On bipartite generalized Ramsey theory

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ABSTRACT. Given graphs G and H, an edge coloring of G is called an (H,q)-coloring if the edges of every copy of $H \subset G$ together receive at least q colors. Let $r(G,H,q)$ denote the minimum number of colors in a (H,q)-coloring of G. In [6] Erdős and Gyárfás studied $r(K_n, K_p, q)$ if p and q are fixed and n tends to infinity. They determined for every fixed p the smallest q for which $r(K_n, K_p, q)$ is linear in n and the smallest q for which $r(K_n, K_p, q)$ is quadratic in n. In [9] we studied what happens between the linear and quadratic orders of magnitude. In [2] Axenovich, Füredi and Mubayi generalized some of the results of [6] to $r(K_{n,n}, K_{p,p}, q)$. In this paper we adapt our results from [9] to the bipartite case, namely we study $r(K_{n,n}, K_{p,p}, q)$ between the linear and quadratic orders of magnitude. In particular we show that that we can have at most $\log p + 1$ values of q which give a linear $r(K_{n,n}, K_{p,p}, q)$.

1 Introduction

1.1 Notation and definitions

For basic graph concepts see the monograph of Bollobás [3]. $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. K_n is the complete graph on n vertices, and $K_{m,m}$ is the complete bipartite graph between two m-sets. In a forest F the set of leaves is denoted by $L(F)$. In this paper $\log n$ denotes the base 2 logarithm.

1.2 Generalized Ramsey theory

In the classical multicoloring Ramsey problem we are looking for the minimum number n such that every k-coloring of the edges of K_n yields a monochromatic K_p. For each n below this threshold, there is a k-coloring such that every K_p is colored with at least 2 colors. A far-reaching generalization of this concept leads to the following definition: given graphs G and H, and an integer $q \leq |E(H)|$, an (H,q)-coloring of G is a coloring of $E(G)$ in which the edges of every copy of $H \subset G$ together receive at least q colors. Let $r(G,H,q)$ denote the minimum number of colors in an (H,q)-coloring of G. Thus for example determining $r(K_n, K_p, 2)$ exactly is hopeless, since it is equivalent to determining the classical Ramsey numbers for multicolorings. The study of $r(G,H,q)$ has received significant attention lately (see [1], [2], [4], [6], [7], [8], [9]). It was first studied in this form by Elekes, Erdős and Füredi for the special case $r(K_n, K_p, q)$ (as described in Section 9 of [5]). Then Erdős and Gyárfás 15 years later returned to the problem in [6]. Among many other interesting results and problems, in [6] using the Local Lemma they gave the general upper bound

$$r(K_n, K_p, q) \leq c_{p,q} n^{\frac{p-2}{\binom{p}{2}-q+1}}. \tag{1}$$

Furthermore, they determined for every p the smallest q ($q_{lin} = \binom{p}{2} - p + 3$) for which $r(K_n, K_p, q)$ is linear in n and the smallest q ($q_{quad} = \binom{p}{2} - \lfloor \frac{p}{2} \rfloor + 2$) for which $r(K_n, K_p, q)$ is quadratic in n.

They raised the striking question if q_{lin} is the only q value which results in a linear $r(K_n, K_p, q)$. In the direction of this question, in [9] we studied the behavior of $r(K_n, K_p, q)$ between the linear and quadratic orders of magnitude, so for $q_{lin} \leq q \leq q_{quad}$. In particular we showed that we can have at most $\log p$ values of q which give a linear $r(K_n, K_p, q)$. The first interesting case is $p = 5$ for which $q_{lin} = 8$. What is the growth rate of $r(K_n, K_5, 9)$? In [1] it is shown by using a construction of Behrend for a set of integers with no 3-term arithmetic progressions that

$$\frac{1+\sqrt{5}}{2} n - 3 \leq r(K_n, K_5, 9) \leq 2n^{1+\frac{c}{\sqrt{\log n}}}.$$

See also [8] for a related article. Another interesting special case is $r(K_n, K_4, 3)$. In [7] (see also [4])) it is shown that

$$r(K_n, K_4, 3) \leq e^{o(\log n)}.$$

The general definition $r(G, H, q)$ given above is introduced in [2] by Axenovich, Füredi and Mubayi. Among other results, in [2] by generalizing the upper bound (1), they showed that if $|V(G)| = n$, $|V(H)| = v$, $|E(H)| = e$

and $1 \leq q \leq e$, then there is a constant $c = c(H,q)$ such that

$$r(G,H,q) \leq cn^{\frac{v-2}{e-q+1}}. \quad (2)$$

Furthermore, for the bipartite case $r(K_{n,n}, K_{p,p}, q)$, they determined for every p the smallest q ($q^b_{lin} = p^2 - 2p + 3$) for which $r(K_{n,n}, K_{p,p}, q)$ is linear in n, and the smallest q ($q^b_{quad} = p^2 - p + 2$) for which $r(K_{n,n}, K_{p,p}, q)$ is quadratic in n.

In this paper we adapt our results from [9] to the bipartite case and study the behavior of the function $r(K_{n,n}, K_{p,p}, q)$ between the linear and quadratic orders of magnitude, so for $q^b_{lin} \leq q \leq q^b_{quad}$. Again we show that we can have at most $\log p + 1$ values of q which give a linear $r(K_{n,n}, K_{p,p}, q)$.

In order to state our results, first we need some definitions. We define the following two strictly decreasing sequences a_i and b_j of positive integers with $a_0 = 2p$, $a_1 = p$. Roughly speaking $a_{i+1} = \lfloor \frac{a_i}{2} \rfloor$ but for every second odd a_i we have to add 1.

The two sequences are defined recursively. Assuming $a_0, a_1, \ldots, a_i$ are already defined, the sequence $b_1, b_2, \ldots, \ldots, b_{i'}$ is just the subsequence consisting of the odd a_i-s which are greater than 1. Then we define

$$a_{i+1} = \begin{cases} \lceil \frac{a_i}{2} \rceil & \text{if } a_i = b_j \text{ for an even } j \\ \lfloor \frac{a_i}{2} \rfloor & \text{otherwise} \end{cases}$$

Furthermore if a_{i+1} is odd and greater than 1, then $b_{i'+1} = a_{i+1}$.

So, for example, if $p = 2^k$, the sequence a_i, $1 \leq i$, is just all the powers of 2 from p to 1, while there are no b_j-s. As another example, if $p = 11$, then $a_0 = 22$, $a_1 = 11$, $a_2 = 5$, $a_3 = 3$, $a_4 = 1$, $b_1 = 11$, $b_2 = 5$ and $b_3 = 3$. Let l_p be the smallest integer for which $a_{l_p} = 1$.

We will need the following simple lemma.

Lemma 1. *For $0 \leq i \leq l_p$, we have*

$$a_i < \frac{p}{2^{i-1}} + 1. \quad (3)$$

The simple inductive proof is given in the next section. This lemma immediately gives the bound

$$l_p \leq \lceil \log p \rceil + 1. \quad (4)$$

Our main result is the following.

Theorem 1. *For positive integers p, $1 \leq k \leq l_p$, if $q \geq q^b_{lin} + a_k + k - 1$, then*

$$r(K_{n,n}, K_{p,p}, q) > \frac{1}{6p^2} n^{\frac{4^k}{4^k - 1}}.$$

Using Lemma 1, we immediately get the following.

Corollary 2. *For positive integers p, $1 \le k \le l_p$, if $q \ge q^b_{lin} + \frac{p}{2^{k-1}} + k$, then*

$$r(K_{n,n}, K_{p,p}, q) > \frac{1}{6p^2} n^{\frac{4^{l_p}}{4^{l_p}-1}}.$$

Another corollary of the lower bound in Theorem 1 ($k = l_p$ and we use (4)) is that we can have at most $\log p+1$ values with a linear $r(K_{n,n}, K_{p,p}, q)$.

Corollary 3. *If $q \ge q^b_{lin} + \log p + 1$, then*

$$r(K_{n,n}, K_{p,p}, q) > \frac{1}{6p^2} n^{4^{l_p} 4^{l_p}-1}.$$

However, it still remains an open problem whether q^b_{lin} is the only q value with a linear $r(K_{n,n}, K_{p,p}, q)$.

2 Preliminaries

Proof of Lemma 1: To prove Lemma 1 we use induction on $i = 1, 2, \ldots, l_p$. It is true for $i = 1$. Assume that it is true for i and then for $i + 1$ from the definition of a_{i+1} we get

$$a_{i+1} \le \frac{a_i + 1}{2} < \frac{\frac{p}{2^{i-1}} + 1 + 1}{2} = \frac{p}{2^i} + 1,$$

and thus proving Lemma 1. □

Let l'_p be the number of b_j-s among $a_1, \ldots, a_{l_p} - 1$, and let $f_p(k)$ be 1 if the cardinality of $\{a_1, \ldots, a_{k-1}\} \cap \{b_1, \ldots, b_{l'_p}\}$ is an even number, and 0 otherwise. Examination of the cases of the definitions of a_k and b_k reveals that $f_p(k+1) = 2a_{k+1} - a_k + f_p(k)$ for all p and k.

Lemma 2. *For any $1 \le k \le l_p$*

$$\sum_{j=1}^{k} a_j = 2p - a_k - 1 + f_p(k).$$

Proof: The lemma is clearly true for $k = 1$, and we assume it is true for arbitrary k.

$$\sum_{j=1}^{k+1} a_j = 2p - a_{k-1} - 1 + (2a_{k+1} - a_k) + f_p(k) = 2p - a_{k+1} - 1 + f_p(k+1).$$

□

3 Proof of Theorem 1

Let $1 \le k \le l_p$ and $q \ge q^b_{lin} + a_k + k - 1$. Denote

$$h = h(n,k) = \frac{1}{6p^2} n^{\frac{4^k}{4^k - 1}}. \tag{5}$$

Assume indirectly that there is a (p,q)-coloring of $K_{n,n}$ with at most h colors. From this assumption we will get a contradiction.

Let us denote the two partite n-sets in $K_{n,n}$ by A and B, so $K_{n,n}$ is a complete bipartite graph between A and B. Consider a fixed (p,q)-coloring of $K_{n,n}$ with at most h colors. We will find a sequence of monochromatic matchings $M_1, M_2, \ldots, M_k$ in $K_{n,n}$.

For any vertices $U \subset V$ we say that color class C is *U-bounded* if every u in U has at most p edges of color C to vertices of $V \backslash U$. If C_i and C_j are two arbitrary color classes (where $C_i = C_j$ is allowed), then either C_i is A-bounded, or C_j is B-bounded, or both. Indeed, otherwise we immediately get a $K_{p,p}$ with fewer than q colors, a contradiction. Consider the color class C_1 with the most edges in it. Then C_1 contains at least n^2/h edges. By the above, C_1 is either A-bounded, or B-bounded, or both. Let us assume without loss of generality that C_1 is A-bounded.

Let us consider first the case when C_1 is B-bounded as well. Then we can easily choose a matching from C_1 of even size at least

$$\frac{n^2}{2ph}.$$

Indeed, we pick the first edge e_1 of the matching arbitrarily. From C_1 remove e_1 and the at most $2(p-1)$ other edges incident to it (C_1 is both A-bounded and B-bounded now). e_2 is an arbitrary edge from the remainder of C_1. We remove e_2 and the at most $2(p-1)$ other edges incident to it. We continue in this fashion until we have no edges left. If we have an odd number of edges in the matching, then remove an arbitrary edge to make the size even. In this case this is the matching M_1.

Assume now that C_1 is not B-bounded. Then every color class C (including C_1) is A-bounded by the above remark. First we construct a forest F of stars in C_1. Take a vertex $v_1 \in B$ that is non-isolated in C_1. v_1 with all of its neighbors in C_1 is the first star S_1 of F. From C_1 we remove S_1 and the at most $p|L(S_1)|$ edges that are incident to the $|L(S_1)|$ leaf vertices of S_1 (C_1 is A-bounded). Take a vertex $v_2 \in B$ that is non-isolated in the remainder of C_1. v_2 with its neighbors in the remainder of C_1 is S_2. We continue in this fashion until we only have isolated vertices left.

Say we constructed stars $S_1, \ldots, S_s$. Then by the construction

$$|\cup_{i=1}^{s} L(S_i)| \ge \frac{n^2}{ph}. \tag{6}$$

If $s > \frac{n^2}{2ph}$, then we can choose the matching M_1 by choosing one edge from each star (and again possibly removing one edge to make the size even).

Thus assume now that $s \leq \frac{n^2}{2ph}$. Consider the following complete bipartite graph (A', B'). For A' we take $\lceil \frac{n^2}{2ph} \rceil$ arbitrary vertices from $\cup_{i=1}^{s} L(S_i)$ (this is possible by (6)). For B' we take $\lceil \frac{n^2}{2ph} \rceil$ arbitrary vertices from B that are not roots for any star.

Then we have the following.

Fact 4. *Every color class C is both A'-bounded and B'-bounded.*

Indeed, C is clearly A'-bounded, since $B' \subset B$, and C is A-bounded. Furthermore, C is also B'-bounded, since otherwise if there is a vertex $v \in B'$ with p neighbors in $A' \subset \cup_{i=1}^{s} L(S_i)$, then there is a $K_{p,p}$ with fewer than q colors, a contradiction. This $K_{p,p}$ contains these p neighbors of v in $\cup_{i=1}^{s} L(S_i)$, the roots of the corresponding stars (except for one root if these stars form a matching), the vertex v and possibly some more vertices from B so that we have exactly p vertices from B.

Thus the above fact is true. Then we can proceed as above when C_1 was both A-bounded and B-bounded. We can choose a monochromatic matching of even size at least

$$\frac{\left(\frac{n^2}{2ph}\right)^2}{2ph} = \frac{n^4}{(2ph)^3}.$$

Thus in each case we can pick a monochromatic matching M_1 (denote its color class by C_1') of even size at least

$$|M_1| \geq \frac{n^4}{(2ph)^3}.$$

Say M_1 is a matching between $A_1 \subset A$ and $B_1 \subset B$. Halve the vertices of A_1 arbitrarily and denote one of the halves by A_1'. Denote by B_1' the set of vertices in B_1 which are not matched to vertices in A_1' by M_1. Consider the complete bipartite graph between A_1' and B_1'. Similarly as above we can choose a monochromatic matching M_2 in color class C_2' with partite sets A_2, B_2 of even size at least

$$\frac{\left(\frac{|M_1|}{2}\right)^4}{(2ph)^3}.$$

We continue in this fashion. Assume that $M_i = (A_i, B_i)$ is already defined. Denote an arbitrary half of the endvertices of M_i in A_i by A_i'. The set of endvertices of the edges of M_i in B_i which are not matched to vertices in A'_i is denoted by B'_i. Consider the complete bipartite graph between A'_i and B'_i. Similarly as above we can choose a monochromatic matching M_{i+1} in color class C'_{i+1} with partite sets A_{i+1}, B_{i+1} of even size at least

$$\frac{\left(\frac{|M_i|}{2}\right)^4}{(2ph)^3}.$$

Then by induction we have

$$|M_i| \geq \frac{n^{4^i}}{(6ph)^{4^i-1}}.$$

Indeed, this is true for $i = 1$

$$|M_1| \geq \frac{n^4}{(2ph)^3} > \frac{n^4}{(6ph)^3}.$$

For $i + 1$ we get

$$|M_{i+1}| \geq \frac{\left(\frac{|M_i|}{2}\right)^4}{(2ph)^3} \geq \frac{\left(\frac{n^{4^i}}{2(6ph)^{4^i-1}}\right)^4}{(2ph)^3} > \frac{n^{4^{i+1}}}{(6ph)^{4^{i+1}-1}}.$$

This and (5) implies that $|M_i| \geq p \geq a_i$, $1 \leq i \leq k$ and thus the matchings $M_1, M_2, \ldots, M_k$ can be chosen. Next using these matchings M_i we choose a $K_{p,p}$ such that it contains at most $q-1$ colors, a contradiction. For this purpose we will find another sequence of matchings M'_i such that $M'_i \subset M_i$, $|M'_i| = a_i$ for $1 \leq i \leq k$, $|\cup_{i=1}^k V(M'_i) \cap A| = p$ and $|\cup_{i=1}^k V(M'_i) \cap B| = p$.

M'_k is just a set of a_k arbitrary edges from M_k. Assume that $M'_k, \ldots, M'_{i+1}$ are already defined and now we define M'_i. We consider the $2a_{i+1}$ vertices in $V(M'_{i+1})$ and the edges of M_i incident to these vertices. We have four cases.

Case 1: If $2a_{i+1} = a_i$ then this is M'_i.

Case 2: If $2a_{i+1} = a_i + 1$, so $a_i = b_j$ for an even j, then we remove one of the edges from this set incident to a vertex in $V(M'_{i+1}) \cap A$ to get M'_i. Furthermore, we mark this vertex in $V(M'_{i+1}) \cap A$ which is not covered by M'_i. This marked vertex is going to be covered only by $M'_{i'}$ if $a_{i'} = b_{j-1}$.

Case 3: If $2a_{i+1} = a_i - 1$ and there is no marked vertex at the moment, then to get M'_i we add one arbitrary edge of M_i to these $2a_{i+1}$ edges.

Case 4: Finally, if $2a_{i+1} = a_i - 1$ and there is a marked vertex then to get M'_i we add to these $2a_{i+1}$ edges the edge of M_i incident to the marked vertex and we "unmark" this vertex.

We.continue in this fashion until $M'_k, \ldots, M'_1$ are defined. Set $A'' = \cup_{i=1}^k V(M'_i) \cap A$ and $B'' = \cup_{i=1}^k V(M'_i) \cap B$. We have $|A''| = |B''| = p$. Consider the complete bipartite graph $K_{p,p}$ between A'' and B''.

By the above construction this $K_{p,p}$ contains a_i edges from the matching M_i (and thus from color class C'_i) for $1 \le i \le k$.

Now since Lemma 2 implies

$$\sum_{j=1}^{k}(a_j - 1) \ge 2p - a_k - 1 - k,$$

thus the number of colors used in this $K_{p,p}$ is at most

$$p^2 - 2p + a_k + k + 1 \le q - 1,$$

a contradiction. This completes the proof of Theorem 1. □

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