Minimal Enclosings of Triple Systems II: Increasing The Index By 1

Spencer P. Hurd, Dept. of Mathematics and Computer Science, The Citadel, Charleston, SC, 29409, email: hurds@citadel.edu
and
Dinesh G. Sarvate, Department of Mathematics, University of Charleston, Charleston, SC, 29424, email: sarvated@cofc.edu

Abstract: In Minimal Enclosings of Triple Systems I, we solved the problem of minimal enclosings of BIBD(v, 3, λ) into BIBD(v+1, 3, λ+m) for $1 \le \lambda \le 6$ with a minimal $m \ge 1$. Here we consider a new problem relating to the existence of enclosings for triple systems for any v, with $1 \le \lambda \le 6$, of BIBD(v, 3, λ) into BIBD(v+s, 3, λ+1) for minimal positive s. The non-existence of enclosings for otherwise suitable parameters is proved, and for the first time the difficult cases for even λ are considered. We completely solve the case for $\lambda \le 3$ and $\lambda = 5$, and partially complete the cases $\lambda = 4$ and 6. In some cases a 1-factorization of a complete graph or complete n-partite graph is used to obtain the minimal enclosing. A list of open cases for $\lambda = 4$ and $\lambda = 6$ is attached.
Key Words: enclosings, embeddings, BIBD, Steiner System, GDD, resolvable, 1-factorization, complete graph, complete n-partite graph, deficiency graph.

1. Introduction

We regard this paper as a continuation of the general program which is to determine the conditions under which a general enclosing of a BIBD can occur. We started our part in this program in [13] by looking at minimal enclosings into BIBD's with one extra point. Here, we explore a new type of minimal enclosing. We must introduce some terminology and notation before going further with this discussion. A balanced incomplete block design, or BIBD(v, b, r, k, λ), is a collection B of b subsets or blocks of a set V of order v such that all blocks have size k, each element appears in r blocks, and each pair of elements appears together in λ blocks. The conditions imply that $bk = vr$ and $\lambda(v - 1) = r(k - 1)$, and we usually write BIBD(v, k, λ) for short. We refer the reader to [8], [15], and [23] for well known facts about BIBD's and triple systems.

A design injection of X = BIBD(v, b_1, r_1, k, λ) into Y = BIBD(v+s, b_2, r_2, k, λ+m) is a mapping φ such that φ is a one-to-one map from V_1 to V_2, and for each block B_i of X, $\varphi(B_i)$ is a block of Y. The injection is an *embedding* if m = 0 and an *enclosing* if m > 0. We will always regard the injection as the inclusion map so that Y is based on the points of X and s additional points. In this paper, k will usually be 3 and m will be 1. Previously, an enclosing of a triple system X = BIBD(v, 3, λ) into a triple system BIBD(v+s, k, λ+m) was said to be *minimal* if s = 1 and X could not be enclosed in any BIBD(v+1, 3, λ+n) with 0 < n < m. We solved the problem for such minimal enclosings having small index in [13], and in the present note deal with a related type of enclosing which may be called a minimal point-enclosing. A design X = BIBD(v, 3, λ) is *minimally point-enclosed* in Y = BIBD(v+s, 3, λ+1) if X can not be enclosed in a BIBD(v+t, 3, λ+1) for 0 < t < s.

It is curious that so apparently slight a change in point of view turns out to make a large difference in results and techniques. A construction in [13] using a PBIBD on the points of X to effect the enclusion into Y was used many times but is not used here. Instead, however, great use is made of embeddings (Lemma 3.1) and graph factorizations (Lemma 3.7). Also, here we will frequently refer to new necessary condition (Corollary 1.2), but a corresponding result in [13] for the case s = 1 played almost no role since the index was usually less or equal to 6.

There is a large body of literature on embeddings, and we refer the reader to [5, 7, 10, 11, and 14]. However, enclosings, the subject of this note, have been studied less extensively, and the reader is referred to [1, 2, 4, 6, and 13].

The previously considered type of enclosings [13] overlap the present case when s = 1. Thus, we will be interested in the present paper only in designs which may not be minimally enclosed into some Y = BIBD(v+1, 3, λ+1). *In other words, s > 1 in all cases here.*

Some cases here overlap results in [1, 2], but there the interest is in faithful enclosings, a very restrictive condition. An enclosing of X into Y is *faithful* if every new block for Y has at least one new point. Also, the only general case considered in [1, 2] is for odd λ. We not only complete these results, as regards minimal enclosings, but begin the consideration of designs with even index. We completely solve the case for index 2 and have nearly complete results for index 4 and 6. Open cases, all for λ = 4 or 6, are listed in Table 3.

To motivate further the type of problem here, we quote the comments in [8], p. 155-156: "Enclosing of partial systems have not been seriously studied; in fact, as we see next, even enclosings of triple systems themselves have not been determined. ... Both the enclosing and the faithful enclosing problems appear to be far from a solution at this point. The situation is worse for simple enclosings."

We will refer to Table 1 below frequently [15, p.50]. It gives necessary and sufficient conditions for the existence of a λ-fold triple system of order v.

Table 1: The λ-v Spectrum of Triple Systems

$\lambda \equiv 0 \pmod 6$	All $v \neq 2$
$\lambda \equiv 1,5 \pmod 6$	All $v \equiv 1, 3 \pmod 6$
$\lambda \equiv 2, 4 \pmod 6$	All $v \equiv 0, 1 \pmod 3$
$\lambda \equiv 3 \pmod 6$	All odd v

In general, we will use the variables x and y to denote the elements added to the set V in order to create the enclosing design. We will refer to a type of *group divisible design, a GDD,* in which the points are partitioned into disjoint sets of equal size called groups. Points within a group will have index zero with each other, i.e., they will not appear in a common block. All pairs of points not in a common group will have the same index λ, i.e., pairs of points not in a common group will appear in exactly λ blocks. We use simplified superscript notation GDD(g^u) to denote a GDD on v elements with u groups of size g, index one, and block size 3, with v/g = u. A design is *resolvable* if its blocks can be partitioned in classes such that each point occurs exactly once in each class. A resolvable GDD is referred to as a RGDD. See [9, 12, 17, 18] for existence results and background.

We next establish a critical necessary condition for general point-enclosings along similar lines as in [13].

Theorem 1.1 *A necessary condition for enclosing X = BIBD(v, k, λ) into Y = BIBD(v+s, k, λ+1) is that*

$$v(v-1) \geq s(\lambda+1)(k-2)\left[(2v-(s-1)(k-1)\right]/2$$

Proof: Suppose that R is the replication number for Y and that $\{a_1, a_2, ..., a_s\}$ are the points of Y which are not in X. We count the "new" blocks of Y which are not in X. The point a_1 will appear in R distinct new blocks. Point a_2 must appear in R-(λ+1) new blocks without a_1. Point a_3 must appear in at least R-2(λ+1) new blocks (that is, without either a_1 or a_2; more new blocks are required if a_1, a_2, and a_3 appear together in the same block), and so on. By adding these, it follows that at least sR – (λ+1)s(s – 1)/2 new blocks are needed. Now let B and b denote the number of blocks in Y and X respectively. Since for any BIBD, vr = bk and λ(v-1) = r(k-1), we get

$$B-b=\frac{(v+s)(\lambda+1)(v+s-1)}{k(k-1)}-\frac{\lambda v(v-1)}{k(k-1)}\geq sR-(\lambda+1)s(s-1)/2\,.$$

The result follows on simplification.

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Corollary 1.2 *A necessary condition for enclosing a triple system (v, 3, λ) into a triple system (v+s, 3, λ+1) is that $v(v-1) \geq s(\lambda+1)(v-s+1)$.*

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In [13], when s = 1, the specialization of Corollary 1.2 led to v - 2 ≥ λ/m, an easily satisfied condition for small index. However, in the present paper, the Corollary 1.2 is vital and will be referred to frequently.

2. λ = 1.

Using Table 1, we see v ≡ 1, 3 (mod 6) when λ = 1. It is known (see [13]) that X = BIBD(6t+3, 3, 1) may be minimally enclosed into Y = BIBD(6t+4, 3, 2). Since this is a minimal point-enclosing as well, there is nothing further to do for v = 6t+3. However, X = BIBD(6t+1, 3, 1) may be minimally enclosed only in Y = BIBD(6t+2, 3, 6) – no enclosings are possible for smaller index, from Table 1. Thus, we need to consider a point-enclosing for BIBD(6t + 1, 3, 1), with t > 0.

Example 2.1 Suppose λ = 1. We give an example of an enclosing of X = BIBD(7, 3, 1) into Y = BIBD(9, 3, 2). We take the points of X as 1, 2, ..., 7 and add points x and y. First add the following blocks to X:

{x, 1, 3}, {x, 1, 4}, {x, 2, 5}, {x, 2, 6}, {x, 3, 5}, {x, 4, 6},
{y, 1, 5}. {y, 1, 6}, {y, 2, 3}, {y, 2, 4}, {y, 4, 5}, {y, 3,6},
{1, 2, 7}, {3, 4, 7}, {5, 6, 7};

next add 2 copies of the block {x, y, 7}. In contrast to this example, Corollary 1.2 can be used to show that no BIBD(7, 3, 3) may be enclosed into a BIBD(9, 3, 4).

Example 2.2 We give an enclosing of X = BIBD(13, 3, λ) into Y = BIBD(15, 3, λ+1) for λ = 1 or 3. First suppose λ = 1. We take the points of X as 1, 2, ..., 13, and add points x and y. To the blocks of X we add the blocks of Z = GDD(4^3) which has groups {1, 2, 3, 4}, {5, 6, 7, 8}, {9, 10, 11, 12} and which we may assume is resolvable [17; Lemma 3.2]. We further add the blocks:

{13, 1, 2}, {13, 3, 4}, {13, 5, 6}, {13, 7, 8}, {13, 9, 10}, {13, 11, 12};
{13, x, y}, {13, x, y}.

Now the index is 2 for point 13. For the element x and the remaining needed pairs from the 3 groups we further add

{x, 1, 3}, {x, 1, 4}, {x, 2, 3}, {x, 2, 4}, {x, 5, 7}, {x, 5, 8},
{x, 6, 7}, {x, 6, 8}, {x, 9, 11}, {x, 9, 12}, {x, 10, 11}, {x, 10, 12}.

Lastly, in one of the resolution classes from Z, replace each block {a, b, c} with the 3 blocks {y, a, b}, {y, a, c}, {y, b, c}. Note this does not raise the index for elements of X, but this creates index 2 for elements y and any s in X. (We call this expanding the resolution class with new point y.)

Now suppose $\lambda = 3$. We use the same construction as for $\lambda = 1$, but we add the following additional blocks. First, we add 2 more copies of block {x, y, 13}. Next we expand a second resolution class from Z with y just as before, and we expand a third resolution class with x. This completes the (15, 3, 4) triple system. The method does not extend to, say, enclosing BIBD(13, 3, 5) into BIBD(15, 3, 6) because Z has only 4 resolution classes.

The conclusion of Theorem 2.3 below also follows as a result of Theorem 2.4 of [2], but the proof here is different and the construction is of independent interest.

Theorem 2.3 *If $t \geq 1$, then any BIBD(6t+1, 3, 1) may be minimally point-enclosed into Y = BIBD(6t+3, 3, 2).*

Proof: We may suppose t > 2 in view of Examples 2.1 and 2.2. Suppose X is based on the points V = {1, 2, ..., 6t+1}. Since t > 2, there exists a GDD, say Z, on 6t points with group size 6. We identify the points of Z with the first 6t points of X arbitrarily. The points of Y are the points $V \cup \{x, y\}$. The blocks of the design Y consist of the blocks of X, the blocks of Z, and the following blocks based on the groups of Z. Suppose $G = \{a_1, a_2, ..., a_6\}$ is any group of Z. Then, by associating point a_i in the group with point i in Example 2.1, for i = 1,..., 6, we add the corresponding blocks indicated by the list for Example 2.1, where, for every group, 6t+1 plays the role of point 7. But note the block {x, y, 6t+1} is added 2 times only, not twice for each group.

3. $\lambda = 2$

When $\lambda = 2$, there are four cases (mod 6) to consider, by Table 1. For three of these cases, triple systems (6t+3, 3, 2), (6t+4, 3, 2), and (6t, 3, 2), all have minimal point-enclosings [13] for t > 0. The general case in this section deals with one of the main results of this paper, the enclosing (6t + 1, 3, 2) $\rightarrow$ (6t + 3, 3, 3). Lemma 3.1 below collects several results which we will use here and in later sections. Lemma 3.1 can be used to construct an enclosing when the parameters are compatible – just add a Steiner triple system to the embedding to increase the index by 1. Further, when the enclosing design has 2v+1 points, Corollary 1.2 is automatically satisfied since the quantity v-s+1 is zero. Part (a) is known but (b) and (c) may be new.

Lemma 3.1 (a) [16, 18] *If there exists a BIBD(v, 3, 2) then it can be embedded into a BIBD(2v+1, 3, 2). If there exists a simple BIBD(v, 3, λ), λ*

$\le v-2$, then it can be embedded into a simple BIBD(2v+1, 3, λ). If there exists a BIBD(v, 3, 6) and $v \neq 10$, then it can be embedded into a BIBD(2v+4, 3, 6). If there exists a BIBD(v, 3, 6), then it can be embedded in a BIBD(2v+2, 3, 6). (b) Any BIBD(v , 3, 4+6r) can be embedded into a BIBD(2v+1, 3, 4+6r) and can be enclosed into a BIBD(2v+1, 3, 5+6r). (c) If a BIBD(v, 3, 2) exists, then a BIBD(v, 3, 6) can be embedded in a BIBD(2v+1, 3, 6) and can be enclosed in a BIBD(2v+1, 3, 7).

Proof: We prove part (b). Let X be the given BIBD(v, 3, 4+6r). Let X_1 be any BIBD(v, 3, 2). Let X_2 denote the design formed from 2+3r copies of X_1. Since X_1 may be embedded into Y_1 = BIBD(2v+1, 3, 2), from part (a), X_2 can be embedded into Y_2 = BIBD(2v+1, 3, 4+6r) which is formed from 2+3r copies of Y_1. Let Y_3 denote the design formed by deleting from Y_2 those blocks also in X_2 and replacing them with the blocks of X. This embeds X into Y_3. Let Y denote the design formed from the blocks of Y_3 augmented with the blocks of a BIBD(2v+1, 3, 1), and X is enclosed into Y. For part (c), apply the argument in part (b).

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Example 3.2 Any design X = BIBD(4, 3, 2) encloses into BIBD(5, 3, 3). Just add blocks {1, 2, 5}, {1, 3, 5}, {1, 4, 5}, {2, 3, 5}, {2, 4, 5}, and {3, 4, 5}. X might be enclosed in some Y = BIBD(7, 3, 3) as well since the necessary conditions in Table 1 are satisfied. However, Corollary 1.1 does not allow this. The next point-enclosing goes into a BIBD(9, 3, 3) by Lemma 3.1.

Example 3.3 We (minimally) enclose BIBD(7, 3, 2) into Y = BIBD(9, 3, 3). The blocks of Y will consist of the blocks of X and the following blocks:

{1, 2, 3}, {x, y, 1}, {x, y, 2}, {x, y, 3}, {x, 1, 4}, {x, 1, 5}, {x, 2, 6}, {x, 2, 7},{x, 3, 4}, {x, 3, 5}, {x, 6, 4}, {x, 7, 5}, {x, 6, 7}, {y, 1, 6}, {y, 1, 7}, {y, 2, 4}, {y, 2, 5}, {y, 3, 6}, {y, 3, 7}, {y, 5, 6}, {y, 4, 7}, {y, 4, 5}.

Example 3.4 We enclose X = BIBD(13, 3, 2) into Y = BIBD(15, 3, 3). The blocks of Y include those of X and those of the resolvable Z = GDD(4^3) constructed in Example 2.2. In addition, we add the blocks of a triple system (7, 3, 1) on the points {1, 2, 3, 4, x, y, 13}, a triple system on the points {5, 6, 7, 8, x, y, 13}, and a triple system on the points {9, 10, 11, 12, x, y, 13}. Finally, we expand a resolution class of Z using the point y (as in Example 2.2) and expand a resolution class using the point x.

Lemma 3.5 *Any X = BIBD(6t+1, 3, 2) may be minimally point-enclosed in some Y = BIBD(6t+3, 3, 3) if there exists a Z = BIBD(6t+3, 3, 1) with a resolution class R_1 and a partial resolution class R_2 with 6t points and which omits the 3 points from precisely one block of R_1.*

Proof: Suppose X is the given BIBD(6t+1, 3, 2) and Z is the required BIBD(6t+3, 3,1). We may suppose R_2 is a partition of {1, 2, ..., 6t} and take β = {6t+1, 6t+2, 6t+3} as the block of the parallel class R_1 omitted by R_2. The blocks of Y are those of X and those of Z supplemented as follows. We add two more copies of block β; we use the partial parallel class R_2 and replace each block with 3 blocks using 6t+2; we use R_1 and replace each block except β with 3 blocks using 6t+3.

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Example 3.6 The previous Lemma can be applied to enclose BIBD(19, 3, 2) into a BIBD(21, 3, 3). We let Z be the Kirkman Triple System(21, 3, 1) whose parallel classes are numbered as π_1 to π_{10} on page 78 of [15]. The blocks of Y are those of X and those of Z supplemented in the following way. In parallel class π_7, we note that {19, 20, 21} is a block. We add two more copies of that block. We use the other blocks of that same class to expand 21, as in Example 2.2. One may note that the set of first blocks in each of the classes π_1 to π_6 together gives a partition of points 1, 2, ..., 18. We use these blocks to expand point 20. Now the index is 3 for points 1 to 21.

We will make use of *Wilson's Construction* [14, p.30] to show that the method of Lemma 3.5 can be made to apply to *every* BIBD(6t+1, 3, 2) for t > 2. First, however, we will give a solution for the special case v = 25 in order to develop a technique that will be applied in later sections and in order to introduce some ideas needed here. Our discussion is taken from Section 3 of [19] using results also in [21]. The complete graph K_n on n vertices consists of all C(n, 2) edges. A one-factorization of K_n is a partition of these edges into one-factors, i.e., sets of edges in which each vertex appears once and only once. A one-factorization of K_{2n} consists of 2n-1 one-factors which are disjoint (as sets of pairs). The edges of K_{2n} also can be put into disjoint classes $P_1, P_2, ..., P_n$, where edge (i, j) is in P_k if and only if $i-j \equiv k \pmod{2n}$.

Lemma 3.7 *With respect to the complete graph K_{2n}, we have:*
(a) The triangles {1+i, 2+i, 4+i} for i = 1, 2, ..., 2n contain exactly the edges from P_1, P_2, and P_3, and the graph K_{2n} may be factored into 2n-1 one-factors such that six of the 1-factors can be combined into 2n triangles. (b) The triangles {1+i, 1+x+i, 1+x+y+i} for i = 1, 2, ..., 2n contain exactly the edges from P_x, P_y, and P_{x+y} where x+y < n. (c) The pairs in P_{2x+1} (for 2x+1 < n) split into two one-factors. The pairs in P_{2x} split into a two-factor if 2x < n. (d) If 2x+1 < n, then $P_{2x} \cup P_{2x+1}$ splits into four one-factors. P_n is a single one-factor. If n is odd, the set $P_{n-1} \cup P_n$ can be split into three one-factors. (e) For the complete graph K_{6s}, the set $P_s \cup P_{2s}$ forms 4s distinct triangles and the set P_{2s} forms 2s distinct triangles.

Example 3.8 To enclose X = BIBD(25, 3, 2) into Y = BIBD(27, 3, 3), we first observe that there are 151 new blocks to add to X to create Y. We use K_{24} and apply Lemma 3.7 in the following way. From K_{24} use $P_4 \cup P_8$ to create 16 triples, $P_5 \cup P_6 \cup P_{11}$ to create 24 triples, $P_1 \cup P_9 \cup P_{10}$ to create 24 blocks, $P_2 \cup P_3$ to create four one-factors, P_7 to create two one-factors, and P_{12} to create one one-factor. By using each P_i, we insure the index is increased by 1 for all points of X except 25. We use the 12 edges in a one-factor to create 12 blocks with v = 25 as the third point. We use three one-factors with x to create 36 blocks and three other one-factors with y and build 36 blocks. We use three copies of the block {25, x, y}. An easy count verifies that the index is 3 for all points of Y.

The *Deficiency Graph* of $(Z_n, +)$, where $n \equiv 1$ or 5 (mod 6), is defined to be the graph G = (V, E) where $V = Z_n \setminus \{0\}$ and $E = \{\{x, -x\}, \{x, -2x\}: x \in V\}$. It is immediately seen that the Deficiency Graph is three-regular and has a 1-factorization into three 1-factors [15, p.26]. Wilson's Construction for a Steiner Triple System on the set {0, 1, 2, ..., 6t, x, y} uses the block {0, x, y} and uses one one-factor with 0, one with x, and one with y. The other blocks are from T, where T denotes the set of blocks {a, b, c} such that $a + b + c \equiv 0 \pmod{6t+1}$ and $1 \le a, b, c \le 6t$. We will use the blocks from T to prove the following theorem.

Theorem 3.9 *Every X = BIBD(6t+1, 3, 2) can be minimally point-enclosed into a Y = BIBD(6t+3, 3, 3).*

Proof: For t = 1 or 2, the examples give a solution. When t > 2, we use Wilson's construction and the blocks of T to build the needed Z as in Lemma 3.5. We form two parallel classes for the points 1, 2, ..., 6t. For the first parallel class, use blocks

{1, 2, 6t-2}, {3, 4, 6t-6}, ..., {2t-1, 2t, 2t+2}, and
{2t+1, 4t+1, 6t}, {2t+3, 4t+3, 6t-4}, ..., {4t-1, 6t-1, 2t+4}.

For the second parallel class, use blocks

{6t, 6t-1, 3}, {6t-2, 6t-3, 7}, ..., {4t+2, 4t+1, 4t-1}, and
{4t, 2t, 1}, {4t-2, 2t-2, 5}, ..., {2t+2, 2, 4t-3}.

We take the first parallel class and the block β = {6t+1, x, y} as the class R_1 and the second partial parallel class is the class R_2. The result follows from Lemma 3.5.

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4. λ = 3

In this case BIBD(v, 3, 3) exist for $v \equiv 1, 3, 5 \pmod 6$. Point-enclosings are known for v = 6t+3 and 6t+5, from [13] for t > 0. The only general case in the section is that for v = 6t+1 and t > 0.

Example 4.1 We investigate the minimal point-enclosing of X = BIBD(7, 3, 3). First, from Table 1, we observe that X may possibly be enclosed into a design Y = BIBD(v, 3, 4) for v = 9, 10, 12, 13. But none of these cases satisfy the inequality in Corollary 1.2. The minimal enclosing occurs for a (15, 3, 4) triple system by Lemma 3.1.

The proof of Theorem 4.2 below is similar to that of Theorem 2.5 of [2].

Theorem 4.2 *Suppose $2 < t$. Any BIBD(6t+1, 3, 3+2m) may be minimally point-enclosed into Y = BIBD(6t+3, 3, 4+2m) provided $m \leq (3t-5)/2$.*

Proof: With the restriction on m, Corollary 1.2 is satisfied. Let Z be a resolvable GDD(2^{3t}) which exists if $t \geq 3$ [8]. Identify 6t elements of X arbitrarily with those of Z, and refer to the remaining element as 6t+1. To form Y we use the blocks of X and of Z. We further add, for each group {g, h} of Z, the blocks {g, h, 6t+1}. Next we add 4+2m copies of {x, y, 6t+1}. We expand the blocks of m+2 resolution classes of Z with point x and do the same for y with a different set of m+2 resolution classes. Z provides 3t-1 resolution classes and 4+2m are needed, but the hypothesis on m and t suffices.

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5. $\lambda = 4$

For $\lambda = 4$, v may be 6t+1, 6t+3, 6t+4 and 6t, by Table 1. Only the v = 6t case is known to have a minimal point-enclosing from [12]. Thus, here we have the following general cases to consider:

(6t + 1, 3, 4) → (6t + 3, 3, 5)

(6t + 3, 3, 4) → (6t + 7, 3, 5)

(6t + 4, 3, 4) → (6t + 7, 3, 5).

We have a general result about v = 6t+4 for t > 3 (Theorem 5.5), but we deal with small t first since each case is different.

Theorem 5.1 *Suppose $v \equiv 4 \pmod 6$. Then the following are minimal point-enclosings:*

(a) BIBD(4, 3, 4) → BIBD(9, 3, 5).

(b) BIBD(10, 3, 4) → BIBD(19, 3, 5).

(c) BIBD(16, 3, 4) → BIBD(19, 3, 5).

(d) BIBD(22, 3, 4) → BIBD(25, 3, 5).

Proof: Part (a) follows immediately from Corollary 1.2 and Lemma 3.1(b). The other cases have interesting constructions.

For part (b), let B = BIBD(10, 3, 4). From Corollary 1.2, the minimal possible s is 9. Any enclosing of B into Y = BIBD(19, 3, 5) will thus be minimal. Let 1, 2, ..., 9 denote the new points to be added to B to form Y. These nine points can be formed into 36 distinct pairs. We divide these pairs into two disjoint classes α and β where α is the following collection of 18 pairs:

$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5,\}, \{2, 7\}, \{2, 8\}, \{2, 9\}, \{3, 4\}, \{3, 5\},$
$\{3, 6\}, \{4, 8\}, \{4, 9\}, \{5, 6\}, \{5, 7\}, \{6, 7\}, \{6, 9\}, \{7, 8\}, \{7, 9\}.$

Let a, b, ..., j denote the points of B. We form blocks for Y by putting each of the points a, b, c, d, and e with each of the pairs of α. We put each of the points f, g, h, i, and j with each of the pairs of β. Each new point appears 5 times in a block with the other new points and 4 times in a block with each point of B. We now use 9 one-factors of K_{10} with vertices labeled with the points of B, and we put each new point 1, ..., 9 with the pairs of one one-factor to form the rest of the blocks for Y.

For part (c), we use K_{16} which gives us 15 one-factors. We use 5 one-factors for each of the 3 new points x, y, and z, and add 5 copies of the block {x, y, z}.

For part (d), let D = BIBD(22, 3, 4) and x, y, and z the three new points. First we use blocks of D and five copies of the block {x, y, z}. We factor the graph K_{22} into 21 one-factors such that some six can be combined to make triples (blocks of Y), applying Lemma 3.7. The other 15 one-factors are used, 5 with each of new points x, y, and z, to make the remaining triples.

■

In the proof of part (b), the enclosing given is faithful, and this is also one of the few cases we have with equality in Corollary 1.2.

Theorem 5.2 *Suppose $v \equiv 1 \pmod 6$. Then the following point-enclosings are minimal.*

(a) BIBD(7, 3, 4) $\to$ BIBD(15, 3, 5).

(b) BIBD(13, 3, 4) $\to$ BIBD(15, 3, 5).

(c) BIBD(19,3, 4) $\to$ BIBD(21, 3, 5).

(d) Suppose $t > 3$ and $t \equiv 0, 2 \pmod 3$ and $m \le t$. If X = BIBD(6t+1, 3, 2m), then there is a minimal point-enclosing Y = BIBD(6t+3, 3, 2m+1).

Proof: Part (a) follows from the Corollary 1.2 and Lemma 3.1. For part (b), Use a 1-factorization of K_{12} into eleven one-factors. Use one one-factor with point 13, and use 5 with each of points x and y. Use 5 copies of block {13, x, y}. For part (c), use a 1-factorization of K_{18} to build 17 one-factors. Use six of them to make triangles (Lemma 3.7, a), use one with point 19, and use 5 each with x and y.

For part (d) the construction is more elaborate. We suppose $t \equiv 0, 2 \pmod 3$ and $t \neq 3$. Since $2t \neq 2, 6$, there exists a resolvable 3-GDD($(2t)^3$). Since $2t+3 \equiv 1, 3 \pmod 6$ a Steiner triple system (2t+3, 3, 1) exists. The blocks for Y = BIBD(6t+3, 3, 2m+1) consist of

i) the blocks of X;

ii) the blocks of Z = 3-RGDD($(2t)^3$) where the groups are {1, 2, ..., 2t}, {2t+1, ..., 4t}, and 4t+1, ..., 6t};
iii) the blocks of three Steiner triple systems (2t+3, 3, 1). The latter designs are based, respectively on the points

{1, 2, ..., 2t, x, y, 6t+1},
{2t+1, 2t+2, ..., 4t, x, y, 6t+1}, and
{4t+1, 4t+2, ..., 6t, x, y, 6t+1};

iv) next, we expand m resolution classes of Z with x and expand m different resolution classes with y. Exactly enough resolution classes are available since $m \leq t$, 2m are needed, and 2t is the number of resolution classes.
v) finally, we add 2m-2 copies of block {x, y, 6t+1}.

■

Although the theorem omits the sequence v = 18s + 7, the construction in Example 6.3 may be used to enclose a (25, 3, 4) into a (27, 3, 5) and a (61, 3, 4) into a (63, 3, 5).

Theorem 5.3 *Suppose $v \equiv 3$ (mod 6). Then the following point-enclosings are minimal:*

(a) BIBD(9, 3, 4) →BIBD(19, 3, 5).

(b) BIBD(15, 3, 4) →BIBD(31, 3, 5).

(c) Suppose $t > 3$ and $t \equiv 1, 2$ (mod 3). If X = BIBD(6t+3, 3, 2m), then there is a minimal point-enclosing into a Y = BIBD(6t+7, 3, 2m+1) if $4m \leq 2t+1$.

Proof: Parts (a) and (b) follow from Corollary 1.2 and Lemma 3.1. For part (c), let Z = RGDD($(2t+1)^3$). The points of Y are those of X and w, x, y, and z. The blocks of Y are those of X, and those of Z, those of three Steiner triple systems based on the point sets below

{1, 2, 3, ..., 2t+1, w, x, y, z},
{2t+2, 2t+3, ..., 4t+2, w, x, y, z}, and
{4t+3, 4t+4, ..., 6t+3, w, x, y, z}.

Also included are the blocks (m-1 times, each):

{w, x, y}, {w, x, z}, {x, y, z}, and {w, y, z}.

Finally, we expand m parallel classes with each of w, x, y, and z.

■

Theorem 5.4 *Suppose $t > 3$ and $t \equiv 1, 2$ (mod 3). If X = BIBD(6t+4, 3, 2m), then there is a minimal point-enclosing of X into a Y = BIBD(6t+7, 3, 2m+1) if $3m \leq 2t+1$.*

Proof: The proof is the same as for Theorem 5.3(c) except we replace w there by 6t+4 (and do not expand 6t+4 with any parallel classes).

■

Theorems 5.3 and 5.4 can not be made to apply to BIBD(18m+3, 3, 4) or to BIBD(18m+4, 3, 4).

6. $\lambda = 5, 6$

For $\lambda = 5$, there are two possibilities, v = 6t+1 and 6t+3, but we need not consider either of them since both systems (6t+1, 3, 5) and (6t+3, 3, 5) are known to have minimal point-enclosings from [13]. Therefore, we move to $\lambda = 6$. Here v may be 0,1,...,5 (mod 6). There are minimal point-enclosings for v = 6t+2 and 6t, from [13]. Thus, there are 4 primary cases for us to consider in this section:

$(6t + 1, 3, 6) \rightarrow (6t + 3, 3, 7)$
$(6t + 3, 3, 6) \rightarrow (6t + 7, 3, 7)$
$(6t + 4, 3, 6) \rightarrow (6t + 7, 3, 7)$
$(6t + 5, 3, 6) \rightarrow (6t + 7, 3, 7)$.

Theorem 6.1 *Suppose v = 3t – 1 and t > 1. Then X = BIBD(v, 3, 6) may be enclosed into Y = BIBD(2v+3, 3, 7).*

Proof: To enclose BIBD(5, 3, 6) into BIBD(13, 3, 7), add the blocks of one BIBD(13, 3, 1). Next we use the edges from 6 copies of K_8, based on the 8 new points, in the following way. The edges from 4 copies of K_8 are used to make 28 one-factors. The edges from 2 of the copies of K_8 are decomposed to get 2 copies of P_1, P_2, and P_3 which are used to make triangles (blocks) and 2 copies of P_4 which give us 2 more one-factors. The 30 one-factors are used, 6 with each original point, to make the remaining blocks. This enclosing for v = 5 is minimal.

For X = BIBD(8, 3, 6) into BIBD(19, 3, 7), add one copy of BIBD(19, 3, 1) and one copy of Z, a cyclic BIBD(11, 3, 6). We expand 8 difference sets from Z, one with each original point (10 difference sets are available). This is not minimal. (A minimal enclosing for BIBD(8, 3, 6) into a BIBD(9, 3, 7) can be obtained by putting each pair of points of X into a block with new point y.)

For a minimal enclosing of BIBD(11, 3, 6) there is a separate argument in Example 6.12. Now, for t > 4, by Lemma 3.1, X may be embedded into a (2v+2, 3, 6) triple system and this may be enclosed into a (2v+3, 3, 7) by Section 7 of [13] in which a (6t, 3, 6) is enclosed into a (6t+1, 3, 7) when t > 1.

■

Theorem 6.2 *Suppose $v \equiv 1$ (mod 6). Then the following point-enclosings are minimal:*

(a) BIBD(7, 3, 6) $\rightarrow$ BIBD(15, 3, 7).

(b) BIBD(13, 3, 6) $\rightarrow$ BIBD(27, 3, 7).

(c) BIBD(19, 3, 6) $\rightarrow$ BIBD(21, 3, 7).

(d) Suppose t >3 and $t \equiv 0, 2$ (mod 3). If X = BIBD(6t+1, 3, 6), then there is a minimal point-enclosing Y = BIBD(6t+3, 3, 7).

Proof: Parts (a) and (b) follow by Corollary 1.2 and Lemma 3.1. Part (c) follows from Theorem 6.9, below. Part (d) follows immediately from Theorem 5.2.

■

This leaves the cases X = BIBD(18s+7, 3, 6) incomplete. However, we have constructions for the first three of these as we now show in Example 6.3 and for some of the remaining cases in Theorem 6.4.

Example 6.3 For v = 18s+7 and s = 1, 2, or 3, there is a minimal point enclosing of X = BIBD(v, 3, 6) into Y = BIBD(v+2, 3, 7). First suppose s = 1, and X = BIBD(25, 3, 6). The blocks of Y are those of X and those of Z, a resolvable GDD(6^4). We may assume the groups are {1, ..., 6}, {7, ...12}, ..., {19, ...24}. We augment each group with 25, x, and y and form four (9, 3, 1) triple systems. We add three copies of the block {25, x, y}, and expand each of x and y with three resolution classes (9 are available). When s = 2 and v = 43, we use Z = RGDD(6^7), and the rest is similar. When s = 3, v = 61, and in this case Z = RGDD(12^5). To continue this idea, Z must have 18s+6 points, the number of groups must be less or equal to seven, and the group size must be 0 or 4 mod 6. This is all accomplished in the next theorem.

Theorem 6.4 *Suppose s ≡ 1 (mod 4). Then X = BIBD(18s+7, 3, 6) can be minimally point-enclosed into Y = BIBD(18s+9, 3, 7).*
Proof: Use RGDD($(3s+1)^6$). Since s is odd, 3s+1 ≡ 4 (mod 6) and the necessary triple systems exist. Since s is 1 mod 4, the RGDD exists (that is, g ≠ 10 mod 12). The rest follows as in Example 6.3. We note that the same proof works for s = 3+4j provided the GDD($(10+12j)^4$) exists.

■

Theorem 6.5 *Suppose v ≡ 3 (mod 6). Then the following point-enclosings are minimal:*

(a) BIBD(9, 3, 6) → BIBD(19, 3, 7).

(b) BIBD(15, 3, 6) → BIBD(31, 3, 7).

(c) BIBD(21, 3, 6) → BIBD(43, 3, 7).

(d) Suppose t > 3 and t ≡ 1, 2 (mod 3). If X = BIBD(6t+3, 3, 6), then there is a minimal point-enclosing into a Y = BIBD(6t+7, 3, 7).

Proof: Parts (a), (b), and (c) follow from Corollary 1.2 and Lemma 3.1. Part (d) follows immediately from Theorem 5.3.

■

This leaves the case X = BIBD(18s+3, 3, 6) incomplete. However, see Example 6.8 and Corollary 6.9, below.

Theorem 6.6 *Suppose* $v \equiv 4 \pmod 6$. *Then the following point-enclosings are minimal.*

(a) $BIBD(4, 3, 6) \to BIBD(9, 3, 7)$.

(b) $BIBD(10, 3, 6) \to BIBD(21, 3, 7)$.

(c) $BIBD(16, 3, 6) \to BIBD(31, 3, 7)$.

(d) $BIBD(22, 3, 6) \to BIBD(25, 3, 7)$.

(e) Suppose $t > 3$ *and* $t \equiv 1, 2 \pmod 3$. *If* $X = BIBD(6t+4, 3, 6)$, *then there is a minimal point-enclosing of* X *into a* $Y = BIBD(6t+7, 3, 7)$.

Proof: For part (a) and part (b) follow from Corollary 1.2. Part (c) follows from Theorem 6.10, below. For part (d), use the 21 one-factors from K_{22}, seven with each of the three new points. Add 7 copies of block {x, y, z}. Part (e) follows immediately from Theorem 5.4.

■

Theorem 6.7 *(a) Suppose that* λ *is even,* $v = g(\lambda+1)$, *and that a* $RGDD(g^{\lambda+1})$ *exists. If* $g+s \equiv 1, 3 \pmod 6$ *and* $g \geq s$, *then* $X = BIBD(v, 3, \lambda)$ *can be enclosed into* $Y = BIBD(v+s, 3, \lambda+1)$.

(b) Suppose that λ *is even,* $v-1 = g(\lambda+1)$, *and that a* $RGDD(g^{\lambda+1})$ *exists. If* $g+s \equiv 1, 3 \pmod 6$ *for* $g \geq s > 0$, *then* $X = BIBD(v, 3, \lambda)$ *can be enclosed into* $Y = BIBD(v+s, 3, \lambda+1)$.

Proof: For part (a), let $Z = RGDD(g^{\lambda+1})$. The blocks of Y are those of X, of Z, and of $\lambda+1$ Steiner triple systems on the groups whose point sets are each augmented by the s points, and each new point is expanded with $\lambda/2$ resolution classes. For part (b), the argument is the same except that point v replaces one of the s points from the part (a) solution.

■

Example 6.8 As an application we might hope to enclose a BIBD(21, 3, 6) into a BIBD(25, 3, 7) using $Z = RGDD(3^7)$ and form 7 Steiner triple systems on the sets {1, 2, 3, w, x, y, z}, ..., {19, 20, 21, w, x, y, z}. Unfortunately, there are not enough resolution classes. In fact, Corollary 1.2 also forbids this enclosing. BIBD(21, 3, 6) minimally encloses into BIBD(43, 3, 7). However, we illustrate part (b) by enclosing BIBD(22, 3, 6) into BIBD(25, 3, 7). The 7 triple systems, adapted from the v = 21 almost-solution, are based on the points {1, 2, 3, 22, x, y, z}, ..., {19, 20, 21, 22, x, y, z}, and so on. Only 9 resolution classes are needed and 11 are available. This may be contrasted by the solution in Theorem 6.6d.

Theorem 6.7 can be applied in several of the otherwise unresolved cases provided the needed GDD exists, and in the next corollary we indicate the appropriate parameters.

Corollary 6.8 *Suppose $n \equiv 1 \pmod 7$ and $n > 1$. Then BIBD(18n+3, 3, 6) and BIBD(18n+4, 3, 6) can be minimally enclosed into Y = BIBD(18n+7, 3, 7).*

Proof: Here 18n+3 = 21+126t = 7(3+18t). Use Z = RGDD($(3+18t)^7$). ■

We note that when u = 7, the necessary conditions for RGDD(g^u) require $g \equiv 0, 3 \pmod 6$. When u = 5, then g must be a multiple of 6. For example, Theorem 6.7 can not be applied to enclose X = BIBD(18k+7, 3, 4) into Y = BIBD(18k+9, 3, 5) if $k \equiv 1 \pmod 5$ since the needed GDD($(5+18t)^5$) does not exist.

Theorem 6.9 *Suppose that v-1 = mt, that a Steiner triple system (m, 3, 1) exists, and that there are 2λ+3 1-factors in the 1-factorization of the complete t-partite graph $K_{m,m,...,m}$. Then X = BIBD(v, 3, λ) can be minimally point-enclosed into Y = BIBD(v+2, 3, λ+1). With these hypotheses, X may be written as X = BIBD(mt+1, 3, [m(t-1)-3]/2).*

Proof: In addition to the blocks of X, we write x with λ+1 1-factors, y with λ+1 1-factors, and v with one 1-factor. We add the block {v, x, y} λ+1 times, and for each partition of size m we add a triple system (m, 3, 1). ■

As an example of the theorem, we may enclose BIBD(19, 3, 6) into BIBD(21, 3, 7). There are 15 1-factors using a 1-factorization of $K_{3,3,3,3,3,3}$. We write x with 7 one-factors, y with 7 one-factors, and 19 with the remaining 1-factor. Now add the block {x, y, 19} seven times, add 6 triple systems (3, 3, 1) that form the partition for the vertices, and add the blocks of X, the original design.

Theorem 6.10(a) below sharpens the enclosing results for some cases also dealt with in Lemma 3.1 and applies to this section since λ = 6. Parts (b) and (c) are more general and take the idea of the proof as far as possible, i.e., for the largest possible index for a point-enclosing. Theorems 6.11 and 6.13 are also quite general but are needed here for small values of n.

Theorem 6.10 *(a) Suppose $v \geq 14$ and $v \equiv 2, 4 \pmod 6$. Then X = BIBD(v, 3, 6) may be enclosed in Y = BIBD(2v-1, 3, 7).*

(b) Suppose $v \geq 16$ and $v \equiv 4 \pmod 6$. Then X = BIBD(v, 3, 2t) can be enclosed into Y = BIBD(2v-1, 3, 2t+1) for $2 \leq 2t \leq (v-2)/2$.

(c) Suppose $v \geq 14$ and $v \equiv 2 \pmod 6$. Then X = BIBD(v, 3, 6t) can be enclosed into Y = BIBD(2v-1, 3, 6t+1) for $6 \leq 6t \leq (v-2)/2$.

Proof: We prove part (a). Since v is even, a 1-factorization of K_v exists and there are v-1 1-factors, say $F_1, F_2, ..., F_{v-1}$. For each edge {a, b} in F_i, we will add block {v+i, a, b} to X. Here the new points are v+i for i = 1,..., v-1.

Suppose $v \equiv 4 \pmod 6$. Then a resolvable $Z = BIBD(v-1, 3, 1)$ exists, and we add 7 copies of Z. Z is based on the new points $\{v+1, v+2, ..., 2v-1\}$. We expand each x in $\{1, 2, ..., v\}$with 3 resolution classes of Z. Now suppose $v \equiv 2 \pmod 6$. Here we use the 1-factors as before, but this time we use a cyclic $Z = BIBD(v-1, 3, 1)$. For each x in $\{1, 2, ..., v\}$ use a different difference set $\{a, b, c\}$ and the blocks developed from it to expand with point x. Since each point of Z will appear in 3 blocks developed from the difference set, each point of X will appear with each new point 6 times using these blocks. The first construction requires the number of resolution classes available, $7(v-2)/2$, to be greater or equal to 3v. In the second we need the number of difference sets, $7(v-2)/6$, to be greater or equal to v. Each is satisfied if $v \geq 14$.

For each of parts (b) and (c), let $v = 2n$. For part (b), we create Y from the blocks and points of X by adding 2n-1 new points and adding several sets of blocks which we will describe. Since $2n-1 \equiv 3 \pmod 6$, there exists a resolvable $Z = BIBD(2n-1, 3, 1)$ based on the new points. We add new blocks from 2t+1 copies of design Z. Each one has $(2n-2)/2 = n-1$ resolution classes. Thus, we have $(2t+1)(n-1) = 2tn+n-2t-1$ resolution classes altogether using all the copies of Z. We expand each point x in X with t resolution classes (as in the proof of Theorem 2.1). This requires 2nt classes, and these classes are available whenever $n-2t-1 \geq 0$. But this happens here because of the hypothesis on v and t. Since each resolution class creates index 2 between x and the points of the class, each point x of X now appears with points of Y in 2t blocks. New points of Y appear with each other in blocks 2t+1 times (since we used 2t+1 copies of Z). To complete the set of blocks needed, we use the 2n-1 one-factors available from a one-factorization of K_{2n} (the points of X). We put each of the new 2n-1 new points with one one-factor. This complete Y.

Now suppose $2n-1 \equiv 1 \pmod 6$. We use one-factors as in the previous case, but in this case we need 6t copies of a cyclic $Z = BIBD(2n-1, 3, 1)$. Since each point appears 3 times in blocks generated by a difference set, Z has r/3 difference sets, where r is the replication number of Z. For any BIBD, $vr = bk$ and $\lambda(v-1) = r(k-1)$, and for Z this means $r = n-1$. Thus, 6t+1 copies of Z give $(6t+1)(n-1)/3$ total difference sets available. Since, for design Y, we have index of 6t+1, we need t difference sets with which to expand each point of X (as described in Section 1). That is, $vt = 2nt$ difference sets are needed. We have them provided $n \geq 6t+1$, and this is equivalent to the condition in the hypothesis for case (c).

■

Theorem 6.11 *Suppose $v \equiv 3, 5 \pmod 6$. Then $X = BIBD(v, 3, (v+1)/2)$ can be enclosed in $Y = BIBD(2v+3, 3, (v+3)/2)$.*

Proof: To the blocks of X we add those of $Z_1 = BIBD(2v+3, 3, 1)$, where Z_1 is based on the points of X and v+3 new points. We now have all the blocks needed which contain pairs from X. We use Z_2 = GDD($2^{(v+3)/2}$), which is based on the v+3 new points. Z_1 and Z_2 exist by the congruence condition on v. The index is 2 for new points with each other (except for points paired with each other in the groups). Let us consider the set of groups as a one-factor. We further create v+2 new one-factors from K_{v+3}, and we make (v-1)/2 copies of these one factors. Now, based on the new points, we have a total of

$$1 + (v+2)(v-1)/2 = v(v+1)/2$$

one-factors available. We use (v+1)/2 of them for each of the v points of X. An easy count shows the new index is (v+3)/2 for all points.

■

Example 6.12 As an example of the previous theorem, we minimally enclose X = BIBD(11, 3, 6) into Y = BIBD(25, 3, 7) - the minimality is from Corollary 1.2. Use the blocks of X, the blocks of Z_1 = BIBD(25, 3, 1), the blocks of Z_2 = GDD(2^7), and the following. We use five one-factorizations of K_{14} giving 65 1-factors. The groups of Z_2 give the 66th needed 1-factor. For each point of X, construct blocks using six 1-factors.

Theorem 6.13 *If the complete graph K_{2n} can be decomposed into $2\lambda+3$ 1-factors and the remaining edges into triangles, then X = BIBD(2n+1, 3, λ) can be minimally enclosed into Y = BIBD(2n+3, 3, $\lambda+1$).*

Proof: Using the decomposition of K_{2n}, one 1-factor is used to make blocks with point 2n+1, and $\lambda+1$ 1-factors are used with each of x and y. Use $\lambda+1$ copies of the block {2n+1, x, y}.

■

Corollary 6.14 *The design X = BIBD(2n+1, 3, n-5) can be minimally point-enclosed into Y = BIBD(2n+3, 3, n-4).*

Proof: The blocks of Y are those of X, n-4 copies of the block {2n+1, x, y}, and the following. Decompose K_{2n} into 2n-7 one-factors and 2n triples (Lemma 3.8). Form new blocks from v = 2n+1 and one one-factor, and from x and y with n-4 one-factors each.

■

Example 6.15 We minimally point-enclose a BIBD(23, 3, 6) into a BIBD(25, 3, 7). Decompose K_{22} into twenty-one 1-factors. Use seven 1-factors with each new point x, y, and one with 23. The rest follows by Theorem 6.14 and Lemma 3.7a.

Example 6.16 We minimally point-enclose X = BIBD(29, 3, 6) into Y = BIBD(31, 3, 7) by applying Theorem 6.13 to K_{28}. The fifteen 1-factors needed come from $P_2 \cup P_3$ and $P_6 \cup P_7$ (eight 1-factors), one from P_{14}, and two each from P_5, P_{11}, and P_{13}. Triples come from $P_4 \cup P_8 \cup P_{12}$ and $P_1 \cup P_9 \cup P_{10}$.

Theorem 6.17 *Suppose $v = 6t+5$ and $t = 2m$. Then $X = BIBD(v, 3, 6)$ may be minimally point-enclosed into $Y = BIBD(v + 2, 3, 7)$.*
Proof: Note $v = 4(3m+1) + 1$. Let Z be a cyclic $GDD(4^{3m+1})$. For each group {a, b, c, d} of Z, we add the following blocks to those of X and Z:

{v, a, b}, {v, c, d}, {x, a, c}, {x, b, d}, {y, a, d}, {y, b, c}.

We expand a difference set (and the blocks developed from it) with x and another one with y. ■

In Table 2, which we hope will be useful for the reader, we list several cases for small v which are either special cases, the first case of a general theorem, not within the scope of some general theorems, or taken from [13] for completeness. Each entry gives the least possible point-enclosing allowed by Table 1 and by Corollary 1.2, and the construction is referenced. Unresolved sequences are listed for $\lambda = 4$ or 6 in Table 3.

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