Nonplanar Sequences of Iterated r-Jump Graphs

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ABSTRACT

For a graph G of size $m \geq 1$ and edge-induced subgraphs F and H of size r $(1 \leq r \leq m)$, the subgraph H is said to be obtained from F by an edge jump if there exist four distinct vertices u, v, w, and x in G such that $uv \in E(F)$, $wx \in E(G) - E(F)$, and $H = F - uv + wx$. The minimum number of edge jumps required to transform F into H is the jump distance from F to H. For a graph G of size $m \geq 1$ and an integer r with $1 \leq r \leq m$, the r-jump graph $J_r(G)$ is that graph whose vertices correspond to the edge-induced subgraphs of size r of G and where two vertices of $J_r(G)$ are adjacent if and only if the jump distance between the corresponding subgraphs is 1. For $k \geq 2$, the kth iterated jump graph $J^k(G)$ is defined as $J_r(J_r^{k-1}(G))$, where $J_r^1(G) = J_r(G)$. An infinite sequence $\{G_i\}$ of graphs is planar if every graph G_i is planar; while the sequence $\{G_i\}$ is nonplanar otherwise. It is shown that if $\{J_2^k(G)\}$ is a nonplanar sequence, then $J_2^k(G)$ is nonplanar for all $k \geq 3$ and there is only one graph G such that $J_2^2(G)$ is planar. Moreover, for each integer $r \geq 3$, if G is a connected graph of size at least $r+2$ for which $\{J_r^k(G)\}$ is a nonplanar sequence, then $J_r^k(G)$ is nonplanar for all $k \geq 3$.

Key Words: jump distance, r-jump graph, planar graph

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1 Introduction

For a graph G of size $m \geq 1$ and edge-induced subgraphs F and H of size r $(1 \leq r \leq m)$, we say that H is obtained from F by an *edge jump* (see [4]) if it can be produced from F by deleting an edge e and adding an edge f to F not adjacent to e (that is, e "jumps" to f). We say that F can be *j-transformed* into H if H can be obtained from F by a sequence of edge jumps. The minimum number of edge jumps required to j-transform F into H is called the *jump distance* $d_j(F, H)$ from F to H. The *r-jump graph* $J_r(G)$ of G is that graph whose vertices are the $\binom{m}{r}$ edge-induced subgraphs of size r in G, and where vertices F and H are adjacent in $J_r(G)$ if and only if $d_j(F, H) = 1$. The graph $J_1(G) = J(G)$ is also called simply the *jump graph* of G. Edge jumps, jump distance, and jump graphs were introduced and studied in [1, 4].

Since the vertices of the r-jump graph of a graph G are the edge-induced subgraphs of size r in G, each such subgraph is completely determined by its edge set. Thus if F is an edge-induced subgraph with $E(F) = \{e_1, e_2, \cdots, e_r\}$, we can represent the vertex F in $J_r(G)$ by $E(F)$ or, more simply, by $e_1e_2 \cdots e_r$. To illustrate these concepts, we show a graph G and its r-jump graphs $J_r(G)$ $(1 \leq r \leq 5)$ in Figure 1.

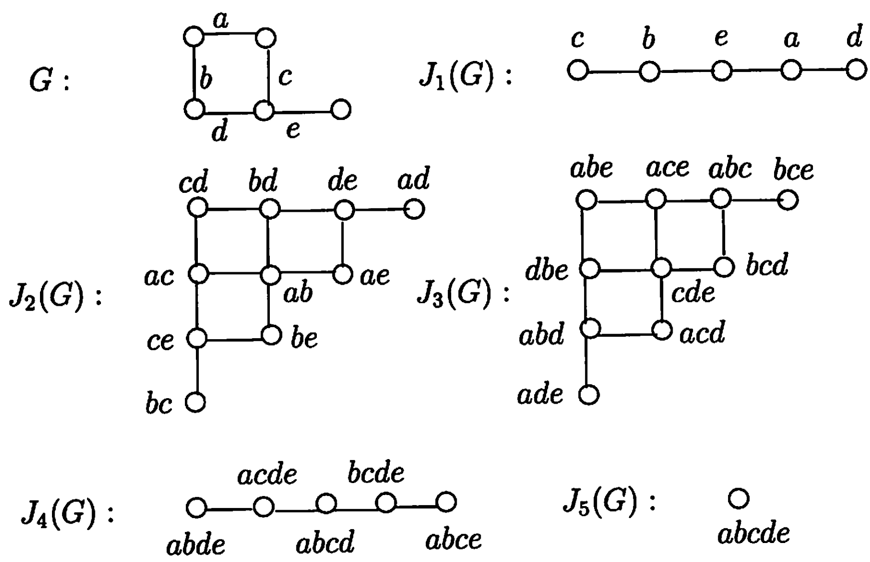

Figure 1: The r-jump $(1 \leq r \leq 5)$ graphs of a graph

For $k \geq 2$ and $r \geq 1$, the kth *iterated r-jump graph* $J_r^k(G)$ is defined as $J_r(J_r^{k-1}(G))$, where $J_r^1(G) = J_r(G)$. An infinite sequence $\{G_k\}$ of graphs is said to *converge* (see [2]) if there exists a graph G and a positive integer N such that G_k is isomorphic to G for all $k \geq N$. The graph G is then called the *limit graph* of the sequence $\{G_k\}$. An infinite sequence that does not converge is said to *diverge*. A finite sequence $\{G_k\}$ is said to *terminate*. An infinite sequence $\{G_k\}$ is called *planar* if G_k is planar for every positive integer k and *nonplanar* if G_k is nonplanar for some positive integer k.

The planarity of iterated jump graphs was studied in [6, 3, 7]. We write P_n and C_n for the path and cycle, respectively, of order n and $cor(K_3)$ for the corona of K_3, obtained by adding a pendant edge at each vertex of K_3. The following result was established in [2, 6, 3, 7].

Theorem A *Let r be a positive integer and let G be a connected graph such that $J_r^k(G)$ is defined for each positive integer k. Then $\{J_r^k(G)\}$ is planar if and only if*

(1) $r = 1$ *and* $G = C_5$ *or* $G = cor(K_3)$,

(2) $r = 2$ *and* $G = C_4$,

(3) $r = 4$ *and* $G = C_5$,

(4) $r = 5$ *and* $G = cor(K_3)$.

By Theorem A(1), if $G \neq C_5$ and $G \neq cor(K_3)$ such that $\{J^k(G)\}$ is infinite, then $\{J^k(G)\}$ is nonplanar. The nonplanar sequences $\{J^k(G)\}$ of connected graphs G were studied in [9], where it was shown that if the sequence $\{J^k(G)\}$ is nonplanar, then $J^k(G)$ is nonplanar for all $k \geq 4$. We state this result as follows.

Theorem B *Let G be a connected graph for which $\{J^k(G)\}$ is infinite. If $\{J^k(G)\}$ is nonplanar, then $J^k(G)$ is nonplanar for all $k \geq 4$.*

The goal of this paper is to study the nonplanar sequence $\{J_r^k(G)\}$ ($r \geq 2$) of iterated r-jump graphs of a connected graph G (which is, of course, an infinite sequence). For $r \geq 2$, we show that if G is a connected graph for which $\{J_r^k(G)\}$ is nonplanar, then nonplanar graphs in the sequence $\{J_r^k(G)\}$ are arrived at quickly as well. Throughout this paper we will use Kuratowski's characterization [8] of planar graphs.

Theorem C *A graph is planar if and only if it contains no subgraph isomorphic to K_5 or $K_{3,3}$ or a subdivision of one of these graphs.*

2 Nonplanar Sequences of Iterated 2-Jump Graphs

By Theorem A(2), if G is a connected graph G such that $J_2^k(G)$ is defined for each positive integer k, then $\{J_2^k(G)\}$ is planar if and only if $G = C_4$. In this section, we show that if $\{J_2^k(G)\}$ is a nonplanar sequence, then $J_2^k(G)$ is nonplanar for all $k \geq 3$. In order to do this, we first present several useful results, the first two of which were established in [7].

Theorem D *Let G be a connected graph that is not a star. Then the graph $J_2(G)$ is planar if and only if G is a subgraph of one of the seven graphs in Figure* 2.

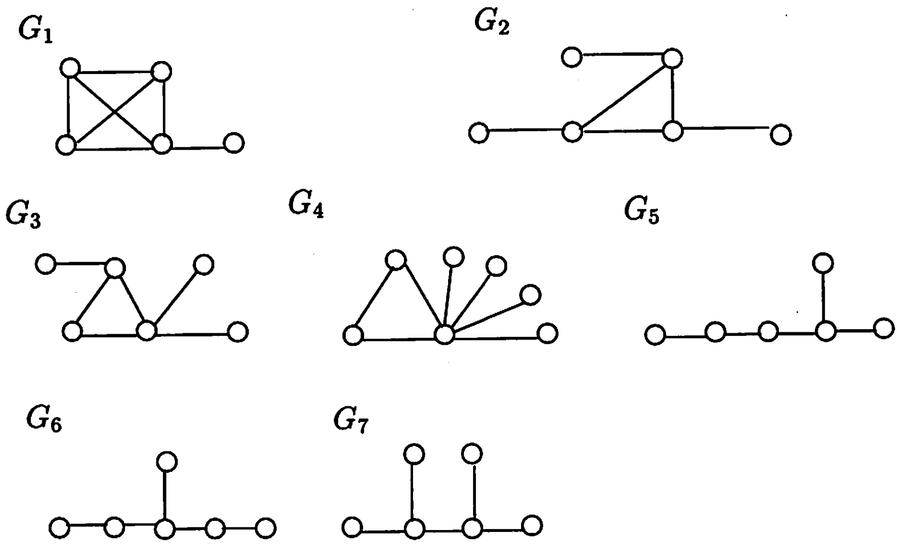

Figure 2: The seven graphs of Theorem C

Lemma E. *If G is a nonplanar graph, then $J_2(G)$ is nonplanar. Moreover, if H is a subdivision of a graph G and $J_2(G)$ is nonplanar, then $J_2(H)$ is nonplanar.*

We write $2H$ to denote the graph consisting of two disjoint copies of a graph H.

Lemma 2.1 *Let G be a graph of order at least* 5.

(*a*) *If G contains two disjoint subgraphs of size 2, then $J_2^2(G)$ is nonplanar.*

(*b*) *If G contains a path of length 4 or more, then $J_2^3(G)$ is nonplanar.*

(*c*) *If G contains cycle of length $n \geq 5$, then $J_2(G)$ is nonplanar.*

Proof. First, we verify (a). Since $J_2(2P_3) = K_{2,4}$, it follows that $J_2(G)$ contains $K_{2,4}$ as a subgraph. By Theorem D, $J_2^2(G)$ is nonplanar and so (a) holds. Next, we establish (b). The 2-jump graphs of P_5 and C_5 are shown in Figure 3. Thus, if G contains a path of length 4, then $J_2(G)$ contains two disjoint subgraphs of size 2 as shown in Figure 3. Hence $J_2^3(G)$ is nonplanar by (a) and so (b) holds. Finally, we verify (c). Since the 2-jump graph of C_5 shown in Figure 3 is a subdivision of K_5, it follows that $J_2(C_5)$ is nonplanar. Then (c) follows from Lemma E. ∎

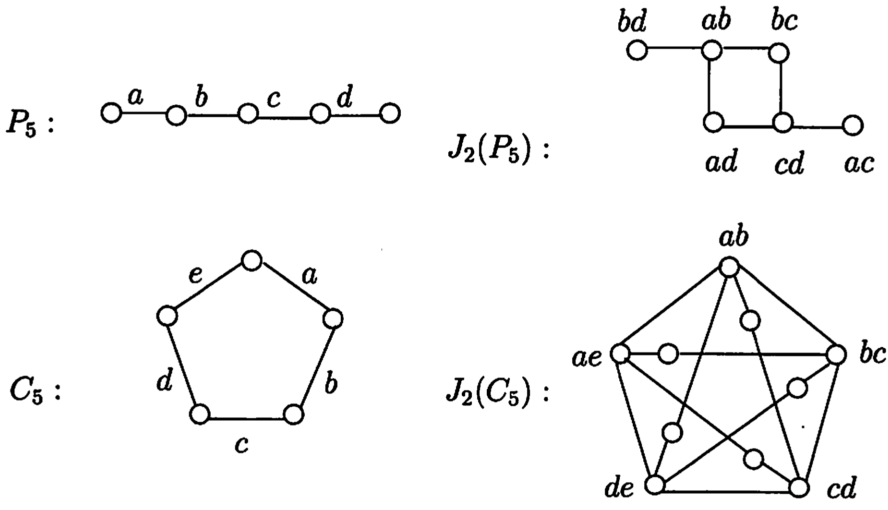

Figure 3: The 2-jump graphs of P_5 and C_5

We are now prepared to present the main result of this section. The length of a longest cycle in a connected graph is called the *circumference* of G and is denoted by $c(G)$. If G is a tree, then we write $c(G) = 0$.

Theorem 2.2 *Let G be a connected graph for which $\{J_2^k(G)\}$ is infinite. If $\{J_2^k(G)\}$ is nonplanar, then $J_2^k(G)$ is nonplanar for all $k \geq 3$. Moreover, $J_2^2(G)$ is planar if and only if G is the graph shown in Figure 4.*

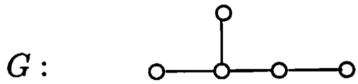

Figure 4: The graph G in Theorem 2.2

Proof. By Theorem A(2), we may assume that $G \neq C_4$. We show that if $\{J_2^k(G)\}$ is nonplanar, then $J_2^k(G)$ is nonplanar for all $k \geq 3$. By Lemma E, it suffices to show that $J_2^3(G)$ is nonplanar. If $J_2(G)$ is nonplanar, then $J_2^k(G)$ is nonplanar for all $k \geq 2$. Thus we may assume that $J_2(G)$ is planar. Then by Theorem D either G is a star or G is a subgraph of one or more of the graphs G_i $(1 \leq i \leq 7)$ of Figure 2. Since for each star G, the jump graph $J_2^2(G)$ does not exist, we may assume that G is a subgraph of some graph G_i $(1 \leq i \leq 7)$ of Figure 2.

First, we make an observation. If G contains a path of length 4 or more, then $J_2^3(G)$ is nonplanar by Lemma 2.1. Thus we may assume that $\operatorname{diam} G \leq 3$. Moreover, if G contains a cycle of length 5 or more, then $J_2(G)$ is nonplanar by Lemma 2.1. Thus we assume that G contains no cycle of order 5 or more. Thus $c(G) = 0$, $c(G) = 3$, or $c(G) = 4$. We consider these three cases.

Case 1. $c(G) = 0$. Then G is a tree with $\operatorname{diam}(G) \leq 3$. If $\operatorname{diam}(G) = 2$, then G is a star and $\{J_2^k(G)\}$ terminates. Thus $\operatorname{diam}(G) = 3$ and, consequently, G is a double star. Let u and v be the vertices of G that are not end-vertices. If $\deg u \geq 3$ and $\deg v \geq 3$, then G contains two disjoint subgraphs of size 2 and so $J_2^2(G)$ is nonplanar by Lemma 2.1. So we may assume that at least one of u and v has degree 2, say $\deg v = 2$. If $\deg u \geq 3$, then G contains the tree T of Figure 4 as a subgraph. Since the 2-jump graph $J_2(T)$ contains the two disjoint subgraphs of size 2 shown in Figure 5, it follows by Lemma 2.1 that $J_2^3(G)$ is nonplanar. Hence $\deg u = \deg v = 2$ and so $G = P_4$. However, $J_2^2(P_4)$ does not exist. Therefore, if $c(G) = 0$ and $\{J_2^k(G)\}$ is nonplanar, then $J_2^3(G)$ is nonplanar. Since $J_2(T) = 2P_3$, it follows that $J_2^2(T) = K_{2,4}$, which is planar. Thus in this case T (shown in Figure 4) is the only connected graph G such that $\{J_2^k(G)\}$ is nonplanar but $J_2^2(G)$ is planar.

Case 2. $c(G) = 3$. Then G contains a triangle but no 4-cycle. Since G contains neither a 4-cycle nor a path of length 4, it follows that G has a unique triangle, say $C_3 : v_1, v_2, v_3, v_1$. If $G = C_3$, then $J_2^2(G)$ does not exist. On the other hand, if at least two of v_1, v_2 and v_3 have degree 3 or more, then G contains a path of length 4 and so $J_3^2(G)$ is nonplanar by Lemma 2.1. So we may assume that exactly one of v_1, v_2 and v_3 has degree 3 or more. Hence G contains a subgraph isomorphic to one of graphs F_1,

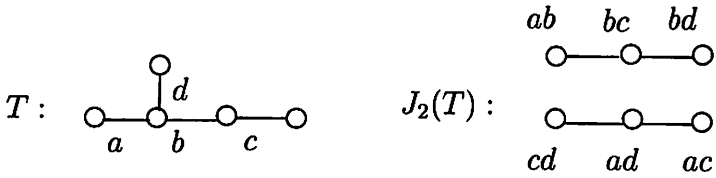

Figure 5: The tree T and $J_2(T)$

F_2, and F_3 in Figure 6. Since (1) $J_2^2(F_1)$ does not exist, (2) F_2 contains a subgraph T of Figure 5 and so $J_2^3(G)$ is nonplanar, and (3) F_3 contains a path of length 4, it follows that $J_2^2(G)$ is nonplanar. Therefore, if $c(G) = 3$ and $\{J_2^k(G)\}$ is nonplanar, $J_2^3(G)$ is nonplanar. In fact, $J_2(F_2) = 3P_3$ and so $J_2^2(F_2)$ is nonplanar. Thus in this case, if $\{J_2^k(G)\}$ is nonplanar, then $J_2^2(G)$ is nonplanar.

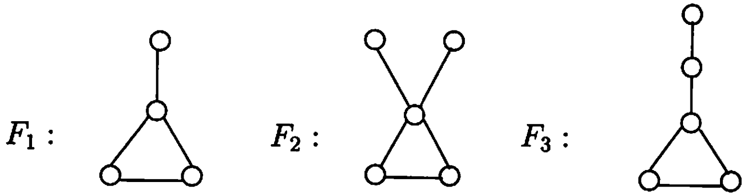

Figure 6: The graphs F_1, F_2 and F_3 in Case 2

Case 3. $c(G) = 4$. Then G contains a 4-cycle. Since $G \neq C_4$, the graph G contains a subgraph isomorphic to one of the graphs H_1 and H_2 of Figure 7. Since H_1 contains a path of length 4, it follows from Lemma 2.1 that $J_2^3(H_1)$ is nonplanar. Moreover, the graph $J_2(H_2)$ shown in Figure 7 contains two disjoint subgraphs of size 2. By Lemma 2.1 $J_2^3(H_2)$ is nonplanar. Therefore, if $c(G) = 4$ and $\{J_2^k(G)\}$ is nonplanar, $J_2^3(G)$ is nonplanar. Moreover, it can be verified that $J_2^2(H_1)$ and $J_2^2(H_2)$ are both nonplanar. Thus, in this case, if $\{J_2^k(G)\}$ is nonplanar, then $J_2^2(G)$ is nonplanar.

Therefore, if G is a connected graph for which $\{J_2^k(G)\}$ is nonplanar, then $J_2^k(G)$ is nonplanar for all $k \geq 3$. Furthermore, the graph G of Figure 4 is the only connected graph such that $\{J_2^k(G)\}$ is nonplanar and $J_2^2(G)$ is planar. ■

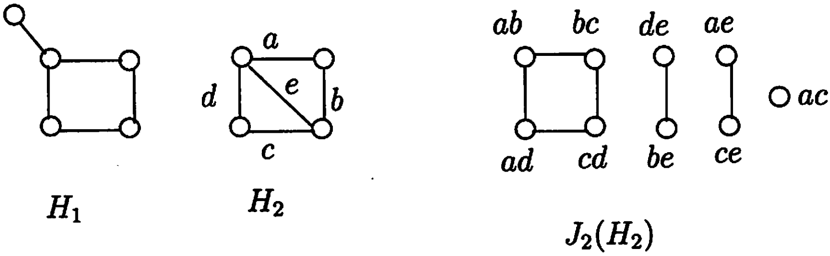

Figure 7: The graph H_1, H_2 and $J_2(H_2)$ in Case 3

3 Nonplanar Sequences of Iterated r-Jump Graphs

In this section, we study the nonplanar sequences $\{J_r^k(G)\}$ of iterated r-jump graphs of a connected graph G for a fixed $r \geq 3$. First we present three useful lemmas which were established in [3].

Lemma F *If n and r are integers with $n \geq r+3 \geq 6$, then $J_r(P_n)$ is nonplanar.*

Lemma G *Let $r \geq 3$ be an integer. If a graph G contains two vertex disjoint subgraphs, one of size $\lfloor \frac{r+2}{2} \rfloor$ and the other of size $\lceil \frac{r+2}{2} \rceil$, then $J_r(G)$ contains P_{r+3} as a subgraph.*

Lemma H *Let $r \geq 3$ be an integer. If a graph G of size at least $r+2$ contains an edge that is not adjacent to two other edges of G, then $J_r(G)$ contains two vertex disjoint subgraphs, one of size $\lfloor \frac{r+2}{2} \rfloor$ and the other of size $\lceil \frac{r+2}{2} \rceil$.*

The following are immediate consequences of Lemmas F, G, and H.

Corollary 3.1 *Let $r \geq 3$ be an integer.*

(*a*) *If a graph G contains two vertex-disjoint subgraphs, one of size $\lfloor \frac{r+2}{2} \rfloor$ and the other of size $\lceil \frac{r+2}{2} \rceil$, then $J_r^2(G)$ is nonplanar.*

(*b*) *If a graph G of size at least $r+2$ contains an edge that is not adjacent to two other edges of G, then $J_r^3(G)$ is nonplanar.*

Corollary 3.2 *Let $r \geq 3$ be an integer and let G be a graph of size at least $r+2$. If $J_r^3(G)$ is planar, then every edge of G is adjacent to all other edges of G, with at most one exception.*

Let G be a connected planar graph of size m such that $\{J_r^k(G)\}$ is infinite and nonplanar. Necessarily, $m \geq r+1$, for otherwise, $\{J_r^k(G)\}$ terminates. We first consider the case when $m \geq r+2$.

Theorem 3.3 *For $r \geq 3$, let G be a connected graph of size $m \geq r+2$ for which $\{J_r^k(G)\}$ is infinite. If $\{J_r^k(G)\}$ is nonplanar, then $J_r^k(G)$ is nonplanar for all $k \geq 3$.*

Proof. Let G be a connected planar graph of size $m \geq r+2$ such that $\{J_r^k(G)\}$ is infinite and nonplanar. Then $G \neq C_5$ and $G \neq cor(K_3)$ by Theorem A. If G contains an edge that is not adjacent to two other edges of G, then $J_r^3(G)$ is nonplanar by Corollary 3.1. Thus we may assume that every edge in G is adjacent to all other edges of G, with at most one exception. If every edge in G is adjacent to all other edges of G, then, since the size of G exceeds 3, G is a star. So we may assume that there exists an edge e that is adjacent to all other edges of G except one edge f. However, then f is adjacent to all other edges of G except e. This implies that each edge in G distinct from e and f is adjacent to both e and f. Since the size of G is at least 5, it follows that $G = K_4 - e$ or $G = K_4$. If G is a star, then $\{J_r^k(G)\}$ terminates for all $r \geq 3$. If $G = K_4 - e$, then $r = 3$; while if $G = K_4$, then $r = 3$ or $r = 4$. Since $J_3(K_4 - e) = C_4 \cup 2K_2$ and $J_3(K_4) = 3C_4$, it follows by Corollary 3.1 that none of $J_3^3(K_4 - e)$, $J_3^3(K_4)$, and $J_4^3(K_4)$ are planar. Thus $J_r^3(G)$ is nonplanar for all $r \geq 3$. Therefore, $J_r^k(G)$ is nonplanar for all $r \geq 3$ and $k \geq 3$. ∎

We now assume that G is a connected planar graph of size $m = r+1$, where $r \geq 3$. If there exists $k \geq 1$ such that $J_r^k(G)$ contains at most r edges, then $\{J_r^k(G)\}$ terminates. Thus we may assume that $J_r^k(G)$ contains at least $r+1$ edges for all $k \geq 1$. The following result was established in [7].

Theorem I Let G be a graph with size $m \geq 2$ and let r be an integer with $1 \leq r < m$. Then $J_r(G) = J_{m-r}(G)$.

By Theorems B and I, we have the following.

Corollary 3.4 *For $r \geq 3$, let G be a connected graph of size $m = r+1$ for which $\{J_r^k(G)\}$ is nonplanar. If $J_r^k(G)$ contains exactly $r+1$ edges for all $k \geq 1$, then $J_r^k(G)$ is nonplanar for all $k \geq 4$.*

Proof. If $J_r^k(G)$ contains exactly $r+1$ edges for all $k \geq 1$, then $J_r^k(G) = J^k(G)$ for all $k \geq 1$ by Theorem I and so $\{J_r^k(G)\} = \{J^k(G)\}$. Since $\{J^k(G)\}$ is nonplanar, it follows from Theorems B that $J^k(G)$ is nonplanar for all $k \geq 4$. Therefore, $J_r^k(G)$ is nonplanar for all $k \geq 4$. ∎

If there exists an integer $k' \geq 1$ such that $J_r^{k'}(G)$ contains $r+2$ or more edges, then $J_r^k(G)$ is nonplanar for all $k \geq k'+3$ by Theorem 3.3. This leaves the following open question.

Problem 3.5 *For $r \geq 3$, let G be a connected graph of size $m = r+1$ for which $\{J_r^k(G)\}$ is infinite and nonplanar. If there exists a positive integer k such that $J_r^k(G)$ contains $r+2$ or more edges, what is the smallest integer k_0 such that $J_r^k(G)$ is nonplanar for all $k \geq k_0$.*

References

[1] G. Chartrand, H. Gavlas, H. Hevia, and M. Johnson, Rotation and jump distance between graphs. *Discuss. Math Graph Theory* **17** (1997) 285-300.

[2] G. Chartrand, H. Gavlas, and M. Schultz, Convergent sequences of iterated H-graphs. *Discrete Math.* **147** (1995) 73-86.

[3] G. Chartrand, H. Gavlas, D. W. VanderJagt and P. Zhang, Which sequences of iterated jump graphs are planar? *Congr. Numer.* **139** (1999) 33-39.

[4] G. Chartrand, H. Hevia, E. B. Jarrett, and M. Schultz, Subgraph distance in graphs defined by edge transfers. *Discrete Math.* **170** (1997) 63-79.

[5] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, third edition. Chapman & Hall, New York (1996).

[6] G. Chartrand, D. W. VanderJagt, and P. Zhang, On the planarity of iterated jump graphs. *Discrete Math.* **226** (2001) 93-106.

[7] H. Hevia, D. W. VanderJagt, and P. Zhang, On the planarity of jump graphs. *Discrete Math.* **220** (2000) 119-129.

[8] K. Kuratowski, Sur le probléme des courbes gauches en topologie. *Fund. Math.* **15** (1930) 270-283.

[9] P. Zhang, A note on nonplanar sequences of iterated jump graphs. *J. Combin. Math. Combin. Comput.* To appear.