
	3		2		
6					1

6.10, size 10

					6
	1		3		
		1			
4					
	3		2	1	
6			5	3	

6.10, size 11

	1		3		5
		1			2
			1		3
5		6			
6	4			3	

6.10, size 12

	1		3		5
		1			2
	6		1		
	3	6		1	
6		2	5		

6.10, size 13

	1		3		5
		1			2
			1		3
	3		2	1	4
6	4			3	

6.10, size 14

	1		3		5
		1			2
			1		3
	3	6	2	1	
	4	2		3	1

6.10, size 15

	1		3		5
		1			2
			1		3
	3		2	1	4
	4	2	5	3	1

6.10, size 16

	1		3	6	
		1	6		
	6	5	1	2	3
	3	6	2	1	
			5	3	1

6.10, size 17

					6
2	1				
	4			1	
				3	
	3		1		
6					2

6.11, size 10

	1				3
				1	5
			2	3	1
5		6			
6					2

6.11, size 11

	1				3
				1	5
			2	3	1
		6	1	2	
6	5				

6.11, size 12

	1				3
				1	5
			2	3	1
5				2	4
6			3	4	

6.11, size 13

	1				3
				1	5
			2	3	1
5				2	4
		1	3	4	2

6.11, size 14

	1				3
				1	5
			2	3	1
	3	6	1	2	
		1	3	4	2

6.11, size 15

	1				3
				1	5
			2	3	1
	3		1	2	4
	5	1	3	4	2

6.11, size 16

	1		5	6	
3	4		6	1	
	6			3	
	3	6	1		
6	5	1	3	4	

6.11, size 17

		1			5
	1	2		6	
				3	
		4		2	3
6			3		

6.12, size 11

		1			5
	1	2		6	
				3	
		4		2	3
6	4	5			

6.12, size 12

		1			5
	1	2		6	
				3	
		4	1	2	3
	4	5			2

6.12, size 13

		1			5
	1	2		6	
				3	
	6	4		2	3
	4		3	1	2

6.12, size 14

		1			5
	1	2		6	
			2		1
		4	1	2	3
	4	5	3		2

6.12, size 15

		1			5
	1	2		6	4
	5	6	2		1
5	6	4	1		
	4	5			

6.12, size 16

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