A Decomposition Theorem for Simply Connected Orthogonal Polygons

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Abstract

ABSTRACT. Let S be a simply connected orthogonal polygon in the plane. Assume that the vertex set of S may be partitioned into sets A, B such that for every pair x, y in A (in B), S contains a staircase path from x to y. Then S is a union of two or three orthogonally convex sets. If S is starshaped via staircase paths, the number two is best, while the number three is best otherwise. Moreover, the simple connectedness requirement cannot be removed. An example shows that the segment visibility analoque of this result is false.

1 Introduction.

We begin with some definitions from [2] and [3]. Let S be a nonempty set in $\mathbb{R}^2$. Set S is called an *orthogonal polygon* (*rectilinear polygon*) if and only if S is a connected union of finitely many convex polygons (possibly degenerate) whose edges are parallel to the coordinate axes. Point q of S is a *point of local nonconvexity* (*lnc point*) of S if and only if for every neighborhood N of $q, N \cap S$ fails to be convex. An edge e of S is a *dent edge* if and only if both endpoints of e are lnc points of $S \cap H$, for H an appropriate closed halfplane determined by the line of e. Set S is said to be *horizontally convex* if and only if for each pair x, y in S with $[x, y]$ horizontal, it follows that $[x, y] \subseteq S$. *Vertically convex* is defined analogously. Set S is an *orthogonally convex polygon* if and only if S is an orthogonal polygon which is both horizontally convex and vertically convex.

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Let λ be a simple polygonal path in $\mathbb{R}^2$ whose edges $[V_{i-1}, V_i], 1 \leq i \leq n$, are parallel to the coordinate axes. Such a path λ is called a *staircase path* if and only if the associated vectors alternate in direction. That is, for i odd the vectors $\overrightarrow{V_{i-1}, V_i}$ have the same direction, and for i even, the vectors $\overrightarrow{V_{i-1}, V_i}$ have the same direction. For points x, y in set S, we say *x sees y via staircase paths* (x is *visible* from y via staircase paths) if and only if there is a staircase path in S containing both x and y. From [7, Lemma 1], it follows that an orthogonal polygon is orthogonally convex if and only if every two of its points see each other via staircase paths. Similarly, set S is *starshaped via staircase paths* (*orthogonally starshaped*) if and only if there is some point p in S such that p sees each point of S via staircase paths. The set of all such points p is called the *staircase kernel* of S, denoted Ker S.

While many results concerning visibility via staircase paths have been motivated by analogous results involving visibility via straight line segments ([1], [2], [3], [7]), sometimes it is possible to obtain a staircase path result which has no corresponding segment visibility predecessor. Theorem 1 of this paper is such a result. For a simply connected orthogonal polygon S, an assignment of its vertices to orthogonally convex subsets S_A, S_B of S induces a decomposition of S into two or three orthogonally convex sets. However, a segment visibility analogue of this theorem fails, at least for unions of less than six convex sets, as an example reveals.

Thoughout the paper, *cl* S, *int* S, and *bdry* S will denote the closure, interior, and boundary, respectively, for set S. The reader may refer to Valentine [8], to Lay [6], to Danzer, Grünbaum, Klee [4], and to Eckhoff [5] for discussions concerning visibility via straight line segments and corresponding convex and starshaped sets.

2 The results.

We begin with the following theorem.

Theorem 1. Let S be a simply connected orthogonal polygon in the plane. Assume that the vertex set of S may be partitioned into two sets A, B such that for every pair x, y in A (in B), S contains a staircase path from x to y. Then S is a union of two or three orthogonally convex sets. If S is starshaped via staircase paths, the bound two is best possible, while the bound three is best otherwise.

Proof. Before we begin, observe that by [3, Lemma 1], each set A, B lies in an orthogonally convex subset of S. If S is orthogonally convex, the result is trivial, so assume that this is not the case. We start with some preliminary remarks. For the moment, assume that the boundary of S is a simple closed curve ordered in a clockwise direction, with corresponding vertices $V_0, V_1, \ldots, V_n = V_0, n \geq 3$. For $1 \leq i \leq n$, we call edge $[V_{i-1}, V_i]$ *north*, *south*, *east*, or *west* according to the direction assigned to the corresponding vector $\overrightarrow{V_{i-1}, V_i}$ by the clockwise ordering.

By our hypothesis, for every three vertices of S, at least two see each other via staircase paths. Hence we may use an argument similar to one in [3, Theorem 2] to conclude that for each of the four directions north, south, east, west there is at most one corresponding dent edge. Since S is not orthogonally convex, S has at least one dent edge, so without loss of generality assume that S has an east dent edge e, with E the corresponding line. For an appropriate labeling of distinct open halfplanes E_1, E_2 determined by E, $E_2 \cap S$ has two components whose closures meet e, while $E, \cap S$ has one such component. Certainly neither $E, \cap S$ nor $E_2 \cap S$ has more than two components. Moreover, it is easy to see that each component of $E_2 \cap S$ has orthogonally convex closure, for otherwise S would have three vertices which we could not assign appropriately to two sets A, B.

In case S has west, north, south dent edges w, n, s, we define halfplanes W_1, N_1, S_1 analogously, and define set $K = cl\ E_1, \cap\ cl\ W_1 \cap\ cl\ N_1 \cap cl\ S_1$. If S as no dent edge in one or more directions, we replace the corresponding closed halfplanes by $\mathbb{R}^2$ in the intersection defining K. Using comments following the proof of Theorem 2 in [2], it is not hard to show that $K \cap S$ is exactly the staircase kernel for set S.

Hence Ker $S \neq \emptyset$ if and only if the associated intersection $K \cap S \neq \emptyset$. In general, there are two ways that this may occur. Either $K = \emptyset$ or $K \neq \emptyset$ while $K \cap S = \emptyset$. For the moment, let us examine the second case for our set S. It is not hard to see that if the dent edges e, n, s exists, then halfplane E_2 contains neither n nor s (since again this would produce three vertices which we could not assign appropriately to two sets A, B). Similar statements hold for W_2, n, s, for N_2, e, w, and for S_2, e, w. However, this implies that when $K \neq \emptyset$, then K contains points of d for every dent edge d of S, so $K \cap S \neq \emptyset$. Therefore, for our set S, if Ker $S = \emptyset$, then $K = \emptyset$. This means that if Ker $S = \emptyset$, then for some pair of dent edges e, w or s, n in S, say e, w, $cl\ E, \cap\ cl\ W, = \emptyset$. That is, the y coordinate of e is less than the y coordinate of w.

To prove Theorem 1 when *bdry* S is a simple closed curve, we consider two cases, determined by whether or not set S is starshaped via staircase paths.

Case 1. Suppose that set S is starshaped via staircase paths. Then Ker $S \neq \emptyset$. By our earlier observations, for each component C of $S \setminus$(Ker S), $C \cup \setminus$(Ker S) is an orthogonal polygon with no dent edges and hence is orthogonally convex by [1, Lemma 1]. If for each C, all the vertices of S in C may be assigned to the same set A or B, then we let $\mathcal{Q}$ denote the collection of C sets with vertices in $A, \mathcal{B}$ the collection with vertices in B. The sets $\cup\{C \cup (\text{Ker } S) : C \text{ in } \mathcal{Q}\}, \cup\{C \cup (\text{Ker } S) : C \text{ in } \mathcal{B}\}$ provide a decomposition of S into two orthogonally convex sets.

It remains to show that this must occur. Assume that the vertices of S have been assigned to sets A, B to satisfy our hypothesis such that the largest possible number $r \geq 0$ of components C of $S \backslash (\text{Ker } S)$ have the required property. That is, for as many components C as possible, all vertices of S in C are assigned to the same set A or B. Assume that the condition fails for some component C_0. That is, some vertices in C_0 are in A, some in B. Also, we cannot remove the C_0 vertices from A and reassign them to B. Using [2, Lemma 1], this implies that for at least one vertex a_0 in C_0, a_0 must remain in A and cannot be moved to B. It follows that for some b_1 in B, b, cannot see a_0 via staircase paths. By a parallel argument, for at least one vertex b_0 in C_0, b_0 cannot be moved from B to A; thus there is an a_1 in A such that a_1 cannot see b_0 via staircase paths.

Since b_1 cannot see a_0, then clearly b_1 and a_0 must be beyond a common dent edge, say n. That is, b_1 and a_0 belong to the corresponding open halfplane N_2 defined previously, in distinct components of $N_2 \cap S$. Similarly, a_1 and b_0 lie beyond a common dent edge d.

For the moment, suppose that $d \neq n$. Since a_0, b_0 both belong to C_0, d is not the south dent edge. Without loss of generality, assume that d is the east dent edge e. Then a_1 and b_0 both be beyond E in distinct components of $E_2 \cap S$. See (Figure 1.)

If C_0 had a vertex u in $N_2 \cap E_3$, then u could not see via staircase paths either a_1 or b_1, and u could not belong to A or to B, impossible. Hence no such vertex u exists. This implies that $C_0 \cap N_2 \cap E_2 = \emptyset$. However, then C_0 cannot be connected, a contradiction. Our supposition is false, and the case $d \neq n$ cannot occur.

It remains to examine the case for $d = n$. Then all four points a_0, b_1, b_0, a_1 must be beyond N, in N_2. There are exactly two components of $N_2 \cap S$, and certainly all points of $C_0 \cap N_2 \cap S$ belong to the same component of $N_2 \cap S$. (See Figure 2.)

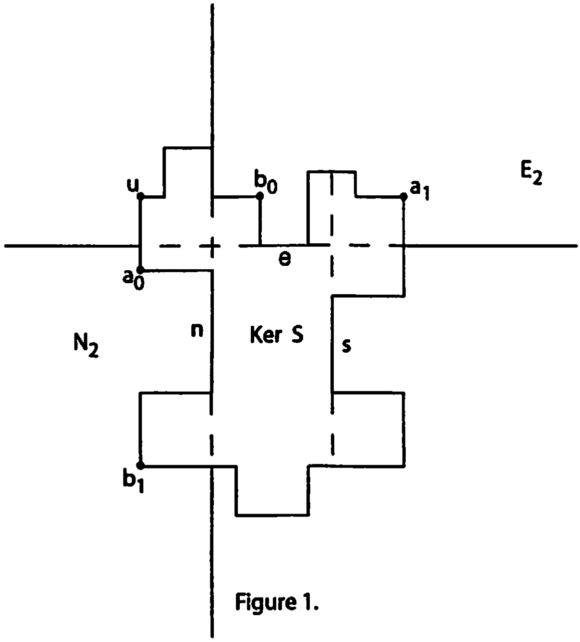

Figure 1.

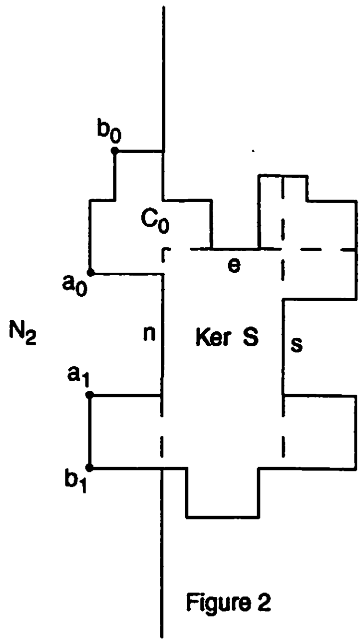

Figure 2

Hence a_0, b_0 belong to the same component of $N_2 \cap S$. However, then a_1, b_1 belong to the remaining component, so a_1 cannot see a_0 via staircase paths, and b_1 cannot see b_0 via staircase paths. Again we have a contradiction, and the case $d = n$ cannot occur either. Thus our original assumption concerning C_0 must be false. It follows that every component C of $S \backslash (\text{Ker } S)$, must have the required property, with all vertices of S in C assigned to the same set A or B. This finishes the argument in Case 1 when the boundary of S is a simple closed curve.

Case 2. Suppose that set S is not starshaped. By our previous comments, $K = \emptyset$, and for an appropriate pair of dent edges, say e and w, the y coordinate of e is less than the y coordinate of w. (When this occurs, we say e is north of w, w south of e.) Also, since we are assuming that the boundary of S is a simple closed curve, without loss of generality we may suppose that e is east of w. That is, every x coordinate assigned to e is larger than every x coordinate assigned to w. As before, assume that the vertices of S are ordered in a clockwise direction along the boundary of S. Relative to our ordering, let a_0 denote the vertex which is the immediate successor of edge e, and let b_0 denote the vertex which is the immediate predecessor of edge w. (See Figure 3.)

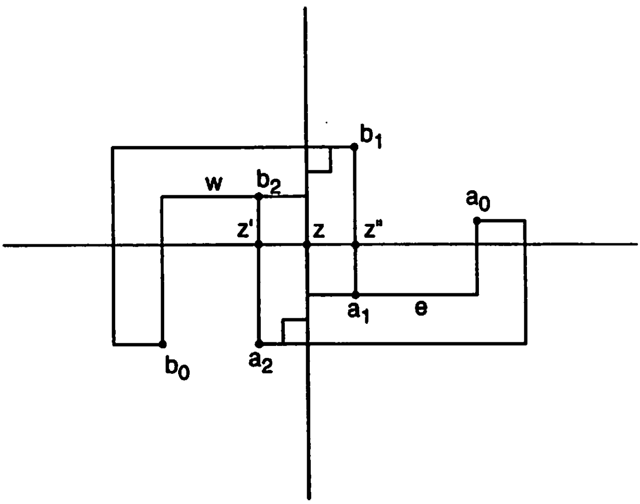

Figure 3.

Certainly a_0, b_0 must be assigned to distinct sets A, B, for any particular pair A, B satisfying our hypothesis. Let L be a line north of e, south of w, and let z be a point of $L \cap (\text{int } S)$ east of w, west of e. If we identify z with the origin, then horizontal and vertical lines at z determine the usual four quadrants. Without loss of generality, assume that z has been chosen so that the corresponding horizontal and vertical lines contain no vertices of S.

Vertices of S in quadrant 2 cannot see a_0 via S and hence must be assigned to the member of $\{A, B\}$, say B, which contains b_0. Similarly, vertices in quadrant 4 must be assigned to A. Consider the intersection of S with quadrant 1, and let C be the corresponding component whose closure contains z. The intersection of S with quadrant 1 consists either of C alone or of C together with exactly one other component. It is clear that vertices of S in C cannot see a_0 via staircase paths and hence must be assigned to B. This means that, in case a second component exists, its vertices must be assigned to A. A parallel statement holds for quadrant 3.

Now consider the component of $L \cap S$, which contains z, say $[z', z'']$, where z' is west of z, z'' east of z. Using comments above, for the edge of S containing z'', the associated first quadrant endpoint b_1 belongs to B, while the associated fourth quadrant endpoint a_1 belongs to A. By a parallel argument, for the edge of S containing z', the second quadrant endpoint b_2 is in B, while the third quadrant endpoint a_2 is in A.

By hypothesis, vertices b_2, b_1 are joined in S by a staircase path $\lambda(b_2, b_1)$, while vertices a_1, a_2 are joined by a staircase path $\mu(a_1, a_2)$. Certainly $\lambda(b_2, b_1), \mu(a_1, a_2)$ be in distinct open halfplanes determined by line L, so they are disjoint. Moreover, in an obvious way these paths separate S into three orthogonal polygons: One of these three is the orthogonally convex subset S_z of S whose boundary is $[b_1, a_1] \cup \mu(a_1, a_2) \cup [a_2, b_2] \cup \lambda(b_2, b_1)$. The remaining two are the closures S_λ, S_μ of the two components of $S \backslash S_z$, where $\lambda(b_2, b_1) \subseteq S_\lambda$ and $\mu(a_1, a_2) \subseteq S_\mu$.

By our earlier comments, it is clear that (relative to our clockwise order on *bdry* S) all vertices of S from a_1 to a_2 are in A. Moreover, by the proof of [3, Lemma 1], A $\cup \mu(a_1, a_2)$ lies in a maximal orthogonally convex subset S_A of S. Certainly $S\mu \subseteq S_A$. Similarly, vertices of S from b_2 to b_1 belong to $B, B \cup \lambda(b_2, b_1)$ lies in a maximal orthogonally convex subset of S_B of S, and $S_\lambda \subseteq S_B$. We conclude that sets S_z, S_A, S_B give a decomposition of S into three orthogonally convex sets, finishing Case 2 and completing the proof of the theorem when *bdry* S is a simple closed curve.

Finally, if *bdry* S is not a simple closed curve, we use a technique from the proof of [3, Theorem 2]. For each natural number n, define $S_n = \{x :$ the horizontal or vertical distance from x to S is at most $\frac{1}{n}\}$. For n sufficiently large, each set n will satisfy our hypothesis, *bdry* S_n will be a simple closed curve, and S_n will be starshaped via staircase paths when S has this property. By an argument from [3, Theorem 2], decompositions for the S_n sets may be used to obtain a corresponding decomposition of S, into two orthogonally convex sets when S is starshaped and into three orthogonallty convex sets otherwise. This completes the proof of the theorem.

Figure 3 above shows that the bound three in Theorem 1 is best possible when Ker $S \neq \emptyset$. Of course, when Ker $S \neq \emptyset$, Case 1 together with closing comments above show that the bound may be reduced to two.

Moreover, it is interesting to observe that, without the simple connectedness condition, the result in Theorem 1 is false and in fact no bound is possible. Consider the following example.

Example 1. For $n \geq 3$, let $R_1, \ldots, R_n$ be a collection of n disjoint rectangular regions, each with two vertices on the line $y = 1$, two on the line $y = -1$. Let R be a rectangular region whose interior contains $\cup\{R_i : 1 \leq i \leq n\}$, and let $S = cl\ [R \backslash (\cup\{R_i : 1 \leq i \leq n\})]$.

(See Figure 4.) If we assign vertices with positive y coordinates to set A, vertices with negative y coordinates to set B, then sets A, B satisfy the hypothesis in Theorem 1. However, we may select a set $\{s_1, \ldots, s_{n+1}\}$ of $n+1$ points of S along the x axis such that no two of the s_i points see each other via staircase paths. Hence S is a union of no fewer than $n+1$ orthogonally convex sets, and in general, no bound independent of n can be established.

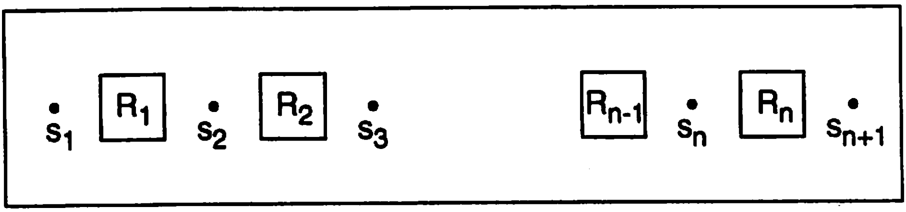

Figure 4.

Because [3] presents a decomposition theorem for orthogonal polygons which satisfy a 3 - convexity property, perhaps it is a good idea to point out that the set S in Theorem 1 above need not be. 3 - convex via staircase paths. That is, there may be three points of S, no two of which see each other via staircase paths. Figure 3 above provides an illustrative example, for three appropriate points may be chosen by selecting one from each component if $L \cap S$. Furthermore, it is possible for a simply connected orthogonal polygon T to be 3 - convex via staircase paths without the vertex set of T satisfying the condition in Theorem 1. Figure 3 in [3, Example 1] provides an example of such a set. Thus the hypothesis of Theorem 1 neither implies nor is implied by the requirement of 3 - convexity.

In conclusion, it is interesting to observe that the segment visibility analogue of Theorem 1 is false, at least for unions of less than six convex sets, as Example 2 reveals.

Example 2. Let S denote the simply connected polygonal region in Figure 5. Every vertex of S lies either in the horizontal rectangle at p_1 or in the vertical rectangle at p_3. However, points $p_i, 1 \leq i \leq 6$ have the property that no two see each other via segments in S, and S is a union of 6 and no fewer convex sets. Thus the bound of three in Theorem 1 fails for segment visibility in polygonal regions.

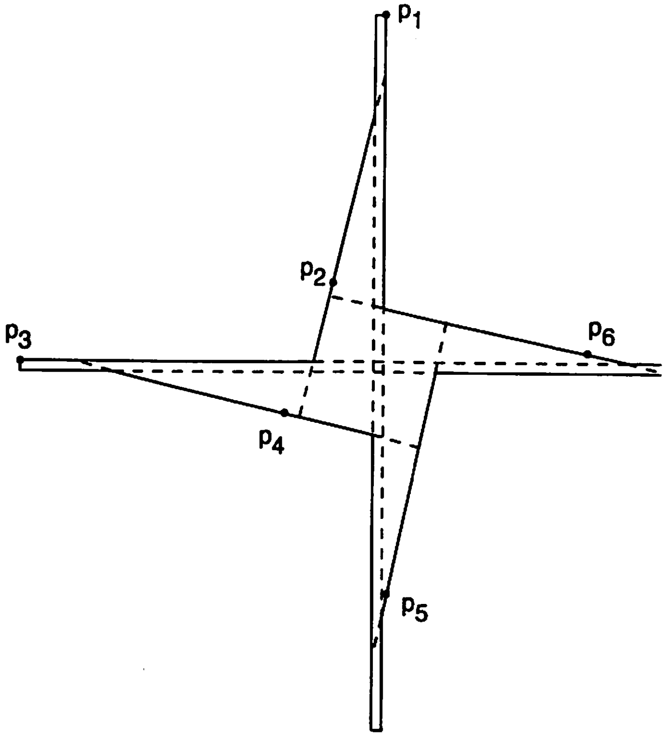

Figure 5.

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