A Note on Dominating Cycles in Tough Graphs

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Abstract

A cycle C in a graph G is said to be a dominating cycle if every vertex of G has a neighbor on C. Strengthening a result of Bondy and Fan [3] for tough graphs, we prove that a k-connected graph G ($k \geq 2$) of order p with $t(G) > k/(k+1)$ has a dominating cycle if $\sum_{x \in S} \deg_G x \geq p - 2k - 2$ for each $S \subset V(G)$ of order $k+1$ in which every pair of vertices in S have distance at least four in G.

Keywords: dominating cycle, toughness

1 Introduction

For a vertex x and a set U of vertices in a graph G, U is said to dominate x if $x \in U$ or $N_G(x) \cap U \neq \emptyset$, where $N_G(x)$ denotes the neighborhood of x in G. A dominating set of G is a set of vertices which dominates every vertex

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in G. A cycle C is said to be a dominating cycle if $V(C)$ is a dominating set in G. Thus, a hamiltonian cycle is a dominating cycle.

There are a number of sufficient conditions for a graph to have a hamiltonian cycle. Since a dominating cycle is a generalization of a hamiltonian cycle, there may be sufficient conditions for a graph to have a dominating cycle, which correspond to known sufficient conditions for hamiltonicity. One such example was obtained by Bondy and Fan [3]. Bondy [2] proved the following theorem concerning degree sums and hamiltonicity. A set S of vertices in a graph G is said to be stable if no pair of vertices in S are joined by an edge. A stable set is also called an independent set.

Theorem A ([2]) *Let k be a positive integer, and let G be a k-connected graph. If $\sum_{x \in S} \deg_G x > \frac{1}{2}(p-1)(k+1)$ for every stable set S in G of order $k+1$, then G has a hamiltonian cycle.*

In [3], Bondy and Fan proved that a much weaker degree sum condition guarantees the existence of a dominating cycle. A set S of vertices in a graph G is said to be *d-stable* if the distance between each pair of distinct vertices in S is at least d.

Theorem B ([3]) *Let $k \geq 2$ and let G be a k-connected graph of order p. If $\sum_{x \in S} \deg_G x \geq p - 2k$ for every 3-stable set S of G of order $k+1$, then G has a dominating cycle.*

In Theorem A and Theorem B, the degree sum conditions are sharp and cannot be relaxed. For example, for Theorem A, $G = K_k + (k+1)K_1$ satisfies $\sum_{x \in S} \deg_G x = \frac{1}{2}(|V(G)| - 1)(k+1)$ for the unique stable set S of order $k+1$ (which is the largest stable set), but G is not hamiltonian. For Theorem B, Bondy and Fan gave the following example. Let $H = K_k + (k+1)K_l$, where $l \geq k$. Then for each K_l, we introduce a new vertex and join each new vertex and every vertex in the corresponding K_l by an edge. Let G be the resulting graph. Then $|V(G)| = k + (k+1)(l+1)$ and G is k-connected. Let $S = V(G) - V(H)$. Since each vertex in S has neighbors only in the corresponding K_l, S is a 3-stable set (actually, S is a 4-stable set). Since each vertex in S has degree l, $\sum_{x \in S} \deg_G x = (k+1)l = |V(G)| - 2k - 1$. However, G has no dominating cycle.

Let $\omega(G)$ denote the number of the components of a graph G. A connected graph G is defined to be t-tough if $|S| \geq t \cdot \omega(G-S)$ for every subset S of $V(G)$ with $\omega(G-S) > 1$. The toughness of G, denoted by $t(G)$, is the maximum value of t for which G is t-tough (taking $t(K_n) = \infty$ for all $n \geq 1$). Note that both sharpness examples in the previous paragraph have toughness $k/(k+1)$. Actually, for several sufficient conditions for hamiltonicity, it is observed that weaker conditions guarantee the existence of a hamiltonian cycle for graphs with large toughness. For example, the case

$k = 1$ of Theorem A, known as Ore's theorem, admits a weaker degree sum condition for 1-tough graphs.

Theorem C ([9]) *Every graph G of order $p \geq 3$ which satisfies $\deg_G x + \deg_G y \geq p$ for every pair of distinct nonadjacent vertices x and y has a hamiltonian cycle.*

Theorem D ([7]) *Every 1-tough graph of order $p \geq 11$ which satisfies $\deg_G x + \deg_G y \geq p - 4$ for every pair of distinct nonadjacent vertices x and y has a hamiltonian cycle.*

A cycle C of a graph G is said to be a D-cycle if every edge in G is incident with $V(C)$. (Note that in some papers, a D-cycle is referred to as a dominating cycle.) Like a hamiltonian cycle, some sufficient conditions for the existence of a D-cycle can be relaxed if we put a further assumption on toughness. One such example is the following.

Theorem E ([2]) *Let G be a 2-connected graph of order p. If $\sum_{x \in S} \deg_G x \geq p + 2$ for every stable subset S of $V(G)$ of order 3, then G has a D-cycle.*

Theorem F ([1]) *Let G be a 1-tough graph of order p. If $\sum_{x \in S} \deg_G x \geq p$ for every stable subset S of $V(G)$ of order 3, then G has a D-cycle.*

The purpose of this paper is to show that Theorem B also admits a similar relaxation under an additional assumption on toughness.

Theorem 1 *Let $k \geq 2$, and let G be a k-connected graph of order p with $t(G) > k/(k+1)$. If $\sum_{x \in S} \deg_G x \geq p - 2k - 2$ for every 4-stable subset S of $V(G)$ of order $k+1$, then G has a dominating cycle.*

The example of sharpness for Theorem B shows that the toughness $k/(k+1)$ in Theorem 1 is sharp. On the other hand, we know the sharpness of degree condition only for $k = 2$. Let $m \geq 3$ and take three complete graphs H_1, H_2 and H_3 of the same order m. Take three distinct vertices x_i, y_i and u_i in each H_i ($1 \leq i \leq 3$). Let K_3 be a complete graph of order three with $V(K_3) = \{v_1, v_2, v_3\}$. Let v_0 be a new vertex. Then construct G by

$$V(G) = V(H_1) \cup V(H_2) \cup V(H_3) \cup V(K_3) \cup \{v_0\}, \quad \text{and}$$
$$E(G) = E(H_1) \cup E(H_2) \cup E(H_3) \cup E(K_3) \cup \{v_0y_1, v_0y_2, v_0y_3, v_1x_1, v_2x_2, v_3x_3\}$$

Then G is a 2-connected graph with $t(G) = 1 > \frac{2}{3}$ and $\sum_{i=1}^{3} \deg_G u_i \geq |G| - 7$ for the 4-stable set S of order three, but G has no dominating cycle. This example shows that even under a stronger condition $t(G) \geq 1$, we cannot relax the condition on the degree sum. For $k \geq 3$, we do not know whether the degree bound $p-2k-2$ is best-possible.

For standard graph-theoretic terminology not explained in this paper, we refer the reader to [5]. For vertices x and y in a graph G, we denote the distance between x and y in G by $d_G(x, y)$. Let X and Y be disjoint subsets of $V(G)$. Then we denote by $e_G(X, Y)$ the number of edges which have one endvertex in X and the other in Y. The neighborhood of X, denoted by $N_G(X)$, is defined by $N_G(X) = \bigcup_{x \in X} N_G(x)$. The subgraph induced by X is denoted by $G[X]$. Let $P = x_0 x_1 \ldots x_k$ and $Q = y_0 y_1 \cdots y_l$ be paths in G. If $x_k = y_0$, the walk $x_0 x_1 \ldots x_k y_1 y_2 \cdots y_l$ is denoted by $x_0 P x_k Q y_l$. We define $\hat{\alpha}_k(G)$ by

$$\hat{\alpha}_k(G) = \max\{|S| \mid S \text{ is a } k\text{-stable set}\}.$$

2 Proof of Theorem 1

To prove Theorem 1, we use two known results. Chvátal-Erdős [6] showed that for $k \geq 2$, a k-connected graph G with independence number at most k and order at least three is hamiltonian. Broersma [4] proved a Chvátal-Erdős type theorem for the existence of a dominating cycle.

Theorem G (Broersma [4]) *Let G be a k-connected graph ($k \geq 2$). If $\hat{\alpha}_4(G) \leq k$, then G has a dominating cycle.*

We also use a degree sum condition for a balanced bipartite graph to be hamiltonian.

Theorem H (Moon and Moser [8]) *Let G be a balanced bipartite graph of order $2n$ with partite sets X and Y ($|X| = |Y| = n$). If $\deg_G x + \deg_G y \geq n+1$ for each $x \in X$ and $y \in Y$ with $xy \notin E(G)$, then G has a hamiltonian cycle.*

Proof of Theorem 1. Assume that G has no dominating cycle. By Theorem G, we have $\hat{\alpha}_4(G) > k$. Let $S = \{x_0, x_1, \ldots, x_k\}$ be a 4-stable set in $V(G)$, and let $B_i = \{x_i\} \cup N_G(x_i)$ ($0 \leq i \leq k$). For each i, j with $0 \leq i < j \leq k$, we have $B_i \cap B_j = \emptyset$ and $e_G(B_i, B_j) = 0$ since $d_G(x_i, x_j) \geq 4$. Let $R = V(G) - \bigcup_{i=0}^{k} B_i$. Then $G - R$ is disconnected, and the components of $G - R$ are $B_0, \ldots, B_k$. Since $t(G) > k/(k+1)$, $|R| \geq k+1$. On the other hand, since $N_G(S) \cap (R \cup \{x_0, \ldots, x_k\}) = \emptyset$, $p - 2k - 2 \leq \sum_{i=0}^{k} \deg_G x_i = |N_G(S)| = p - |R| - (k+1)$, which implies $|R| \leq k+1$. Therefore, we have $|R| = k+1$. Let $R = \{y_0, y_1, \ldots, y_k\}$.

Now contract each B_i into a single vertex b_i ($0 \leq i \leq k$) and remove all the edges in $G[R]$. Let H be the resulting graph. Then H is a balanced bipartite graph with partite sets $\{b_0, \ldots, b_k\}$ and R. Since G is k-connected, $|N_G(B_i) \cap R| \geq k$. This implies $\deg_H b_i \geq k$. If $\deg_H y_i \leq 1$ for some i, $0 \leq i \leq k$, then $N_G(y_i) \subset R \cup B_j$ for some j with $0 \leq j \leq k$. Without loss of generality, we may assume $j = 0$. Let $R' = R - \{y_i\}$. Then $G - R'$ has $k+1$ components $B_0 \cup \{y_i\}$, $B_1, \ldots, B_k$. Since $|R'| = k$, this contradicts the assumption that $t(G) > k/(k+1)$. Thus, we have $\deg_H y_i \geq 2$ for each y_i $(0 \leq i \leq k)$.

Now since we have $\deg_H b_i + \deg_H y_j \geq k+2$ for each i, j with $1 \leq i, j \leq k$, H has a hamiltonian cycle C' by Theorem H. Without loss of generality, we may assume $C' = y_0 b_0 y_1 b_1 \ldots y_k b_k y_0$. For each i, $0 \leq i \leq k$, and each $y_i b_i$ and $b_i y_{i+1}$, there exist vertices $z_i^{(1)}$, $z_i^{(2)} \in B_i$ with $y_i z_i^{(1)}$, $z_i^{(2)} y_{i+1} \in E(G)$ (indices modulo k). If $z_i^{(1)} \neq z_i^{(2)}$, let $P_i = y_i z_i^{(1)} x_i z_i^{(2)} y_{i+1}$. If $z_i^{(1)} = z_i^{(2)}$, let $P_i = y_i z_i^{(1)} y_{i+1}$. Then $C = y_0 P_0 y_1 P_1 \ldots y_k P_k y_0$ is a cycle in G.

We claim that C is a dominating cycle of G. Assume, to the contrary, that there exists a vertex v in G which is not dominated by C. Since $R \subset V(C)$ and $N_G(x_i) \cap V(C) \neq \emptyset$ for each i, $0 \leq i \leq k$, every vertex in $R \cup \{x_0, \ldots x_k\}$ is dominated by C. Therefore, $v \in N_G(x_{i_0})$ for some i_0, $0 \leq i_0 \leq k$. Since v is not dominated by C, $x_{i_0} \notin V(C)$ and hence $z_{i_0}^{(1)} = z_{i_0}^{(2)}$. Since $R \cup \{z_{i_0}^{(1)}\} \subset V(C)$, $N_G(v) \cap (R \cup \{z_{i_0}^{(1)}\}) = \emptyset$. This implies $N_G(v) \subset B_{i_0} - \{z_{i_0}^{(1)}\}$ and, if $i \neq i_0$, $d_G(v, x_i) \geq 4$. Let $S' = (S - \{x_{i_0}\}) \cup \{v\}$. Then S' is a 4-stable set of G of order $k+1$ and since $N_G(S') \cap (R \cup S' \cup \{z_{i_0}^{(1)}\}) = \emptyset$, we have $\sum_{x \in S'} \deg_G x \leq p - |R| - (k+1) - 1 = p - 2k - 3$. This contradicts the assumption, and hence G has a dominating cycle C of G. □

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