Regular Factors in Connected Regular Graphs

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Abstract

In [5] Pila presented best possible sufficient conditions for a regular σ- connected graph to have a 1-factor, extending a result of Wallis [7]. Here we present best possible sufficient conditions for a σ-connected regular graph to have a k-factor for any $k \geq 2$.

1 Introduction

All graphs considered are finite, undirected and simple. For a graph $G = (V, E)$ let $n(G) = |V(G)|$ denote the order of G. For any vertex $x \in V(G)$ let $d(x, G)$ denote the degree. We call G d-regular, if $d(x, G) = d$ for every vertex x. For disjoint $X, Y \subseteq V(G)$ let $e_G(X, Y)$ denote the number of edges in G with one endvertex in X and one endvertex in Y. A graph G is c-connected if $G - Y$ is connected for any $Y \subseteq V(G)$ with $|Y| < c$. The connectivity $\sigma(G)$ of G is defined as the maximum c such that G is c-connected. A k-factor of a graph G is a k-regular spanning subgraph of G.

Wallis [7] presented graphs of minimum order without a 1-factor, thus giving sufficient conditions for a regular graph to have a 1-factor. Two years later Pila [5] improved these conditions in case the connectivity of the graph is known.

Theorem 1.1 (Pila [5]) *Let n, d, σ be integers with $n > d > 1$, n even and $d \geq \sigma \geq 1$. Define $\sigma^* \in \{\sigma, \sigma + 1\}$ such that $\sigma^* \equiv d \pmod 2$. A d-regular graph of order n and connectivity σ has a 1-factor if*

- *d even and*
  - *(i) $d = \sigma^* + 2$ and $n < \sigma^* + (\sigma^* + 2)(d + 1)$;*
  - *(ii) $\sigma = 1$, $d \geq \sigma^* + 4$ and $n < 1 + 3(d + 1)$;*
  - *(iii) $\sigma = 2$, $d \geq \sigma^* + 4$ and $n < 2 + 4(d + 1)$;*

(iv) $\sigma = 3$, $d \in \{\sigma^* + 4, \sigma^* + 6\}$ *and* $n < 3 + 5(d+1)$*;*

(v) $\sigma \geq 3$, $d \geq \sigma^* + 8$, *or* $d \geq \sigma^* + 4$ *for* $\sigma \geq 4$, *and* $n < 2d - l + (l + 2)(d+1)$*;*

- *d odd and*

(i) $\sigma = 1$, $d \geq \sigma^* + 2$ *and* $n < \sigma + (\sigma + 2)(d + 2)$*;*

(ii) $\sigma = 2$, $d \in \{\sigma^* + 2, \sigma^* + 4\}$ *and* $n < \sigma + (\sigma + 2)(d + 2)$*;*

(iii) $\sigma = 2$, $d \geq \sigma^* + 6$ *and* $n < 3(d + 2)$*;*

(iv) $\sigma \geq 3$, $d = \sigma^* + 2$ *and* $n < \sigma^* + (\sigma^* + 2)(d + 2)$*;*

(v) $\sigma \in \{3, 4\}$, $d = \sigma^* + 4$ *and* $n < \sigma + (l + 2)(d + 2) + d$*;*

(vi) $\sigma \geq 3$, $d \geq \sigma^* + 6$, *or* $d \geq \sigma^* + 4$ *for* $\sigma \geq 5$, *and* $n < 2d - l + (l + 2)(d + 2)$,

with $l := \left\lceil \frac{2\sigma^*}{d - \sigma^*} \right\rceil$.

These conditions are best possible.

In our work we extend Pila's result to k-factors with $k \geq 2$. Our main result is

Theorem 1.2 *For integers* n, d, k, σ *with* $d - 1 > k > 1$ *and* $n > d \geq \sigma \geq 1$ *such that* nd *and* nk *are even, let* G *be a* d*-regular graph of order* n *and connectivity* σ*. Define* $\sigma^* \in \{\sigma, \sigma + 1\}$ *such that* $\sigma^* \equiv d \pmod 2$, $p \in \{1, 2\}$ *such that* $p \not\equiv d \pmod 2$ *and*

$$\hat{k} = \begin{cases} \min\{k, d-k\}, & \text{for } d \text{ even and } k \text{ odd}; \\ k, & \text{for } d \text{ and } k \text{ odd}; \\ d - k, & \text{for } d \text{ odd and } k \text{ even}. \end{cases}$$

The graph G *has a* k*-factor if*

- *d and k even, or else*
- *if either* $d \leq \hat{k}\sigma^*$, *or*

(i) $d = \hat{k}\sigma^* + 2$ *and* $n < \sigma^* + (\hat{k}\sigma^* + 2)(d + p)$*;*

(ii) $\sigma = 1$, $d \geq \hat{k}\sigma^* + 4$ *and* $n < 1 + (\hat{k} + 2)(d + p)$*;*

(iii) $\sigma = 2$, $d = 3\hat{k} + 4$ *and* $n < 2 + (2\hat{k} + 2)(d + p)$*;*

(iv) $\sigma = 2$, $d = 2\hat{k} + 4$ *and* $n < 2d - l + (\hat{k}l + 2)(d + p)$*;*

(v) $\sigma \geq 2$, $d \geq \hat{k}\sigma^* + 6$, *or* $d \geq \hat{k}\sigma^* + 4$ *for* $\sigma \geq 3$, *and* $n < 2d - l + (\hat{k}l + 2)(d + p)$,

where $l := \left\lceil \frac{2\sigma^*}{d - \hat{k}\sigma^*} \right\rceil$.

These conditions are best possible.

We restrict ourselves to connected graphs only, since the disconnected case presents no new results. Further, we exclude the cases that nd or nk are odd where either G or the k-factor cannot exist. The restriction $d-1 > k > 1$ can be made because of Pila's result on 1–factors and the fact that a d-regular graph has a k-factor if and only if it has a $(d-k)$-factor.

Theorem 1.2 bears close resemblance to a result of Niessen and Randerath. In [3] they determined all quadruples (n, d, k, λ) for which a d-regular graph of order n and edge-connectivity λ has a k-factor. Due to the fact that the connectivity of a graph is always less than or equal to its edge-connectivity, it is plausible that our conditions will allow higher orders in most cases. Note that neither result can be used to prove the other, since there exist regular graphs which have arbitrarily high edge-connectivity but are only one-connected.

2 Proof of the Main Theorem

The proof of Theorem 1.2 uses the k-factor Theorem of Belck [1] and Tutte [6], which we cite in its version for regular graphs.

Theorem 2.1 *The d-regular graph G has a k-factor if and only if*

$$\Theta(X, Y, k) := k|X| - k|Y| + d|Y| - e_G(X, Y) - q_G(X, Y, k) \geq 0$$

for all disjoint subsets X, Y *of* $V(G)$. *Here* $q_G(X, Y, k)$ *denotes the number of components* C *of* $G - (X \cup Y)$ *satisfying*

$$e_G(Y, V(C)) + k|V(C)| \equiv 1 \pmod 2.$$

We simply call these components odd.
It always holds $\Theta(X, Y, k) \equiv k|V(G)| \pmod 2$ *for all disjoint subsets* X, Y *of* $V(G)$, *wether* G *has a k-factor or not.*

We further need the following Theorem of Menger.

Theorem 2.2 (Menger [2]) *A graph G has connectivity* $\sigma(G) = c$ *if and only if there exist c paths between any two vertices* x, y *of* G, *which only have* x *and* y *in common.*

Proof of Theorem 1.2. By the well known Theorem of Petersen [4] every regular graph of even degree has a 2-factor, and thus the case that d and k are even is proved. In the remaining cases we may assume without loss of generality that $k = \hat{k} \geq 3$ odd. This holds since G has a k-factor if and only if G has a $(d-k)$-factor. We proceed by presenting lower bounds on the order of G, if G does not have a k-factor.

If G does not have a k-factor, then by Theorem 2.1 there exist disjoint subsets X, Y of $V(G)$ such that

$$k|X| - k|Y| + d|Y| - e_G(X,Y) + 2 \leq q_G(X,Y,k) =: q \tag{1}$$

We follow the ideas of [3], [5] and [7] and call an odd component C of $Z := G - (X \cup Y)$ a small component, if $|V(C)| \leq d$. Let s denote the number of small components of Z. It is easy to see that

- $|V(C)| \geq d + p$ for every odd component C of Z which is no small component;
- $e_G(X \cup Y, V(C)) \geq \sigma^*$ for every odd component C of Z, and even
- $e_G(X \cup Y, V(C)) \geq d$ for every small component C.

This leads to

$$e_G(X \cup Y, V(Z)) \geq sd + (q-s)\sigma^* = q\sigma^* + (d - \sigma^*)s. \tag{2}$$

Counting the edges between X and Y in two different ways leads to

$$2e_G(X,Y) \leq d|X| + d|Y| - e_G(X \cup Y, V(Z)). \tag{3}$$

(2) and (3) yield

$$d|X| + d|Y| - 2e_G(X,Y) \geq q\sigma^* + (d - \sigma^*)s.$$

With (1) it follows

$$(d - 2k)(|X| - |Y|) \geq q(\sigma^* - 2) + (d - \sigma^*)s + 4. \tag{4}$$

We now consider two cases.

Case 1. $\sigma^* \geq 2$.

Claim: $|X| - |Y| > 0$.

Proof of the Claim. The righthand-side of (4) is positive, so for even d we get $|X| - |Y| > 0$.

If d is odd, we use the fact that for every odd component C it holds

$$\begin{aligned} e_G(X, V(C)) &= d|V(C)| - e_G(Y, V(C)) - 2|E(C)| \\ &\equiv k|V(C)| + e_G(Y, V(C)) \equiv 1 \pmod 2 \end{aligned}$$

and in particular $e_G(X, V(C)) \geq 1$. This gives us

$$e_G(X,Y) \leq d|X| - e_G(X, V(Z)) \leq d|X| - q$$

which together with (1) results in $|X| - |Y| > 0$. This completes the proof of the claim. □

Due to (1) we have

$$q \geq k(|X| - |Y|) + 2 \geq k + 2 \tag{5}$$

and with (4)

$$(d - k\sigma^*)(|X| - |Y|) \geq 2\sigma^* + (d - \sigma^*)s > 0. \tag{6}$$

Since $|X| - |Y| > 0$ we get $d \geq k\sigma^* + 2$, proving the statement that G has a k-factor if $d \leq k\sigma^*$. We can rewrite (6) as

$$\begin{aligned} |X| - |Y| &\geq \frac{2\sigma^*}{d - k\sigma^*} + \frac{d - \sigma^*}{d - k\sigma^*}s \qquad (7) \\ \Rightarrow \qquad |X| - |Y| &\geq \left\lceil \frac{2\sigma^*}{d - k\sigma^*} \right\rceil =: l \geq 1. \end{aligned}$$

The next two subcases complete our discussion of Case 1.

Subcase 1.A. $d = k\sigma^* + 2$, or $d \geq k\sigma^* + 2$ for $\sigma = 1$, or $d = 3k + 4$ for $\sigma = 2$.

In this subcase we have $l \geq \sigma$. We get

$$\begin{aligned} n(G) &= |X| + |Y| + |V(Z)| \\ &\geq l + s + (q - s)(d + p) \\ &\overset{(5)+(7)}{\geq} l + \left(\frac{k}{d - k\sigma^*}(2\sigma^* + (d - \sigma^*)s) + 2 - s\right)(d + p) \\ &\geq l + \left(2\frac{k\sigma^*}{d - k\sigma^*} + 2\right)(d + p) \\ \Rightarrow \qquad n(G) &\geq l + (kl + 2)(d + p). \qquad (8) \end{aligned}$$

If $d = k\sigma^* + 2$ than $l = \sigma^*$ and thus $n(G) \geq \sigma^* + (k\sigma^* + 2)(d + p)$, proving statement (i) of our theorem. If $\sigma = 1$ and $d \geq 2k + 4$, we have $l + (kl + 2)(d + p) \geq 1 + (k + 2)(d + p)$, proving (ii), if d is even (d odd and $\sigma = 1$ yield Case 2). If $\sigma = 2$ and $d = 3k + 4$, than $l = 2 = \sigma$ and it follows $n(G) \geq 2 + (2k + 2)(d + p)$, giving us (iii).

Subcase 1.B. $\sigma \geq 2$ and $d \geq k\sigma^* + 6$, or $d \geq k\sigma^* + 4$ for $\sigma \geq 3$, or $\sigma = 2$ and $d = 2k + 4$.

As in Subcase 1.A we have

$$\begin{aligned}
|V(Z)| &\geq s+(q-s)(d+p) \\
&\overset{(*)}{\geq} s+\left(k\frac{2\sigma^*}{d-k\sigma^*}+2s+2\right)(d+p) \\
\Rightarrow \qquad |V(Z)| &\geq s+(kl+2s+2)(d+p)
\end{aligned}$$

Inequality $(*)$ holds, since $\frac{k(d-\sigma^*)}{d-k\sigma^*}s - s \geq 2s$ for $k \geq 3$.
If there exists at least one small component, $|V(Z)| \geq 2d+3+(kl+2)(d+p)$ and

$$\begin{aligned}
n(G) &= |X|+|Y|+|V(Z)| \\
&\overset{(5)}{\geq} \sigma+2d+3+(kl+2)(d+p) \\
&> 2d-l+(kl+2)(d+p).
\end{aligned}$$

If Z does not have any small component, then

$$|V(Z)| \geq q(d+p) \overset{(1)}{\geq} \Big(k(|X|-|Y|)+d|Y|-e_G(X,Y)+2\Big)(d+p).$$

If $d|Y|-e_G(X,Y) \geq 2$, than

$$\begin{aligned}
n(G) &\geq l+2(d+p)+(kl+2)(d+p) \\
&> 2d-l+(kl+2)(d+p).
\end{aligned}$$

If $e_G(X,Y) = d|Y|-1$, than $|X| \geq d-1$ and we get

$$\begin{aligned}
n(G) &\geq |X|+|Y|+(d+p)+(kl+2)(d+p) \\
&> 2d-l+(kl+2)(d+p).
\end{aligned}$$

It remains the case $d|Y| = e_G(X,Y)$, which can only occur for either

- $|Y| = 0$ and $|X| \geq \sigma$; or
- $1 \leq |Y| \leq |X|-l$ with $|X| \geq d$.

Since $\sigma > l$ for Subcase 1.B, a short calculation shows $2d-l+(kl+2)(d+p) < \sigma+(k\sigma+2)(d+p)$. So $|X| = d$ and $|Y| = d-l$ yields the lowest possible case, and we get

$$n(G) \geq 2d-l+(kl+2)(d+p).$$

These cases prove (iv) to (vi) of our theorem and complete the discussion of Case 1.

Case 2. $\sigma^* = 1$.

Here we have $\sigma^* = \sigma = 1$ and d odd. In this case we make use of the result of Niessen and Randerath [3] and get $d \geq k + 2$ as well as $n > 1 + (k + 2)(d + p)$. This proves (ii) for odd d.

In Cases 1 and 2 we have given lower bounds for the order of G, under the assumption that G does not have a k-factor. It remains to show that we can construct d-regular σ-connected graphs of these orders which do not have a k-factor.

Analogous to Pila [5] we first construct graphs $C(d, h)$ on $d + p$ vertices with h vertices of degree $d - 1$ and $d + p - h$ vertices of degree d. These graphs will function as our odd components of $G - (X \cup Y)$. For this let $1 \leq h \leq d$ such that $h \equiv d \pmod 2$.
If d is even, then $p = 1$ and let $V(C(d, h)) = \{x, y_1, \dots, y_d\}$ where $\{y_1, \dots, y_d\}$ induces a complete graph with a matching of size $h/2$ removed. Further let x be connected to y_i for every $1 \leq i \leq d$.
If d is odd, then $p = 2$ and let $V(C(d, h)) = \{x, z, y_1, \dots, y_d\}$ where $\{y_1, \dots, y_d\}$ induces a complete graph with a cycle of length h removed. Further let x and z be connected to y_i for every $1 \leq i \leq d$.
In both cases $C(d, h)$ is $(d - 2)$-connected with h vertices of degree $d - 1$.

The following Cases A to D are exhaustive:

Case A: $d = k\sigma^* + 2$:

Take a set X of σ^* independent vertices and d copies of $C(d, \sigma^*)$. Connect each vertex $x \in X$ with vertices of degree $d-1$ such that $m(x, U) = 1$ holds for every vertex x and every copy U of $C(d, \sigma^*)$. By Theorem 2.2 the resulting graph is σ-connected and has order $n = \sigma^* + (k\sigma^* + 2)(d + p)$. It has no k-factor, since $\Theta(X, \emptyset, k) = -2$.

Case B: $\sigma = 2$ and $d = 3k + 4$:

Here the following construction is possible (for which we need $d \geq k\sigma^* + 4$): Take a set $X = \{x_1, x_2\}$ of 2 independent vertices, $k + 2$ copies of $C(d, 2)$ and $k + 2$ copies of $C(d, 4)$. Connect x_1, x_2 to the vertices of degree $d - 1$ such that $m(x_i, C(d, 2)) = 1$ and $m(x_i, C(d, 4)) = 2$ for $i = 1, 2$. The resulting graph is d-regular, 2-connected with order $2 + (2k + 2)(d + p)$ and has no k-factor since $\Theta(X, \emptyset, k) = -2$.

Case C: $\sigma = 1$ and $d \geq k\sigma^* + 4$:

Here an analogous construction to that in Case B is possible: Take a vertex x, $k+1$ copies of $C(d,\sigma^*)$ and one copy of $C(d,h)$ with $1 \leq h = d-(k+1)\sigma^* < d$. Note that $h \equiv d \pmod 2$. Connect x to every vertex of degree $d-1$ with one edge. The resulting graph is connected, d-regular of order $1+(k+2)(d+p)$ and has no k-factor, since $\Theta(\{x\},\emptyset,k) = -2$.

Case D: $\sigma \geq 2$ and $d \geq k\sigma^* + 6$, or $d \geq k\sigma^* + 4$ for $\sigma \geq 3$, or $\sigma = 2$ and $d = 2k+4$:

Under the conditions of Case D it holds $d \geq \sigma > l$. Construct a graph G as follows: Take a complete bipartite graph with partitions X and Y such that $|X| = d$ and $|Y| = d-l > 0$. Take $lk+1$ copies of $C(d,\sigma^*)$ and one copy of $C(d,h)$ with $h = dl - \sigma^*(kl+1)$. Note that $d \geq h \geq \sigma^*$ due to the definition of l. Connect the vertices of X with the vertices of degree $d-1$ in such a way, that each copy of $C(d,\sigma^*)$ is joined to exactly σ^* vertices of X. By Theorem 2.2 the resulting graph is σ-connected. It is d-regular of order $2d-l+(kl+2)(d+p)$ and has no k-factor, since $\Theta(X,Y,k) = -2$.
This concludes the proof of our theorem. □

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References

[1] H.B. Belck, Reguläre Faktoren von Graphen, *J. Reine Angew. Math.* **188** (1950) 228–252.

[2] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.* **10** (1927) 96–115.

[3] T. Niessen and B. Randerath, Regular factors of simple regular graphs and factor-spectra, *Discrete Math.* **185** (1998) 89–103.

[4] J. Petersen, Die Theorie der regulären graphs, *Acta Math.* **15** (1891) 193–220.

[5] J. Pila, Connected Regular Graphs Without One-Factors, *Ars Combin.* **18** (1983) 161–172.

[6] W.T. Tutte, The factors of graphs, *Canad. J. Math.* **4** (1952) 314–328.

[7] W.D. Wallis, The Smallest Regular Graphs Without One-Factors, *Ars Combin.* **11** (1981) 295–300.

Halin's Theorem for the Möbius Strip

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Abstract

Halin's Theorem characterizes those locally finite infinite graphs that embed in the plane without accumulation points by giving a set of six topologically-excluded subgraphs. We prove the analogous theorem for graphs that embed in an open Möbius strip without accumulation points. There are 153 such obstructions under the ray ordering defined herein. There are 350 obstructions under the minor ordering. There are 1225 obstructions under the topological ordering. The relationship between these graphs and the obstructions to embedding in the projective plane is similar to the relationship between Halin's graphs and $\{K_5, K_{3,3}\}$.[1]

1 Introduction

A fundamental result in graph theory is Kuratowski's Theorem [13], which says that a finite graph embeds in the plane if and only if it does not contain a subdivision of either K_5 or $K_{3,3}$. We are concerned here with embeddings of locally-finite infinite graphs. For the requisite background on embedding finite and infinite graphs see [15] and [6] respectively (see [7] for related work on embedding infinite graphs).

Halin [12] proved that a locally finite, infinite graph embeds in the plane without an accumulation point if and only if it does not contain a non-isomorphic

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