On a Problem on Generalised Fibonacci Cubes

Titus Hilberdink
Reading University, Whiteknights, PO Box 217, Reading
Berkshire RG6 2AH, U.K.
e-mail t.w.hilberdink@reading.ac.uk

Carol Whitehead
Goldsmiths College, London SE14 6NW, U.K.
e-mail:c.whitehead@gold.ac.uk

Norma Zagaglia Salvi
Politecnico di Milano, P.za L. da Vinci 32, 20133 Milano, Italy
e-mail:norzag@mate.polimi.it

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Abstract

A *Fibonacci string of order* n is a binary string of length n with no two consecutive ones. The *Fibonacci cube* Γ_n is the subgraph of the hypercube Q_n induced by the set of Fibonacci strings of order n. For positive integers i, n, with $n \geq i$, the ith *extended Fibonacci cube* is the vertex induced subgraph of Q_n for which $V(\Gamma_n^i) = V_n^i$ is defined recursively by

$$V_{n+2}^i = 0V_{n+1}^i + 10V_n^i,$$

with initial conditions $V_i^i = B_i$, $V_{i+1}^i = B_{i+1}$, where B_k denotes the set of binary strings of length k. In this study, we answer in the affirmative a conjecture of Wu [10] that the sequences $\{| V_n^i |\}_{n=i+2}^{\infty}$ are pairwise disjoint for all $i \geq 0$, where $V_n^0 = V(\Gamma_n)$.

1 Introduction and notation

The hypercube (or n-cube) Q_n is the graph with 2^n vertices, each corresponding to a unique binary string of length n, where two vertices are adjacent if and only if the corresponding binary strings differ in exactly one bit. These graphs have been used extensively as architectural models for parallel processors, where each vertex represents a processor and each edge represents a direct link between two processors. For the purposes of message routing, Q_n has the advantages of being Hamiltonian and of containing a Hamming path connecting every pair of vertices. Further, the degree of each vertex is only n, so there are relatively few links into each processor.

However, the hypercubes have the disadvantage that they allow only a limited choice for the number of vertices. In recent years, various subgraphs of Q_n have been proposed as alternative models. Among these are the *Fibonacci cubes* proposed by Hsu [5], and the *extended Fibonacci cubes* proposed by Wu [10], which are the subject of this article.

A *Fibonacci string of order* n is a binary string of length n with no two consecutive ones. The *Fibonacci cube* Γ_n is the subgraph of Q_n induced by the set of Fibonacci strings of order n.

We use the following notation. Let α, β be two binary strings. Denote by $\alpha\beta$ the string obtained by concatenating α and β. More generally, if S is any non-empty set of binary strings, then $\alpha S\beta = \{\alpha\sigma\beta : \sigma \in S\}$, and $\alpha\emptyset\beta = \alpha\beta$. Let B_n denote the set of binary strings of length n and let V_n be the subset of B_n containing all strings with no two consecutive ones, so that $V(Q_n) = B_n$ and $V(\Gamma_n) = V_n$. For $n \geq 0$, the set V_n satisfies the recursive relation

$$V_{n+2} = 0V_{n+1} + 10V_n, \tag{1}$$

with initial conditions $V_0 = \{\emptyset\}$, $V_1 = \{0, 1\}$. Let $G_n = |V_n|$, $n \geq 0$. Then it follows from (1) that the sequence $\{G_n\}_{n=0}^{\infty}$ satisfies

$$G_{n+2} = G_{n+1} + G_n,$$

and hence it is a generalised Fibonacci sequence, with initial terms $G_0 = 1$, $G_1 = 2$. Clearly $\{G_n\}_{n=0}^{\infty}$ is a subsequence of the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ with initial terms $F_0 = 0$, $F_1 = 1$, where $G_n = F_{n+2}$. The Fibonacci cubes Γ_n, $n = 0, 1, 2, 3$ are illustrated in Figure 1.1 below.

The *extended Fibonacci cubes* are constructed by the same recursive relation as the Fibonacci cube, but with different initial conditions. For positive integers i, n, with $n \geq i$, the ith *extended Fibonacci cube of order* n, denoted by Γ_n^i, is a vertex induced subgraph of Q_n, where $V(\Gamma_n^i) = V_n^i$ is defined recursively by the relation

$$V_{n+2}^i = 0V_{n+1}^i + 10V_n^i, \tag{2}$$

with intitial conditions $V_i^i = B_i$, $V_{i+1}^i = B_{i+1}$. Thus $\Gamma_i^i = Q_i$, $\Gamma_{i+1}^i = Q_{i+1}$ and in general, when $n \geq i+2$, the vertices of Γ_n^i are (0,1)-strings in which the last $i+1$ positions are vertices of Q_{i+1} and the first $n-i$ positions are vertices of Γ_{n-i}. The extended Fibonacci cubes Γ_n^1, for $n = 1, 2, 3, 4$, are illustrated in Figure 1.1 below.

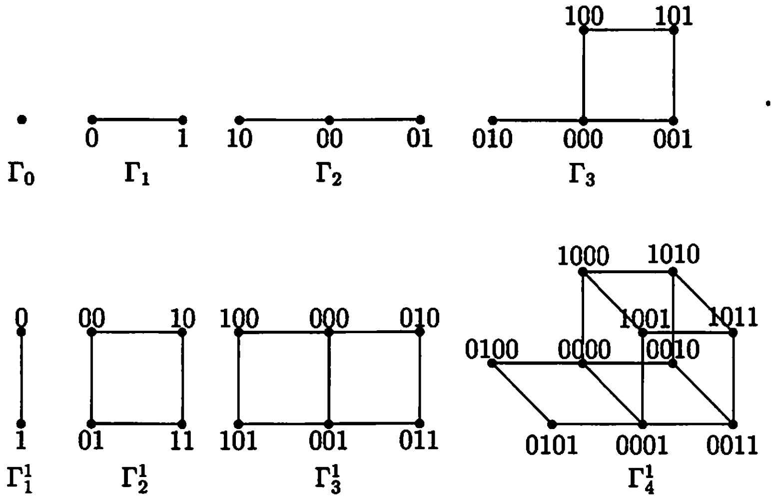

Figure 1.1

Let G_n^i denote $|V(\Gamma_n^i)|$, for $n \geq i \geq 0$, where we define $\Gamma_n^0 = \Gamma_n$, $V_n^0 = V_n$ and $G_n^0 = G_n$. Then for fixed $i \geq 0$, the sequence $\{G_n^i\}_{n=i}^{\infty}$ satisfies the Fibonacci recurrence relation

$$G_{n+2}^i = G_{n+1}^i + G_n^i \tag{3}$$

for $n \geq i$, with initial conditions $G_i^i = 2^i$, $G_{i+1}^i = 2^{i+1}$. For other definitions and notation, the reader is referred to [2].

Wu [10] has shown that the sequence $\{G_n\}_{n=2}^{\infty}$ and $\{G_n^1\}_{n=3}^{\infty}$ are disjoint, so that the graphs Γ_n and Γ_n^1 together considerably widen the choice for the number of vertices in an interconnection structure. He conjectured that the sequences $\{G_n^i\}_{n=i+2}^{\infty}$ are disjoint for all $i \geq 0$, and in Section 3 of this study we prove this conjecture is correct.

The structural properties of Fibonacci cubes have been studied in [1] and [7]. Although they have many of the useful properties of Q_n for modelling an interconnection structure, including the Hamming path property, it is shown in [3] and [6] that less than one third of Fibonacci cubes are Hamiltonian. However, all extended Fibonacci cubes have been shown by Wu [10] to be Hamiltonian. Other structural properties of these graphs are discussed in [9] and their applications in [8].

2 Structure of the extended Fibonacci cubes

In this section, we establish two decompositions of the extended Fibonacci cube Γ_n^i. The first of these gives a recursive construction for the sequence of extended Fibonacci cubes, starting from the Fibonacci cube. The construction of Γ_n^1 from Γ_{n-1} by this method, for $n = 1, 2, 3, 4$, is illustrated in Figure 1.1.

Theorem 2.1 *Let i, n be positive integers with $n \geq i$. Then*

$$\Gamma_n^i = \Gamma_{n-1}^{i-1} \times K_2,$$

where $\Gamma_n^0 = \Gamma_n$.

Proof. We show that for all $n \geq i \geq 1$, we have

$$V_n^i = V_{n-1}^{i-1}0 + V_{n-1}^{i-1}1.$$

The case when $n = i = 1$ is trivial, so we assume $n \geq 2$ and proceed by induction. The result clearly holds when $n = i$ and $n = i + 1$, since $V_k^k = B_k$ and $V_{k+1}^k = B_{k+1}$ for all $k \geq 1$, by definition. From the induction hypothesis and (2), we have

$$\begin{aligned} V_n^i &= 0V_{n-1}^i + 10V_{n-2}^i \\ &= 0\{V_{n-2}^{i-1}0 + V_{n-2}^{i-1}1\} + 10\{V_{n-3}^{i-1}0 + V_{n-3}^{i-1}1\} \\ &= \{0V_{n-2}^{i-1} + 10V_{n-3}^{i-1}\}0 + \{0V_{n-2}^{i-1} + 10V_{n-3}^{i-1}\}1 \\ &= V_{n-1}^{i-1}0 + V_{n-1}^{i-1}1, \end{aligned}$$

and the result follows. ■

Corollary 2.2 *Let i, n be integers such that $n \geq i \geq j \geq 1$. Then*

(i) $\Gamma_n^i = \Gamma_{n-j}^{i-j} \times Q_j$;

(ii) $G_n^i = 2^i G_{n-i} = 2^i F_{n+2-i}$. ■

Lemma 2.3 *Let i, n be integers such that $i \geq 0$ and $n \geq i + 3$. Then*

$$V_n^i = V_{n-i-2}0V_{i+1}^i + V_{n-i-3}010V_i^i.$$

Proof. Using the recurrence relation (2), we have

$$\begin{aligned} V_{i+3}^i &= 0V_{i+2}^i + 10V_{i+1}^i \\ &= \{00 + 10\}V_{i+1}^i + 010V_i^i \\ &= V_1 0V_{i+1}^i + V_0 010V_i^i, \end{aligned}$$

where $V_0 = \{\emptyset\}$. Using this expression for V_{i+3}^i, a similar analysis gives

$$\begin{aligned} V_{i+4}^i &= \{00+01+10\}0V_{i+1}^i + \{0+1\}010V_i^i \\ &= V_2 0 V_{i+1}^i + V_1 010 V_i^i. \end{aligned}$$

Hence the result is true when $n = i+3$ and $i+4$. So we assume $n \geq i+5$, and again proceed by induction. Thus

$$\begin{aligned} V_n^i &= 0V_{n-1}^i + 10V_{n-2}^i \\ &= \{0V_{n-i-3} + 10V_{n-i-4}\}0V_{i+1}^i + \{0V_{n-i-4} + 10V_{n-i-5}\}010V_i^i \\ &= V_{n-i-2}0V_{i+1}^i + V_{n-i-3}010V_i^i, \end{aligned}$$

and the result follows. ■

Theorem 2.4 *Let i, r, n be integers such that $i \geq 0$, $n \geq i+3$ and $2 \leq r \leq n-2$. Then*

$$V_n^i = V_{r-1}0V_{n-r}^i + V_{r-2}010V_{n-r-1}^i.$$

Proof. Let $v \in V_n^i$. It follows from Lemma 2.3 that given r such that $2 \leq r \leq n-2$, we can write $v = xy$, where x is a Fibonacci string of length $r-1$ and $y \in V_{n-r+1}^i$. Two cases arise: if the first bit of y is 0, then $y \in 0V_{n-r}^i$; otherwise, the first bit of y is 1 and then the last bit of x and the second bit of y are both 0. In this case, $v \in V_{r-2}010V_{n-r-1}^i$. Hence $V_n^i \subseteq V_{r-1}0V_{n-r}^i + V_{r-2}010V_{n-r-1}^i$. It is immediate from Lemma 2.3 that $V_{r-1}0V_{n-r}^i + V_{r-2}010V_{n-r-1}^i \subseteq V_n^i$, establishing the result. ■

Lemma 2.3 and Theorem 2.4 yield relations between the numbers G_n^i and G_n. In particular, from Lemma 2.3, we have

$$\begin{aligned} G_n^i &= G_{n-i-2} \cdot G_{i+1}^i + G_{n-i-3} \cdot G_i^i \\ &= 2^i(2G_{n-i-2} + G_{n-i-3}), \end{aligned}$$

using Corollary 2.2. Putting $n - i = s$, this reduces to the relation

$$G_s = 2G_{s-2} + G_{s-3},$$

which is of course an immediate deduction from the basic Fibonacci recurrence relation. Similarly, Theorem 2.4 implies

$$G_n^i = G_{r-1} \cdot G_{n-r}^i + G_{r-2} \cdot G_{n-r-1}^i,$$

which, on using Corollary 2.2 and putting $n - i = s$ as before, reduces to the following well-known relation for generalised Fibonacci numbers:

$$G_s = G_{r-1} \cdot G_{s-r} + G_{r-2} \cdot G_{s-r-1}.$$

This process can of course be reversed, so that multiplying through a relation for generalised Fibonacci numbers by 2^i, for $i \in \mathbf{Z}^+$, and applying Corollary 2.2(i) yields a relation between the numbers G_n^i and G_n.

3 Wu's conjecture

Wu [10] has conjectured that the sequences $\{G_n^i\}_{n=i+2}^{\infty}$ are disjoint for all $i \geq 0$, and in this section we prove this is true.

In view of the relation $G_n^i = 2^i G_{n-i} = 2^i F_{n+2-i}$, for $n \geq i \geq 1$, established in Corollary 2.2(ii), it suffices to show that the only instance of a relation of the form

$$F_N = 2^k F_n$$

with $N \geq n \geq 3$, is when $N = 6$, $n = 3$ and $k = 2$. The proof uses some well-known properties of the Fibonacci sequence and a relation connecting the Fibonacci and Lucas sequences, which we give below as Remark 3.1 and Remark 3.4 respectively. Proofs of can be found in [4], for example.

Recall that the Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

with initial terms $F_0 = 0$ and $F_1 = 1$, has the following properties.

Remark 3.1 *For all n,*

(i) $(F_n, F_{n+1}) = 1$;

(ii) *for all d, n, with $d \geq 3$, $F_d \mid F_n$, if and only if $d \mid n$;*

(iii) *if $(m, n) = d$, then $(F_m, F_n) = F_d$.*

The following Lemma can be deduced from Remark 3.1, using the facts that $F_3 = 2$ and $F_6 = 8$, and the Fibonacci recurrence relation. However, the shortest proof is to note that $\{F_n\}$ (mod 4) is the cyclic sequence

$$0, 1, 1, 2, 3, 1, 0, 1, \ldots$$

with period 6.

Lemma 3.2 *Let $n \in \mathbf{Z}^+$. Then*

(i) $F_{3n\pm1} \equiv 1 \pmod 2$;

(ii) $F_{6n} \equiv 0 \pmod 4$; $\quad F_{6n+3} \equiv 2 \pmod 4$.

Recall that the Lucas sequence $\{L_n\}$ is the generalised Fibonacci sequence

$$2, 1, 3, 4, 7, 11, 18, 29, \ldots$$

with intitial terms $L_0 = 2$ and $L_1 = 1$.

Lemma 3.3 *Let $n \in \mathbf{Z}^+$. Then*

(i) $L_n \not\equiv 0 \pmod 8$;

(ii) $L_{3n\pm1} \equiv 1 \pmod 2$;

(iii) $L_{6n} \equiv 2 \pmod 8$; $L_{6n+3} \equiv 4 \pmod 8$.

Proof. We note that $\{L_n\}$ (mod 8) is the cyclic sequence

$$2, 1, 3, 4, 7, 3, 2, 5, 7, 4, 3, 7, 2, 1, \ldots$$

with period 12. ■

Remark 3.4 *For all n,* $F_{2n} = F_n \cdot L_n$.

Lemma 3.5 *Let m be an odd integer and let t be a non-negative integer. Then*

(i) $F_{2^t(3m\pm1)}$ *is odd;*

(ii) $F_{2^t\cdot 3m} = \begin{cases} 2h & \text{if } t=0 \\ 2^{t+2}h & \text{otherwise} \end{cases}$,

where h is an odd integer.

Proof. (i) This result follows from Lemma 3.2(i).
(ii) When $t = 0$, the result follows from Lemma 3.2(ii). When $t \geq 1$, repeated application of Remark 3.4 gives

$$F_{2^t\cdot 3m} = F_{2^{t-1}\cdot 3m} \cdot L_{2^{t-1}\cdot 3m} = F_{3m} \cdot L_{3m} \cdot L_{2\cdot 3m} \cdot L_{2^2\cdot 3m} \cdots L_{2^{t-1}\cdot 3m}.$$

For $1 \leq i \leq t-1$, each term of this product of the form $L_{2^i\cdot 3m}$ is congruent to $2 \pmod 8$ and $L_{3m} \equiv 4 \pmod 8$, by Lemma 3.3; $F_{3m} \equiv 2 \pmod 4$, by Lemma 3.2. Hence $F_{2^t\cdot 3m} = 2^{t+2}h$, for some odd integer h. ■

Theorem 3.6 *Given integers n, N, with $3 \leq n \leq N$, there exists a positive integer k such that*

$$F_N = 2^k F_n$$

only when $N = 6$, $n = 3$ and $k = 2$.

Proof. Since $F_n \mid F_N$ with $n \geq 3$, we have $n \mid N$, by Remark 3.1(ii). In particular,

$$n \leq N/2. \tag{4}$$

Since F_N is even , we have $3 \mid N$, by Lemma 3.2. Hence $N = 3 \cdot 2^s m$, for some m, s where m is odd and $s \geq 0$. Thus by Lemma 3.5, $F_N = 2^{s+2}h$, for some odd integer h. Then $F_n = 2^{s+2-k}h$, where $s + 2 \geq k$.

Case (a) Assume $k \geq 3$. This implies $s \geq 1$ and hence $6 \mid N$ by Lemma 3.2. Then

$$F_N = F_{N/2} \cdot L_{N/2} = 2^k F_n \leq 2^k F_{N/2},$$

from equation (4). Hence

$$L_{N/2} \leq 2^k. \tag{5}$$

However, $N/2 \geq 3 \cdot 2^{s-1} \geq 3 \cdot 2^{k-3}$. It is easily verified that $L_r > \frac{8r}{3}$ when $r \geq 6$, and hence $L_{N/2} \geq L_{3 \cdot 2^{k-3}} > 2^k$, when $k \geq 4$. But this contradicts equation (5) and hence there is no solution when $k \geq 4$.

We may thus assume $k = 3$. Then from equation (5), $L_{N/2} \leq 8$ and hence $N \leq 8$. Since $6 \mid N$, this gives $N = 6$. But $n \geq 3$ and hence there is no solution in this case either.

Case (b) Assume $k \leq 2$. From equation (4), we have $F_N \geq F_{2n}$, and hence $F_N \geq F_n \cdot L_n$. Thus $L_n \leq 2^k$.

When $k = 2$, this gives $L_n \leq 4$ and hence $n \leq 3$. Thus the only possibility is $n = 3$, yielding the solution $F_6 = 2^2 \cdot F_3$.

When $k = 1$, we have $L_n \leq 2$, giving $n = 1$ and hence there is no solution in this case. ■

Corollary 3.7 *The sequences* $\{G_n^i\}_{n=i+2}^{\infty}$ *are disjoint for all* $i \geq 0$.

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