Two General Results on Harmonious Labelings

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Abstract

In this paper, we show that if G is a harmonious graph, then (2n+1)G (the disjoint union of 2n+1 copies of G) and $G^{(2n+1)}$ (the graph consisting of 2n+1 copies of G with one fixed vertex in common) are harmonious for all $n \geq 0$.

1. Introduction

Graham and Sloane [4] defined a graph G of vertex set V(G) and edge set E(G) and of q edges to be harmonious if there exists an injective function f, called a harmonious labeling,

$$f : V(G) \rightarrow Z_q \quad \text{(the group of integers modulo q)}$$

such that the induced function

$$f^* : E(G) \rightarrow Z_q$$

defined by

$$f^*(xy) = (f(x) + f(y)) \pmod{q}, \quad \text{for all } xy \in E(G)$$

is an injection

According to Gallian's surveys [1], [2] and [3], there are few general results on graph labelings. Among of these, a necessary condition for certain families of graphs to be harmonious due to Graham and Sloane [4, Theorem 11] which state : If a harmonious graph has an even number q of edges and the degree of every vertex is divisible by 2^k $(k \geq 1)$, then q is divisible by 2^{k+1}. This necessary condition is called the harmonious parity condition. In this paper, we give two general results on the harmonious labelings.

More details and results on harmonious labelings can be found in ([1], [2], [3]).

2. Two General Results

In this section, we show that if G is a harmonious graph, then (2n+1)G (the disjoint union of 2n+1 copies of G) and $G^{(2n+1)}$ (the graph consisting of 2n+1 copies of G with one fixed vertex in common) are harmonious for all $n \geq 0$.

Theorem 2.1

If G is a harmonious graph, then mG is harmonious for all odd $m \geq 1$.

Proof.

Let $p = |V(G)|$, $q = |E(G)|$,

$V(G) = \{v_1, v_2, \ldots, v_p\}$ and $V(mG) = \{ v_1^i, v_2^i, \ldots, v_p^i : 1 \leq i \leq m\}$ such that for $1 \leq i \leq m$, the function

$$c_i : V(G) \to V(mG)$$

defined by

$$c_i(v_j) = v_j^i, \qquad 1 \leq j \leq p$$

defines a graph isomorphism of G onto the induced subgraph in mG by the image of c_i.

Let $f : V(G) \to Z_q$ be a harmonious lebeling of G. Define

$$g : V(mG) \to Z_{mq}$$

by

$$g(v_j^i) = f(v_j) + (i-1)q, \qquad 1 \leq i \leq m, \ 1 \leq j \leq p.$$

Then g is injective. Now we have to show that the induced function

$$g^* : E(mG) \to Z_{mq} \qquad \text{is also injective.}$$

Suppose that $v_j^i \, v_{j'}^i$, $v_s^r \, v_{s'}^r \in E(mG)$, where $1 \leq j < j' \leq p$ and $1 \leq s < s' \leq p$ and $g^*(v_j^i \, v_{j'}^i) \equiv g^*(v_s^r \, v_{s'}^r) \pmod{mq}$, then

$$f(v_j) + f(v_{j'}) + 2(i-1)q \equiv (f(v_s) + f(v_{s'}) + 2(r-1)q) \pmod{mq} \qquad (1)$$

then $f(v_j) + f(v_{j'}) \equiv (f(v_s) + f(v_{s'})) \pmod q$ and $v_j v_{j'} = v_s v_{s'}$ in G, since f is a harmonious labeling of G and from equation (1), we get

$$2(r-i)q \equiv 0 \pmod{mq}$$

which gives that $r = i$, since m is odd, hence g^* is injective as desired. □

Note that Theorem 2.1 is not valid if m is even since $2m\, C_{2n+1}$ is not harmonious for all $m, n \geq 1$ [7], although C_{2n+1} is harmonious for $n \geq 1$ [4].

Corollary 2.2

(i) $(2m+1)K_3$ is harmonious for all $m \geq 0$ [6].

(ii) $(2m+1)K_4$ is harmonious for all $m \geq 0$

(iii) $(2m-1)C_{2n+1}$ is harmonious for all $m, n \geq 1$ [7].

(iv) $(2m+1)W_n$ is harmonious for all $m \geq 0$ and $n \geq 3$.

Proof

K_3, K_4, C_{2n+1} $(n \geq 1)$ and W_n $(n \geq 3)$ are harmonious [4] and the corollary follows from Theorem 2.1. □

We note that, Corollary 2.2 (ii) gives a partial answer to the harmoniousness of the graph mK_4.

Figure 1, shows that the graph consisting of the disjoint union of three copies of the Petersen graph, is harmonious.

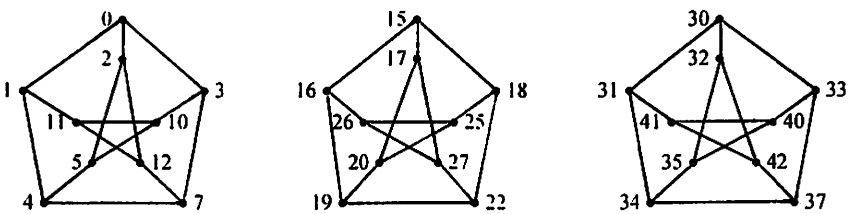

Figure 1.

Figure 2, shows that $5K_4$ is harmonious.

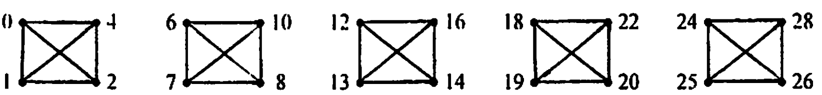

Figure 2.

Theorem 2.3

If G is a harmonious graph, then $G^{(m)}$ (the graph consisting of m copies of G with one fixed vertex in common) is harmonious for all odd $m \geq 1$.

Proof

Let $p = |V(G)|$, $q = |E(G)|$ and $V(G) = \{v_1, v_2, \ldots, v_p\}$. Let the fixed vertex be v_1. Since any vertex in a harmonious graph can be assigned the label

0 [4, Theorem 8], then we may assume that $f : V(G) \to Z_q$ is a harmonious labeling of G with $f(v_1) = 0$.

Let $V(G^{(m)}) = \{v_0, v_1^i, v_2^i, \ldots, v_{p-1}^i : 1 \le i \le m\}$ such that for $1 \le i \le m$, the function

$$c_i : V(G) \to V(G^{(m)})$$

defined by

$$c_i(v_1) = v_0$$

$$c_i(v_j) = v_{j-1}^i, \qquad 2 \le j \le p$$

defines a graph isomorphism of G onto the induced subgraph in $G^{(m)}$ by the image of c_i.

Define

$$g : V(G^{(m)}) \to Z_{mq}$$

as follows

$$g(v_0) = 0$$

$$g(v_j^i) = f(v_{j+1}) + (i-1)q, \quad 1 < i \le m, \qquad 1 \le j \le p-1.$$

Then g is injective. Now we have to show that the induced function $g^* : E(G^{(m)}) \to Z_{mq}$ is also injective. We have 3 cases to consider :

Case 1 $v_0 v_j^i, v_0 v_s^r \in E(G^{(m)})$, where $1 \le j, s \le p-1$

and $g^*(v_0 v_j^i) \equiv g^*(v_0 v_s^r) \pmod{mq}$, then

$$f(v_{j+1}) + (i-1)q = (f(v_{s+1}) + (r-1)q) \pmod{mq}$$

and

$$f(v_{j+1}) \equiv f(v_{s+1}) \pmod q$$

and this gives $j = s$ and $i = r$.

Case 2 $v_j^i v_{j'}^i, v_0 v_s^r \in E(G^{(m)})$, where

$1 \le j < j' \le p-1, \ 1 \le s \le p-1$ and

$g^*(v_j^i v_{j'}^i) \equiv g^*(v_0 v_s^r) \pmod{mq}$, then

$$f(v_{j+1}) + f(v_{j'+1}) + 2(i-1)q \equiv (f(v_{s+1}) + (r-1)q) \pmod{mq}$$

and

$$f(v_{j+1}) + f(v_{j'+1}) \equiv f(v_{s+1}) \pmod q$$

and this gives $v_{j+1}v_{j'+1} = v_1 v_{s+1}$ in G, which is absurd.

Case 3 $v_j^i v_{j'}^i,\ v_s^r v_{s'}^r \in E(G^{(m)})$, where

$1 \le j < j' \le p-1,\ 1 \le s < s' \le p-1$ and

$g^*(v_j^i v_{j'}^i) \equiv g^*(v_s^r v_{s'}^r) \pmod{mq}$, then

$$f(v_{j+1}) + f(v_{j'+1}) + 2(i-1)q \equiv (f(v_{s+1}) + f(v_{s'+1}) + 2(r-1)q) \pmod{mq}$$

and

$$f(v_{j+1}) + f(v_{j'+1}) \equiv (f(v_{s+1}) + f(v_{s'+1})) \pmod{q}$$

and this gives $j = s$ and $j' = s'$. Hence g^* is injective as desired. □

Note that Theorem 2.3 is not valid if $m \equiv 2 \pmod 4$, since $G_{2n+1}^{(4m+2)}$ is not harmonious by using the harmonious parity condition [4, Theorem 11].

The following corollary comes directly from Theorem 2.3.

Corollary 2.4

(i) $K_3^{(2m+1)}$ is harmonious for all $m \ge 0$ [4].

(ii) $K_4^{(2m+1)}$ is harmonious for all $m \ge 0$ [5].

(iii) C_{2n+1}^{2m-1} is harmonious for all $m, n \ge 1$ [8].

Figure 3, shows that $K_4^{(3)}$ is harmonious.

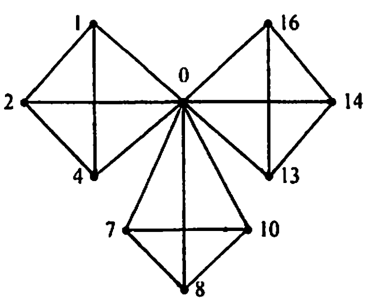

Figure 3.

Graham and Sloane [4] showed that $C_3^{(m)}$ is harmonious if and only if $m \equiv 2 \pmod 4$, this motivates the following question.

Question : If G is a harmonious graph, is $G^{(m)}$ harmonious if $m \equiv 0 \pmod 4$?.

References

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