A Decomposition for Simple Polygons

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Abstract

ABSTRACT. Let S be a simple polygon in the plane whose vertices may be partitioned into sets A', B', such that for every two points of A' (of B'), the corresponding segment is in S. Then S is a union of 6 (or possibly fewer) convex sets. The number 6 is best possible. Moreover, the simple connectedness requirement for set S cannot be removed.

1 Introduction.

We begin with some familiar definitions. Let S be a set in the plane. For points x, y in S we say x *sees* y via S (x is *visible* from y via S) if and only if the corresponding segment $[x, y]$ lies in S. Of course S is *convex* if and only if for every pair x, y in S, x sees y. Set S is *starshaped* if and only if for some point p in S, p sees each point of S, and the set of all such points p is the (convex) *kernel* of S, denoted ker S. Set S is called a *simple polygon* if and only if S is a connected, simply connected union of convex polygons. Clearly the boundary of S will be a closed polygonal curve λ, and we consider the vertices of S to be the vertices of λ together with those points at which S fails to be locally convex.

In case the edges of the simple polygon S are parallel to the coordinate axes, set S is called an *orthogonal polygon*. Moreover, replacing the usual notion of segment visibility above with the idea of staircase path visibility (see [7],[2],[3]), we may define sets which are convex or starshaped relative to staircase paths. In particular, set S is called *orthogonally convex* if and only if every two of its points lie on a common staircase path in S.

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It is fairly common for results concerning visibility via segments to motivate analogous results concerning visibility via staircase paths. However, here we have the situation reversed. In [1] it was proved that for a simply connected orthogonal polygon S, an assignment of vertices of S to orthogonally convex subsets A, B of S induces a decomposition of S into two or three orthogonally convex sets. While the bound three fails for segment visibility, the analogous result holds when the bound is raised to six. That is, if S is a simple polygon whose vertices may be assigned to convex subsets A, B of S, then S is a union of six (or possibly fewer) convex sets.

Throughout the paper, cl S, conv S, and ker S will denote the closure, convex hull, and kernel, respectively, for set S. For distinct points x and y, $L(x, y)$ will be their corresponding line. The reader may refer to Valentine [8], to Lay [6], to Danzer, Grünbaum, Klee [4], and to Eckhoff [5] for discussions concerning visibility via segments and corresponding convex and starshaped sets.

2 The Results

. We will establish the following theorem.

Theorem 1. Assume that S is a simple polygon in the plane whose vertices may be partitioned into sets A', B' such that for every two points of A' (of B'), the corresponding segment is in S. Then S is a union of 6 (or possible fewer) convex sets. The number 6 is best possible.

Proof. If S is convex, there is nothing to show, so assume that S is not convex and hence A', B' are nonempty. Since S is simply connected, clearly sets $A \equiv \text{conv } A'$ and $B \equiv \text{conv } B'$ lie in S. There are two cases to consider.

Case 1. Suppose that A and B are not disjoint. Observe that for each point p in $K \equiv A \cap B, p$ sees via S each vertex of S, and hence it is easy to show that $p \in \text{ker } S$. Certainly sets $A \backslash B, B \backslash A$ are nonempty. Let A_1 be a component of $A \backslash K$. Fix p in K, let R_1 be a ray from p which meets A_1, and order the rays at p in a clockwise direction, beginning at R_1. Relative to our clockwise order, these rays impose an order on the components of $A \backslash K$ and $B \backslash K$, alternately meeting component A_1 of $A \backslash K$, component B_2 of $B \backslash K$, component A_3 of $A \backslash K$, and so on.

Moreover, for any component C of $S \backslash (A \cup B)$ cl C will meet cl A_i and cl B_j for some A_i, B_j which are consecutive relative to our clockwise order. We assume that components of $S \backslash (A \cup B)$ exist, for otherwise S will be a union of two convex sets, finishing the argument. For convenience of notation, we let C_1 denote the C set whose closure meets cl A_1 and cl B_2, if it exists. Otherwise, let $C_1 = \phi$. Let C_2 denote the C set whose closure meets cl B_2 and cl A_3, if it exists. Otherwise, let $C_2 = \phi$. In this way we define $A_1, C_1, B_2, C_2, \dots, C_n, A_1$, where cl C_n meets cl B_n and cl A_1 (or is empty) and where n is even, $n \geq 2$. Observe that each nonempty set cl C_i is a triangular region having one vertex in $A \backslash K$, one in $B \backslash K$, one in $A \cap B$. Hence each cl C_i and each C_i will be convex. For future reference, we define the distance between sets C_i, C_j to be the shortest distance between their subscripts when the subscripts are adjusted modulo n.

To prove the theorem, we will assign every C_i set to one of four collections, each having its convex hull in S. As a preliminary result, we show that when the distance from C_i to C_j is at least three, then conv $(C_i \cup C_j) \subseteq S$. For convenience of notation, we let $i = 1$, $j = 4$, where $n \geq 6$. Since S is simply connected, clearly it suffices to show that for c_i in $C_i \neq \phi, i = 1, 4, [c_1, c_4] \subseteq S$. In case point p lies on the line $L = L(c_1, c_4)$, the result is immediate. Hence we assume that p lies in one of the corresponding open half planes L_1 or L_2 say L_1. Certainly relative to our clockwise ordering there are points from sets B_2, A_3, and B_4 which follow C_1 and precede C_4, while there are points from A_{n-1}, B_n, and A_1 which follow C_4 and precede C_1. (See Figure 1.) Thus if $p \in L_1$, then for one of the triples B_2, A_3, B_4 or A_{n-1}, B_n, A_1, each set has points which lie in L_2. Since the situations are symmetric, without loss of generality assume that B_2, A_3, B_4 all contain points in L_2, and select b_2, a_3, b_4 in B_2, A_3, B_4, respectively, each in L_2. Since $a_3 \notin B, a_3 \notin$ conv $\{b_2, b_4, p\}$, so by our clockwise ordering $[a_3, p]$ meets $[b_2, b_4]$ at some point p' in $(A \cap B) \cap L_2$. Hence $[p', p] \subseteq A \cap B \subseteq \ker S$. Moreover, again by our ordering $[p', p]$ meets $[c_1, c_4]$, say at p'', and since $p'' \in \ker S$, it follows that $[c_1, c_4] \subseteq S$, establishing our preliminary result.

We are ready to assign sets $C_i, 1 \leq i \leq n$, to collections $T_j, 1 \leq j \leq 4$. If $n \leq 4$, the procedure is trivial, so assume that $n > 4$. Since n is even, this implies that $n \geq 6$. There are several cases to consider:

If $n = 3k$ (for some $k \geq 2$), define

$$\begin{aligned} T_1 &= \{C_i : i \equiv 1 \bmod 3\}, \\ T_2 &= \{C_i : i \equiv 2 \bmod 3\}, \\ T_3 &= \{C_i : i \equiv 0 \bmod 3\}, \\ T_4 &= \phi. \end{aligned}$$

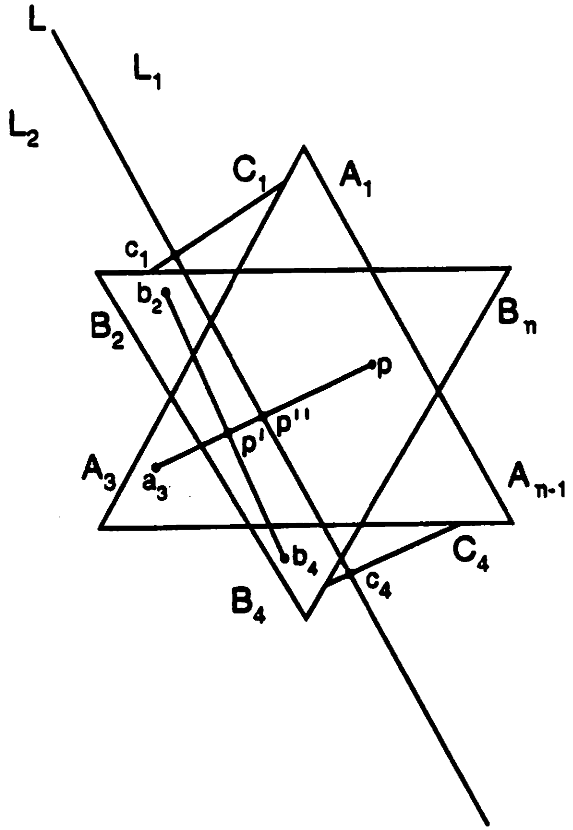

Figure 1.

If $n = 3k + 1$ (for some $k \geq 2$), define

$$\begin{aligned} T_1 &= \{C_i : i \equiv 1 \bmod 3, i < 3k+1\}, \\ T_2 &= \{C_i : i \equiv 2 \bmod 3\}, \\ T_3 &= \{C_i : i \equiv 0 \bmod 3\}, \\ T_4 &= \{C_{3k+1}\}. \end{aligned}$$

If $n = 3k + 2$ (for some $k \geq 2$) and 4 divides n, define

$$\begin{aligned} T_1 &= \{C_i : i \equiv 1 \bmod 4\}, \\ T_2 &= \{C_i : i \equiv 2 \bmod 4\}, \\ T_3 &= \{C_i : i \equiv 3 \bmod 4\}, \\ T_4 &= \{C_i : i \equiv 0 \bmod 4\}. \end{aligned}$$

Finally, if $n = 3k + 2$ (for some $k \geq 2$) and 4 fails to divide n, then since n is even, $3k + 2 \equiv 2 \bmod 4$. We define

$$\begin{aligned} T_1 &= \{C_i : i \equiv 1 \bmod 4, i < 3k + 1\}, \\ T_2 &= \{C_i : i \equiv 2 \bmod 4, i < 3k + 2\} \cup \{C_{3k+1}\}, \\ T_3 &= \{C_i : i \equiv 3 \bmod 4\} \cup \{C_{3k+2}\}, \\ T_4 &= \{C_i : i \equiv 0 \bmod 4\}. \end{aligned}$$

It is easy to check that every C_i is assigned to some collection T_j. Moreover, for each T_j, any two corresponding C sets are at least distance two apart. Using our preliminary result together with the simple connectedness of S, it follows that conv $T_j \subseteq S, 1 \leq j \leq 4$. These four sets, together with A and B, provide a decomposition of S into 6 (or possibly fewer) convex sets, finishing Case 1.

Case 2. Suppose that sets A and B are disjoint. Then for some line L, A and B lie in distinct open halfplanes determined by L. For any edge of S, either both endpoints lie in the same set A or B, or one vertex lies in A, one in B. Edges neither in A nor in B must be of the second type, and since S is simply connected, clearly S has either one or two such edges, say $[a, b]$ and $[a', b']$. Region D bounded by the closed curve $[a, a'] \cup [a', b'] \cup [b', b] \cup [b, a]$ is either convex or a union of two convex sets, and every point of $S \backslash (A \cup B)$ is in D. Hence S is a union of four (or possibly fewer) convex sets. This finishes Case 2 and completes the proof of the theorem.

Example 2 in [1] demonstrates that the bound in Theorem 1 is best possible, for the corresponding set satisfies our hypothesis and is a union of no fewer than 6 convex sets. Using segment visibility instead of staircase visibility, the set in [1, Example 1] shows that no bound is possible when the simple connectedness condition is removed.

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