On Bases of the Cycle and Cut Spaces in Digraphs

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Abstract. We find a maximal number of directed circuits (directed cocircuits) in a base of a cycle (cut) space of a digraph. We show that this space has a base composed from directed circuits (directed cocircuits) if and only if the digraph is totally cyclic (acyclic). Furthermore, this basis can be considered as an ordered set so that each element of the basis has an arc not contained in the previous elements.

1. Preliminaries

We consider graphs with multiple edges and loops. Endowing each edge of a graph G by an orientation, we get a digraph D which is called an *orientation* of G. The vertex and arc sets of D are denoted by $V(D)$ and $E(D)$, respectively. A subgraph D' of D is called a *component* of D if it is an orientation of a component of G. The arcs of D arising from loops and bridges of G are called *loops* and *bridges* of D, respectively.

By a *circuit* C in D we mean any subgraph which is an orientation of a connected 2-regular subgraph of G. Furthermore, if no two distinct arcs of C are directed to the same vertex, then C is called a *directed circuit.* Let F be a minimal cut in G (i. e., there is no cut $F' \subsetneq F$). Then the set B of arcs of D associated with the edges of F is called a *cocircuit* in D. Furthermore let B contain only arcs directed out from a new component arising from D after deleting the arcs of B. Then B is called a *directed cocircuit.* Every arc-disjoint union of (directed) circuits and (directed) cocircuits are called (*directed*) *cycle* and (*directed*) *cocycle*, respectively.

Let $\mathbb{F}_2$ denote the field of two elements. It is well known (see, e. g., [1, 2]), that the function $E(D) \to \mathbb{F}_2$ form the edge space $\mathcal{E}(G)$ of D (its elements are subsets of $E(D)$, vector addition amounts to symmetric difference, $\emptyset \subseteq E(D)$ is the zero, and $X = -X$ for every $X \subseteq E(D)$). The sets of all cycles and cocycles in D are subspaces of $\mathcal{E}(D)$ called *cycle* and *cut* (or *cocycle*) spaces of D and denoted by $\mathcal{C}(D)$ and $\mathcal{C}^*(D)$, respectively (for simplicity, we shall not distinguish between the subgraphs of D and their edge sets in this paper). It is known that $\mathcal{C}(D)^\perp = \mathcal{C}^*(D)$, $\mathcal{C}(D)$ has dimension $m(D) = |E(D)| - |V(D)| + c(D)$, $\mathcal{C}^*(D)$ has dimension $r(D) = |V(D)| - c(D)$, where $c(D)$ denotes the number of components of D (note that $r(D)$ is the number of edges in a spanning forest of D, i. e., a maximal subgraph of D not containing a circuit, and $m(D)$ is the number of edges of its complement). It is known that $\mathcal{C}(D)$ and $\mathcal{C}^*(D)$ have basis consisting of circuits and cocircuits, respectively. Now we study how many of them can be directed.

Let $D_{\mathcal{C}}$ ($D_{\mathcal{A}}$) be the subgraphs of D such that $V(D_{\mathcal{C}}) = V(D_{\mathcal{A}}) = V(D)$ and $e \in E(D_{\mathcal{C}})$ ($e \in E(D_{\mathcal{A}})$) iff e is contained in a directed circuit (directed cocircuit) in D. By Minty [5] (see also [1]), $E(D_{\mathcal{C}}) \cup E(D_{\mathcal{A}}) = E(D)$ and $E(D_{\mathcal{C}}) \cap E(D_{\mathcal{A}}) = \emptyset$. D is called *totally cyclic* (*acyclic*) if $D = D_{\mathcal{C}}$ ($D = D_{\mathcal{A}}$).

Lemma 1. *A digraph D is totally cyclic iff for every pair of distinct vertices (u, v) from one component of D there exist a directed u-v-path in D.*

Proof: Sufficiency is trivial. If D is totally cyclic, then there exists an directed u-v-trial in D, thus also a directed u-v-path. □

Contract in D all arcs from $D_{\mathcal{C}}$ (i. e., contract in D the subgraphs $D_1, \dots D_s$ which are components of $D_{\mathcal{C}}$ to vertices $v_1, \dots, v_s$, respectively). We get an acyclic digraph D' called a *condensation* of D. Clearly $D = D'$ iff D is acyclic. Usually we shall not distinguish between the arcs from D' and the arcs from $D_{\mathcal{A}}$ in this paper.

2. Bases in cycle and cut spaces

An n-tuple $(D_1, \dots, D_n)$ of subgraphs of a digraph D is called *triangular* if every D_i contains an arc e_i ($i = 1, \dots, n$) such that $e_i \notin C_j$ for each $j < i$. Clearly, the graphs $D_1, \dots, D_n$ are independent in the edge space $\mathcal{E}(D)$.

Lemma 2. *Let D be a totally cyclic digraph. Then there exists a triangular $m(D)$-tuple $(C_1, \dots, C_{m(D)})$ of directed circuits in D.*

Proof: We use induction on $m(D)$. The statement holds if $m(D) = 0$ (i. e., if $E(D) = \emptyset$). Let e be an arc of D, $D_1, \dots, D_s$ be the components of $(D - e)_{\mathcal{C}}$, and D' be the digraph arising from D after contracting $D_1, \dots, D_s$ to vertices $v_1, \dots, v_s$, respectively (note that $D' - e$ is the condensation of $D - e$). If we take spanning forests in $D_1, \dots, D_s$, and D', then their union is a spanning forest in D (e cannot be a bridge, because D is totally cyclic), whence $m(D) = m(D_1) + \cdots + m(D_s) + m(D')$ and $m(D_1), \dots, m(D_s), m(D') < m(D)$. Thus, by the induction hypothesis, $D_1, \dots, D_s$, and D' have the required sets $(C_{1,1}, \dots, C_{1,m(D_1)})$, $\dots$, $(C_{s,1}, \dots, C_{s,m(D_s)})$, and $(C'_1, \dots, C'_{m(D')})$, respectively. Directed circuits in $D_1, \dots, D_s$ are directed circuits in D. By Lemma 1, every directed circuit C'_i in D' can be extended to a directed circuit C_i in D ($i = 1, \dots, m(D')$). Thus the $m(D)$-tuple $(C_{1,1}, \dots, C_{1,m(D_1)}, \dots, C_{s,1}, \dots, C_{s,m(D_s)}, C_1, \dots, C_{m(D')})$ has the required properties. □

Lemma 3. *Let D be an acyclic digraph. Then there exists a triangular $r(D)$-tuple $(C_1, \dots, C_{r(D)})$ of directed cocircuits in D.*

Proof: Without abuse of generality we can suppose that D has one component. We use induction on $r(D)$. The statement is trivial if $r(D) = 0$ (i. e., $E(G) = \emptyset$). Otherwise D has a sink v (i. e., there is no arc directed out from v). Let $D_1, \dots, D_t$ be the components of $D - v$. Then $r(D) = t + r(D_1) + \dots + r(D_t)$. Let C_i be the set of arcs directed from $V(D_i)$ to v ($i = 1, \dots, t$). Then C_i is a directed cocircuit in D. By the induction hypothesis, there exists a triangular $r(D_i)$-tuple of directed cocircuits $(C_{i,1}, \dots, C_{i,r(D_i)})$ in D_i. Let $D_{i,j}$ be the component of $D_i - C_{i,j}$ so that the arcs of $C_{i,j}$ are directed out from $V(D_{i,j})$ ($i = 1, \dots, t$, $j = 1, \dots, m(D_i)$). Let $C'_{i,j}$ be the set of arcs of D directed out from $V(D_{i,j})$. Then $C'_{i,j}$ is a directed cocircuit, $C_{i,j} \subseteq C'_{i,j}$, $C'_{i,j} \setminus C_{i,j} \subseteq C_i$, and the $m(G)$-tuple of directed cocircuits $(C_1, \dots, C_t, C'_{1,1}, \dots, C'_{1,r(D_1)}, \dots, C'_{t,1}, \dots, C'_{t,r(D_t)})$ has the required properties. □

Note that Lemma 2 cannot be improved so that every circuit C_i contains an arc e_i satisfying $e_i \notin C_j$ for all $j \neq i$. It suffices to consider a digraph D having two vertices u and v and four arcs, two of them directed to u and the other two directed to v. Then $m(D) = 3$, and if we take three arcs e_1, e_2, e_3 of D, then two of them, say e_1, e_2, are directed to the same vertex, and every directed circuit in D must contain at least one of e_1, e_2. Since D is planar, we can apply duality (see [1, 2]) and show that Lemma 3 cannot have a similar improvement either.

Theorem 1. *Let D be a digraph and D' be the condensation of D. Then the maximal number of directed cycles (resp. directed cocycles) in a base of $\mathcal{C}(D)$ (resp. $\mathcal{C}^*(D)$) is $m(D_C)$ (resp. $r(D')$). On the other hand there exists a base of $\mathcal{C}(D)$ (resp. $\mathcal{C}^*(D)$) containing a triangular $m(D_C)$-tuple (resp. $r(D')$-tuple) of directed circuits (resp. directed cocircuits).*

Proof: Let $C_1, \dots, C_{m(D)}$ be a base of $\mathcal{C}(D)$ such that $C_1, \dots, C_s$ are directed cycles. The union of $C_1, \dots, C_s$ is a subset of D_C. Thus $s \leq m(D_C)$. Conversely, by Lemma 2, there is a triangular $m(D_C)$-tuple of circuits $(C_1, \dots, C_{m(D_C)})$ in D_C (thus also in D) which can be extended to a base in $\mathcal{C}(D)$.

Let $C_1, \dots, C_{r(D)}$ be a base of $\mathcal{C}^*(D)$ such that $C_1, \dots, C_t$ are directed cocycles. The union of $C_1, \dots, C_t$ is a subset of $D_{\mathcal{A}}$. Furthermore, C is a directed cocircuit in D iff it is a directed cocircuit in D'. Thus $t \leq r(D')$. Conversely, by Lemma 3, there is a triangular $r(D')$-tuple of circuits $(C_1, \dots, C_{r(D')})$ in D' (thus also in D) which can be extended to a base in $\mathcal{C}^*(D)$. This proves the statement. □

Theorem 2. *Let D be a digraph and $b(D)$ ($l(D)$) be the digraph arising from D after deleting all bridges (loops). Then the cyclic (cut) space of D has a basis consisting of directed circuits (directed cocircuits) iff $b(D)$ ($l(D)$) is totally cyclic (acyclic). Furthermore, the base can be a triangular $m(D)$-tuple ($r(D)$-tuple).*

Proof: Sufficiency holds by Lemmas 2, 3, and the fact that $\mathcal{C}(D) = \mathcal{C}(b(D))$ ($\mathcal{C}^*(D) = \mathcal{C}^*(l(D))$). Suppose that $b(D)$ ($l(D)$) is not totally cyclic (acyclic). Then $m(b(D)_{\mathcal{C}}) < m(D)$ ($r(D') < r(D)$, where D' is the condensation of $l(D)$). Thus the necessity holds by Theorem 1. □

Note that we cannot replace D' by $D_{\mathcal{A}}$ in Theorem 1. For example, let D be an orientation of K_4 so that $D_{\mathcal{A}}$ has exactly 3 arcs which are directed to one vertex and let D' be the condensation of D. Then $r(D_{\mathcal{A}}) = 3$, $r(D') = 1$, $D_{\mathcal{A}}$ has three cocircuits of cardinality one and these are not cocircuits in D. The arcs of $D_{\mathcal{A}}$ form a directed cocircuit in D which is a directed cocycle (but no directed cocircuit) in $D_{\mathcal{A}}$.

In accompanied papers [3, 4], we use that, by Lemmas 2 and 3, the minimal number of directed cycles (directed cocycles) in a totally cyclic (acyclic) digraph D is $m(D)+1$ ($r(D)+1$). This estimate is the best possible in general. For example take two trees T_1 and T_2 with two distinguished vertices v_1 and v_2, respectively. Orient the edges of T_1 (T_2) in the direction towards v_1 (away from v_2), add n new arcs $e_1, \dots, e_n$ directed from $V(T_2)$ to $V(T_1)$ and a new arc e directed from v_1 to v_2. We get a totally cyclic digraph D_n so that $m(D_n) = n$, $D_n - e$ is acyclic, and every arc e_i ($i = 1, \dots, n$) is contained in exactly one directed cycle in D_n. Thus D_n has exactly $m(D_n) + 1$ directed cycles ($\emptyset \subseteq E(D_n)$ is one of them). Furthermore, if D_n is planar, we can apply duality and get a digraph D_n^* having exactly $n + 1 = r(D_n^*) + 1$ directed cocycles.

Finally note that similarly as in Theorem 1 we can show that every digraph D has at least $m(D_{\mathcal{C}}) + 1$ directed cycles and $r(D') + 1$ directed cocycles, where D' is the condensation of D.

References

1. Berge C., "Graphs", 2nd ed., North-Holland, Amsterdam, 1985.
2. Diestel R., "Graph Theory", Springer-Verlag, New York, 1997.
3. Kochol M., *Polynomials associated with nowhere-zero flows in graphs*, manuscript.
4. Kochol M., *Tension polynomials of graphs*, manuscript.
5. Minty G. J., *Monotone networks*, Proc. R. Soc. Lond., Ser. A **257** (1960), 194-212.