# Notes on Two Properties of the Generalized Sequences $\{W_n\}$ Relevant to Recurring Decimal

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## 1 Introduction

Let $a$, $b$, $p$, $q$ are arbitrary integers. The generalized sequences $W_n = W_n(a, b; p, q)$ are defined by

$$W_n = pW_{n-1} + qW_{n-2}, \quad (n \geq 2) \tag{1}$$

$$W_0 = a, \quad W_1 = b.$$

If $\alpha$ and $\beta$, assumed distinct, are the roots of

$$\lambda^2 - p\lambda - q = 0,$$

i.e.,

$$\alpha = \frac{p + \sqrt{p^2 + 4q}}{2} \quad and \quad \beta = \frac{p - \sqrt{p^2 + 4q}}{2},$$

we have the Binet form

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{2}$$

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in which $A = b - a\beta$ and $B = b - a\alpha$ (see [1]).

Special cases of $\{W_n\}$ which interest us are

- the Fibonacci sequence $\{F_n\}$: $p = 1$, $q = 1$, $a = 0$, $b = 1$;
- the Lucas sequence $\{L_n\}$: $p = 1$, $q = 1$, $a = 2$, $b = 1$;
- the Pell sequence $\{P_n\}$: $p = 2$, $q = 1$, $a = 0$, $b = 1$;
- the Pell-Lucas sequence $\{Q_n\}$: $p = 2$, $q = 1$, $a = 2$, $b = 2$.

Recently, Chengheng Zhang obtain some identities on $F_n$, $L_n$, $P_n$ and $Q_n$. The purpose of this note is to generalize these results. Furthermore, we give some convergent conditions on those kinds of identities.

## 2 The Main Results

We also denote

$$U_n = W_n(0,1;p,q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \; V_n = W_n(2,p;p,q) = \alpha^n + \beta^n. \quad (3)$$

**Lemma**

$$\sum_{n=0}^{\infty} W_{mn} x^n = \frac{a + (bU_m - aU_{m+1})x}{1 - V_m x + (-1)^m q^m x^2}. \quad (4)$$

*Proof.* Noting (2) and (3) we have

$$\sum_{n=0}^{\infty} W_{mn} x^n$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{A\alpha^{mn} - B\beta^{mn}}{\alpha-\beta} x^n = \frac{1}{\alpha-\beta}\left\{\frac{A}{1-\alpha^m x} - \frac{B}{1-\beta^m x}\right\} \\
&= \frac{1}{\alpha-\beta}\left\{\frac{b-a\beta}{1-\alpha^m x} - \frac{b-a\alpha}{1-\beta^m x}\right\} \\
&= \frac{1}{\alpha-\beta}\frac{(b-a\beta)(1-\beta^m x)-(b-a\alpha)(1-\alpha^m x)}{1-(\alpha^m+\beta^m)x+(\alpha\beta)^m x^2)} \\
&= \frac{1}{\alpha-\beta}\frac{a(\alpha-\beta)+\left[b(\alpha^m-\beta^m)-a(\alpha^{m+1}-\beta^{m+1})\right]x}{1-V_m x+(-1)^m q^m x^2} \\
&= \frac{a+(bU_m-aU_{m+1})x}{1-V_m x+(-1)^m q^m x^2}.
\end{aligned}
$$

**Proposition.**

$$
\sum_{k=0}^{\infty} \frac{W_{mk}}{\delta^{k+1}} = \frac{a\delta + bU_m - aU_{m+1}}{\delta^2 - V_m\delta + (-1)^m q^m}, \tag{5}
$$

where $\delta > A\alpha^m$, i.e., $m < \frac{\ln(\frac{\delta}{A})}{\ln\alpha}$.

*Proof.* In equation (4), putting $x = \frac{1}{\delta}$, we get

$$
\sum_{k=0}^{\infty} \frac{W_{mk}}{\delta^{k}} = \frac{a\delta^2 + (bU_m - aU_{m+1})\delta}{\delta^2 - V_m\delta + (-1)^m q^m}.
$$

And dividing both sides of the equation by $\delta$, we have

$$
\sum_{k=0}^{\infty} \frac{W_{mk}}{\delta^{k+1}} = \frac{a\delta + bU_m - aU_{m+1}}{\delta^2 - V_m\delta + (-1)^m q^m}.
$$

**Theorem 1.**

$$
\sum_{k=0}^{\infty} \frac{F_{mk}}{\delta^{k+1}} = \frac{F_m}{\delta^2 - L_m\delta + (-1)^m}, \tag{6}
$$

where $\delta > \left(\frac{1+\sqrt{5}}{2}\right)^m$, i.e., $m < \ln\delta / \ln\left(\frac{1+\sqrt{5}}{2}\right)$.

*Proof.* Theorem 1 follows by taking $p = 1,\ q = 1,\ a = 0,\ b = 1$ in Proposition.

**Corollary 1.**

$\sum_{k=0}^{\infty} \frac{F_k}{\delta^{k+1}} = \frac{1}{\delta^2-\delta-1}$, where $\delta > \frac{1+\sqrt{5}}{2}$;

$\sum_{k=0}^{\infty} \frac{F_{2k}}{\delta^{k+1}} = \frac{1}{\delta^2-3\delta+1}$, where $\delta > \frac{3+\sqrt{5}}{2}$;

$\sum_{k=0}^{\infty} \frac{F_{3k}}{\delta^{k+1}} = \frac{2}{\delta^2-4\delta-1}$, where $\delta > 2+\sqrt{5}$;

$\sum_{k=0}^{\infty} \frac{F_{4k}}{\delta^{k+1}} = \frac{3}{\delta^2-7\delta+1}$, where $\delta > \frac{7+3\sqrt{5}}{2}$.

**Corollary 2.** We have

$$\sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}} = \frac{1}{89}, \qquad \sum_{k=0}^{\infty} \frac{F_{2k}}{10^{k+1}} = \frac{1}{71},$$

$$\sum_{k=0}^{\infty} \frac{F_{3k}}{10^{k+1}} = \frac{2}{59}, \qquad \sum_{k=0}^{\infty} \frac{F_{4k}}{10^{k+1}} = \frac{3}{31}.$$

But when $m \geq 5$, $\sum_{k=0}^{\infty} \frac{F_{mk}}{10^{k+1}}$ does not converge.

**Theorem 2.**

$$\sum_{k=0}^{\infty} \frac{L_{mk}}{\delta^{k+1}} = \frac{2\delta - L_m}{\delta^2 - L_m\delta + (-1)^m}, \tag{7}$$

where $\delta > \left(\frac{1+\sqrt{5}}{2}\right)^m$, i.e., $m < \ln\delta / \ln\left(\frac{1+\sqrt{5}}{2}\right)$.

*Proof.* Theorem 2 follows by taking $p=1,\ q=1,\ a=2,\ b=1$ in Proposition.

**Corollary 3.**

$\sum_{k=0}^{\infty} \frac{L_k}{\delta^{k+1}} = \frac{2\delta-1}{\delta^2-\delta-1}$, where $\delta > \frac{1+\sqrt{5}}{2}$;

$\sum_{k=0}^{\infty} \frac{L_{2k}}{\delta^{k+1}} = \frac{2\delta-3}{\delta^2-3\delta+1}$, where $\delta > \frac{3+\sqrt{5}}{2}$;

$\sum_{k=0}^{\infty} \frac{L_{3k}}{\delta^{k+1}} = \frac{2\delta-4}{\delta^2-4\delta-1}$, where $\delta > 2+\sqrt{5}$;

$\sum_{k=0}^{\infty} \frac{L_{4k}}{\delta^{k+1}} = \frac{2\delta-7}{\delta^2-7\delta+1}$, where $\delta > \frac{7+3\sqrt{5}}{2}$.

**Corollary 4.** We have

$$\sum_{k=0}^{\infty} \frac{L_k}{10^{k+1}} = \frac{19}{89}, \qquad sum_{k=0}^{\infty} \frac{L_{2k}}{10^{k+1}} = \frac{17}{71},$$

$$\sum_{k=0}^{\infty} \frac{L_{3k}}{10^{k+1}} = \frac{16}{59}, \qquad \sum_{k=0}^{\infty} \frac{L_{4k}}{10^{k+1}} = \frac{13}{31}.$$

But when $m \geq 5$, $\sum_{k=0}^{\infty} \frac{L_{mk}}{10^{k+1}}$ does not converge.

**Corollary 5.** $\sum_{k=0}^{\infty} \frac{F_{mk}}{10^{2k+2}}$ and $\sum_{k=0}^{\infty} \frac{L_{mk}}{10^{2k+2}}$ converge when $m = 1, 2, 3, \ldots, 9$. But $\sum_{k=0}^{\infty} \frac{F_{mk}}{10^{2k+2}}$ and $\sum_{k=0}^{\infty} \frac{L_{mk}}{10^{2k+2}}$ do not converge when $m \geq 10$.

Similarly, we have

**Theorem 3.**

$$\sum_{k=0}^{\infty} \frac{P_{mk}}{\delta^{k+1}} = \frac{P_m}{\delta^2 - Q_m\delta + (-1)^m}, \tag{8}$$

where $\delta > \left(1+\sqrt{2}\right)^m$, i.e., $m < \ln\delta / \ln\left(1+\sqrt{2}\right)$.

*Proof.* Theorem 3 follows by taking $p = 2,\ q = 1,\ a = 0,\ b = 1$ in Proposition.

**Theorem 4.**

$$\sum_{k=0}^{\infty} \frac{Q_{mk}}{\delta^{k+1}} = \frac{2(\delta + P_m - P_{m+1})}{\delta^2 - Q_m\delta + (-1)^m}, \tag{9}$$

where $\delta > \left(1+\sqrt{2}\right)^m$, i.e., $m < \ln\delta / \ln\left(1+\sqrt{2}\right)$.

*Proof.* Theorem 4 follows by taking $p = 2,\ q = 1,\ a = 2,\ b = 2$ in Proposition.

**Corollary 6.** We have

$$\sum_{k=0}^{\infty}\frac{P_k}{10^{k+1}}=\frac{1}{79},\qquad \sum_{k=0}^{\infty}\frac{P_{2k}}{10^{k+1}}=\frac{2}{41},$$

$$\sum_{k=0}^{\infty}\frac{Q_k}{10^{k+1}}=\frac{18}{79},\qquad \sum_{k=0}^{\infty}\frac{Q_{2k}}{10^{k+1}}=\frac{14}{41}.$$

But when $m \geq 3$, $\sum_{k=0}^{\infty}\frac{P_{mk}}{10^{k+1}}$ and $\sum_{k=0}^{\infty}\frac{Q_{mk}}{10^{k+1}}$ do not converge.

**Corollary 7.** $\sum_{k=0}^{\infty}\frac{P_{mk}}{10^{2k+2}}$ and $\sum_{k=0}^{\infty}\frac{Q_{mk}}{10^{2k+2}}$ converge when $m = 1,2,3,4,5$. But $\sum_{k=0}^{\infty}\frac{P_{mk}}{10^{2k+2}}$ and $\sum_{k=0}^{\infty}\frac{Q_{mk}}{10^{2k+2}}$ do not converge when $m \geq 6$.

# References


[1] A.F.Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers". The Fibonacci Quarterly 3.2(1965): 161-76.

[2] Chengheng Zhang, Two properties of the generalized sequence $\{W_n\}$ relevant to recurring decimal, ARS Combinatoria, 57(2000): 193-199.


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