Generalized Fibonacci polynomial of graph

by

Iwona Włoch
Department of Mathematics
Technical University of Rzeszów
ul. W.Pola 2, 35-959 Rzeszów, Poland
e-mail:iwloch@ewa.prz.rzeszow.pl

ABSTRACT: In this paper we define the concept of generalized Fibonacci polynomial of a graph G which gives the total number of all k - stable sets in generalized lexicographical products of graphs. This concept generalize the Fibonacci polynomial of graph introduced by G.Hopkins and W.Staton in [3].

Keywords: stable set, Fibonacci numbers, Fibonacci polynomial
1991 MSC:05C20

1.Introduction

By a graph G we mean a finite, undirected, connected graph without loops and multiple edges. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The length of the shortest path joining vertices x and y in G we will denote by $d_G(x, y)$. Recall that the length of the path is the number of edges in it. By P_n and C_n, for $n \geq 2$ we mean graphs with the vertex sets $V(P_n) = V(C_n) = \{t_1, ..., t_n\}$ and the edge sets $E(P_n) = \{\{t_i, t_{i+1}\}; i = 1, ..., n-1\}$ and $E(C_n) = E(P_n) \cup \{t_n, t_1\}$, respectively. In addition $C_1 = P_1$, where P_1 is a graph consists the only one vertex. Let G be a graph on $V(G) = \{t_1, .., t_n\}, n \geq 2$ and $H_i, i = 1, ..., n$ are graphs on $V(H_i) = V = \{y_1, ..., y_x\}, x \geq 1$. By generalized lexicographical product of G and $H_1, ..., H_n$, $n \geq 2$, we mean a graph $G[H_1, ..., H_n]$ such that $V(G[H_1, ..., H_n]) = V(G) \times V$ and $E(G[H_1, ..., H_n]) = \{\{(t_i, y_p), (t_j, y_q)\}; (t_i = t_j \text{ and } \{y_p, y_q\} \in E(H_i)) \text{ or } \{t_i, t_j\} \in E(G)\}$. If $H_i = H, i = 1, ..., n$ then $G[H_1, ..., H_n] = G[H]$ where $G[H]$ is a lexicographical product of two graphs.

Let k be a fixed integer, $k \geq 2$. A subset $S \subseteq V(G)$ is said to be a k - stable set of G if for each two distinct vertices $x, y \in S$, $d_G(x, y) \geq k$. In addition a subset containing only one vertex and the empty set also is meant as a k - stable set of G. Note that for $k = 2$ the definition reduces to the definition of a stable set of the graph G. By $F_k(G)$ we denote the number of all k - stable sets of G and we put $F_2(G) = F(G)$. The number $F(G)$ also is named as Fibonacci number of graph G. Moreover by $f_G(k, n, p)$ we denote the number off all p - elements, $p \geq 0$, k - stable sets of a graph G on n vertices and also we put $f_G(2, n, p) = f_G(n, p)$. Consequently

$F_k(G) = \sum_{p\geq 0} f_G(k,n,p)$.

For $n \geq 0$ we define the set X as follows: if $n = 0$ then $X = \emptyset$, if $n \geq 1$ then $X = \{1, ..., n\}$. Let $Y \subseteq X$ where Y does not contain two consecutive integers. By $f(n,p)$ we denote the number of all subsets Y having exactly p elements and

(1) $\qquad f(n,p) = \binom{n-p+1}{p}$

The number $F_n = \sum_{p\geq 0} f(n,p)$ is called the Fibonacci number, see [1]. In a graph interpretation, given in [5], the number F_n, for $n \geq 0$ is equal to the number of all stable sets $S \subseteq V(P_n)$, i.e.

(2) $\qquad F_n = F(P_n)$, and also

(3) $\qquad f(n,p) = f_{P_n}(n,p)$.

For graph interpretation of the number F_0 we introduce the empty graph P_0 having a unique stable set $X = \emptyset$.

Let $Y^* \subseteq X$ such that Y^* does not contain either two consecutive integers or both 1 and n simultaneously. The number of all subsets Y^* having exactly p elements is denoted by $f^*(n,p)$.

Moreover, for $n \geq 3$ it holds, see [1].

(4) $\qquad f^*(n,p) = f(n-3,p-1) + f(n-1,p) = \frac{n}{p}\binom{n-p-1}{p-1}$.

Of course $f^*(n,p) = f(n,p)$, for $n = 0,1,2$. The number $F_n^* = \sum_{p\geq 0} f^*(n,p)$ is called the Lucas number, see [1], and in the graph interpretation, given in [5], we have

(5) $\qquad F_n^* = F(C_n)$ and also

(6) $\qquad f^*(n,p) = f_{C_n}(n,p)$.

In [4] it was given the generalized Fibonacci and Lucas number. Let $k \geq 2$ be an integer and let the set X is defined as above. Let $Y \subseteq X$ such that $i, j \in Y$ if and only if $|i-j| < k$. By $f(k,n,p)$ we denote the number of all such subsets Y having exactly p elements and further let $F(k,n) = \sum_{p\geq 0} f(k,n,p)$. The number $F(k,n)$ we called the generalized Fibonacci number. It easy to see that for $k = 2$ we obtain $f(2,n,p) = f(n,p)$ and $F(2,n) = F_n$. It has been proved:

Theorem 1.[4].*Let k, n, p be integers, $k \geq 2$, $n \geq 0$, $0 \leq p \leq n$. Then*

$$f(k,n,p) = \binom{n-p-(p-1)(k-2)+1}{p}.$$

Remark 1. $f(2,n,p) = \binom{n-p+1}{p} = f(n,p)$.

Let $Y^* \subseteq X$ such that $i, j \in Y^*$ if and only if $k \leq |i-j| \leq n-k$. Further we denote by $f^*(k,n,p)$ the number of all subsets Y^* on p elements and we put $F^*(k,n) = \sum_{p\geq 0} f^*(k,n,p)$. The number $F^*(k,n)$ we called the generalized Lucas number. It easy to see that for $k = 2$ we obtain $f^*(2,n,p) = f^*(n,p)$ and $F^*(2,n) = F_n^*$. It has been proved:

Theorem 2.[4]. *Let $k \geq 2$ and $0 \leq p \leq n$ be integers. If $n \geq 2k$ and $p \geq 2$ then we have*

$$f^*(k,n,p) = (k-1)f(k,n-(2k-1),p-1) + f(k,n-(k-1),p).$$

If $n \leq 2k$ then $f^*(k,n,1) = n$, $f^*(k,n,0) = 1$.

Using the Theorems 1 and 2 we can write for $n \geq 2k$ and $p \geq 2$ that

$$(7) \qquad f^*(k,n,p) = \frac{n}{p}\binom{n-p(k-1)-1}{p-1}.$$

For others classes of graphs the total number of stable sets and k - stable sets were determined, see [4],[5],[6].

In [3] G.Hopkins and W.Staton introduced the concept of the Fibonacci polynomial of a graph which gives the total number of stable sets of the lexicographical product of two graphs. They define the Fibonacci polynomial $F_G(x)$ of the graph G by $F_G(x) = F(G[K_x])$, for integer x, where K_x is a complete graph on x vertices.

It has been proved:

Theorem 3.[3]. *For an arbitrary graph G on n vertices $F_G(x) = \sum_{p\geq 0} f_G(n,p)x^p$.*

Consequently in case G is a graph P_n by (1),(3) and Theorem 3 they give

$$(8)\ F_{P_n}(x) = \sum_{p\geq 0} f_{P_n}(n,p)x^p = \sum_{p\geq 0}\binom{n-p+1}{p}x^p.$$

In case G is a graph C_n by (6) and Theorem 3 they give

$$(9)\ F_{C_n}(x) = \sum_{p\geq 0} f_{C_n}(n,p)x^p = 1 + \sum_{p\geq 1}\frac{n}{p}\binom{n-p-1}{p-1}x^p.$$

Evidently the degree of $F_G(x)$ is the stability number of G. Moreover for an establish integer x, $x \geq 1$

$$(10) \qquad F(P_n[K_x]) = \sum_{p\geq 0}\binom{n-p+1}{p}x^p \text{ and}$$

$$(11) \qquad F(C_n[K_x]) = 1 + \sum_{p\geq 1}\frac{n}{p}\binom{n-p-1}{p-1}x^p.$$

2. Generalizations

In [7] it was proved the following theorem:

Theorem 4.[7]. *Let $(t_i, y_p), (t_j, y_q) \in V(G[H_1, ..., H_n])$. Then*

$$d_{G[H_1,...,H_n]}((t_i,y_p),(t_j,y_q)) =$$

$$\begin{cases} d_G(t_i,t_j) & \text{for} \quad i \neq j, \\ 1 & \text{for} \quad i = j \quad \text{and} \quad d_{H_i}(y_p,y_q) = 1, \\ 2 & \text{otherwise.} \end{cases}$$

This theorem gives that we propose the following generalizations of the Fibonacci polynomial:

For an arbitrary integers $k \geq 3$, $x \geq 1$ we define the generalized Fibonacci polynomial $F_G(k,x)$ of the graph G on n vertices, $n \geq 2$, by $F_G(k,x) = F_k(G[H_1, ..., H_n])$, where $H_1, ..., H_n$ is an arbitrary sequence of graphs on $|V(H_i)| = |V| = x$.

Theorem 5. *Let $k \geq 3$, $x \geq 1$ be integers. Then for an arbitrary graph G on n, $n \geq 2$ vertices, $F_G(k,x) = \sum\limits_{p\geq 0} f_G(k,n,p)x^p$.*

Proof: Let G be a given graph on n vertices. We shall show that if $k \geq 3$ then for an arbitrary sequence of $H_1, ..., H_n$ the generalized Fibonacci polynomial $F_G(k,x) = \sum\limits_{p\geq 0} f_G(k,n,p)x^p$. It sufficies to calculate the number $F_k(G[H_1, ..., H_n])$. From the definition of the graph $G[H_1, ..., H_n]$ and by Theorem 4 we deduce that to obtain a p - elements, $p \geq 0$, k - stable set of $G[H_1, ..., H_n]$ first we have to choose a p - elements k - stable set of the graph G. Of course we can do it on $f_G(k,n,p)$ ways. Next we have to choose one of the x vertices in each of the p choosen copies of H_i, $i = 1, ..., n$. Evidently, from Theorem 4 and by $k \geq 3$ we have that for an arbitrary graph H_i, $i = 1, ..., n$ only one vertex from its copy can be choosen to a k - stable set. Because every vertex of p - copies can be choose on x ways, so we have $f_G(k,n,p)x^p$ k - stable sets having exactly p elements in $G[H_1, ..., H_n]$. Hence $F_k(G[H_1, ..., H_n]) = \sum\limits_{p\geq 0} f_G(k,n,p)x^p$.

Thus the theorem is proved.

Note that to study of the generalized Fibonacci polynomial it suffices to study the coefficients of $F_G(k,x)$. For example the constant coefficient of $F_G(k,x)$ is 1, the linear is n. The degree of $F_G(k,x)$ is the cardinality of the largest k - stable set of G.

Hence if $\delta(G) \leq k-1$ then $F_G(k,x) = 1+nx$, where $\delta(G) = \max\limits_{x,y\in V(G)} d_G(x,y)$.

Using the Theorems 1,5 and by (7) we obtain

Theorem 6. *Let $k \geq 3$, $x \geq 1$, $n \geq 2$. Then*

$$F_{P_n}(k,x) = \sum_{p\geq 0} \binom{n-p-(p-1)(k-2)+1}{p} x^p.$$

Theorem 7. *Let $k \geq 3$, $x \geq 1$, $n \geq 2$. Then*

$$F_{C_n}(k,x) = 1 + nx + \sum_{p\geq 2} \frac{n}{p}\binom{n-p(k-1)-1}{p-1} x^p.$$

Using the definition of $F_G(k,x)$ and by the above Theorems we have:

Corollary 1. *Let $k \geq 3$, $x \geq 1$, $n \geq 2$. Then for an arbitrary sequence of graphs $H_1, ..., H_n$ we have:*

$$F_k(P_n[H_1, ..., H_n]) = \sum_{p\geq 0} \binom{n-p-(p-1)(k-2)+1}{p} x^p \text{ and}$$

$$F_k(C_n[H_1, ..., H_n]) = 1 + nx + \sum_{p\geq 2} \frac{n}{p}\binom{n-p(k-1)-1}{p-1} x^p.$$

3. The total number of k-stable sets of $P_n[H_1, ..., H_n]$ and $C_n[H_1, ..., H_n]$.

Now we present numbers $F_k(P_n[H_1, ..., H_n])$ and $F_k(C_n[H_1, ..., H_n])$ by the linear recurrence relations.

Theorem 8. *Let $k \geq 3$, $n \geq 2$, $x \geq 1$ be integers. Then for an arbitrary sequence of graphs $H_1, ..., H_n$ on $|V(H_i)| = |V| = x$, $i = 1, ..., n$ the number $F_k(P_n[H_1, ..., H_n])$ satisfy the following reccurence relations:*
$F_k(P_n[H_1, ..., H_n]) = F_k(P_{n-1}[H_1, ..., H_{n-1}]) + xF_k(P_{n-k}[H_1, ..., H_{n-k}])$, *for* $n \geq k+2$
with the initial conditions:
$F_k(P_n[H_1, ..., H_n]) = nx + 1$, $n = 2, ..., k$ and
$F_k(P_{k+1}[H_1, ..., H_{k+1}]) = x^2 + (k+1)x + 1$.

Proof: Let k, n, x be as it was mentioned in the statement of the theorem. Let $n = 2, ..., k$. Then every vertex of $V(P_n[H_1, ..., H_n])$ and the empty set is a k - stable set of the graph $P_n[H_1, ..., H_n]$. Moreover there no exist a k - stable set of $P_n[H_1, ..., H_n]$ having at least two elements. This implies that $F_k(P_n[H_1, ..., H_n]) = nx + 1$.
If $n = k+1$ then in this case we have also k- stable sets having exactly two elements. Every two elements k - stable sets has the form $\{(t_1, y_j), (t_{k+1}, y_q)\}$, where $1 \leq j \leq x$ and $1 \leq q \leq x$. So we have x^2 such subsets and consequently $F_k(P_{k+1}[H_1, ..., H_n]) = x^2 + (k+1)x + 1$.
Now suppose that $n \geq k+2$ and let S be an arbitrary k - stable set of $P_n[H_1, ..., H_n]$. Because at most one vertex from each copy of H_i, $i = 1, ..., n$ can belong to the k - stable set of $P_n[H_1, ..., H_n]$, by Theorem 4 and $k \geq 3$, so two case can occur now:
Case 1. for each $j = 1, ..., x$ holds $(t_n, y_j) \notin S$.
If $\mathcal{S}_1$ is the family of all such sets S, then its cardinality $|\mathcal{S}_1|$ is equal to the total number of k - stable sets of the graph $P_n[H_1, ..., H_n] - \bigcup_{j=1}^{x} (t_n, y_j)$, $j = 1, ..., x$ isomorphic to $P_{n-1}[H_1, ..., H_{n-1}]$. In other words we obtain $|\mathcal{S}_1| = F_k(P_{n-1}[H_1, ..., H_{n-1}])$.
Case 2. there exists $1 \leq j \leq x$ such that $(t_n, y_j) \in S$.
Then by the definition of the graph $P_n[H_1, ..., H_n]$ we have $(t_{n-i}, y_j) \notin S$, for each $i = 1, ..., k-1$ and $j = 1, ..., x$. This implies that $S = S^* \cup \{(t_n, y_j)\}$, where S^* is an arbitrary k - stable set of the graph $P_n[H_1, ..., H_n] - \bigcup_{i=0}^{k-1} \bigcup_{j=1}^{x} (t_{n-i}, y_j)$ isomorphic to $P_{n-k}[H_1, ..., H_{n-k}]$. Consequently we have $F_k(P_{n-k}[H_1, .., H_{n-k}])$ sets S^*. Moreover because vertex (t_n, y_j) can be taken among of x vertices, so if we denote by $\mathcal{S}_2$ the family of all k - stable sets such that the condition in Case 2 is fulfilled, then $|\mathcal{S}_2| =$

$xF_k(P_{n-k}[H_1,...,H_{n-k}])$. Consequently for the number $F_k(P_n[H_1,...,H_n])$ we have the linear recurrence $F_k(P_n[H_1,...,H_n]) = F_k(P_{n-1}[H_1,...,H_{n-1}])+ xF_k(P_{n-k}[H_1,...,H_{n-k}])$, which completes the proof.

Theorem 9. *Let $k \geq 3$, $n \geq 2$, $x \geq 1$ be an integers. Then for an arbitrary sequence of graphs $H_1,...,H_n$ on $|V(H_i)| = |V| = x$, $i = 1,...,n$ the number $F_k(C_n[H_1,...,H_n])$ satisfy the following reccurence relations:*
$F_k(C_n[H_1,...,H_n]) = x(k-1)F_k(P_{n-(2k-1)}[H_1,...,H_{n-(2k-1)}])+$
$F_k(P_{n-(k-1)}[H_1,...,H_{n-(k-1)}])$, for $n \geq 2k+1$
with the initial conditions
$F_k(C_n[H_1,...,H_n]) = nx+1$, $n = 2,...,2k-1$ and
$F_k(C_{2k}[H_1,...,H_{2k}]) = kx^2+2kx+1$.

Proof: Let k, n, x be as it was mentioned in the statement of the theorem and let $H_1,...,H_n$ be an arbitrary sequence of graphs. Suppose that $n = 2,...,2k-1$. Then every vertex of $V(C_n[H_1,...,H_n])$ and the empty set is a k - stable set of the graph $C_n[H_1,...,H_n]$. Moreover there no exist a k - stable set of $C_n[H_1,...,H_n]$ having at least two elements. This implies that $F_k(C_n[H_1,...,H_n]) = nx+1$ in this case.
If $n = 2k$ then every vertex of $V(C_{2k}[H_1,...,H_{2k}])$, the empty set and also sets of the form $\{(t_i,y_j),(t_{i+k},y_q)\}$, where $1 \leq j \leq x$, $1 \leq q \leq x$ and $i = 1,...,k$ is a k - stable sets of $C_{2k}[H_1,...,H_{2k}]$. Evidently we have that $F_k(P_{2k}[H_1,...,H_{2k}]) = kx^2+2kx+1$.
Now suppose that $n \geq 2k+1$ and let S be an arbitrary k - stable set of $C_n[H_1,...,H_n]$. Because at most one vertex from each copy of H_i, $i = 1,...,n$ can belong to the k - stable set of $C_n[H_1,...,H_n]$, by Theorem 4 and $k \geq 3$, so two case can occur now:
Case 1. for each $i = 1,...,k-1$ and $j = 1,...,x$ holds $(t_i,y_j) \notin S$.
If $\mathcal{S}_1$ is the family of all such sets S, then the cardinality $|\mathcal{S}_1|$ is equal to the total number of k - stable sets of the graph $C_n[H_1,...,H_n] - \bigcup_{j=1}^{x}\bigcup_{i=1}^{k-1}(t_i,y_j)$ isomorphic to $P_{n-(k-1)}[H_1,...,H_{n-(k-1)}]$. In other words using Theorem 8 we obtain that $|\mathcal{S}_1| = F_k(P_{n-(k-1)}[H_1,...,H_{n-(k-1)}])$.
Case 2. there exists $1 \leq i \leq k-1$ and $1 \leq j \leq x$ such that $(t_i,y_j) \in S$.
Then by the definition of the graph $C_n[H_1,...,H_n]$ we have $(t_p,y_q) \notin S$, where $p = 1,...,i-1,i+1,...,i+k-1$ and $p = n-k+i+1,...,n$ and $q = 1,...,x$. This means that $S = S^* \cup \{(t_i,y_j)\}$, where S^* is an arbitrary k - stable set of the graph $C_n[H_1,...,H_n] - (\bigcup_{j=1}^{x}(\bigcup_{p=1}^{i+k-1}(t_p,y_j) \cup \bigcup_{r=0}^{k-i-1}(t_{n-r},y_j)))$ isomorphic to $P_{n-(2k-1)}[H_1,...,H_{n-(2k-1)}]$. Because vertex (t_i,y_j) can be taken among of $x(k-1)$ vertices, so if we denote by $\mathcal{S}_2$ the family of all k - stable sets such that the condition in Case 2 is fulfilled, then $|\mathcal{S}_2| =$

$x(k-1)F_k(P_{n-(2k-1)}[H_1, ..., H_{n-(2k-1)}])$. In a consequence, for the numbers $F_k(C_n[H_1, ..., H_n])$ we have the recurrence relation
$F_k(C_n[H_1, ..., H_n]) = F_k(P_{n-(k-1)}[H_1, ..., H_{n-(k-1)}])+$
$x(k-1)F_k(P_{n-(2k-1)}[H_1, ..., H_{n-(2k-1)}])$, which completes the proof.

References

[1]. B.Berge, *Principles of combinatorics*, Academic Press New York and London 1971.

[2]. F.Harrary, *Graph Theory*, Addison Wesley, Reading, Mass.1969.

[3]. G.Hopkins, W.Staton, *Some idenities arising from The Fibonacci numbers of certains graphs*, The Fibonacci Quarterly (1984), 225-228.

[4]. M.Kwaśnik, I.Włoch, *The total number of generalized stable sets and kernels of graphs*, Ars Combinatoria, 55(2000), 139-146.

[5]. H.Prodinger, R.F.Tichy, *Fibonacci numbers of graphs*, The Fibonacci Quarterly 20, (1982) 16-21.

[6]. B.E.Sagan, *A note on independent sets in trees*, SIAM J.Alg.Discrete Math. Vol 1, No 1, February (1988) 105-108.

[7]. A.Włoch, I.Włoch, *On (k,l)-kernels in generalized products of graphs*, Discrete Math. 164, (1996) 295-301.