On The Hamiltonian Decomposition of Special Product of Graphs

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Abstract

We show that (a) the special product of two cycles is Hamiltonian decomposable, and (b) if G_1 and G_2 are two Hamiltonian decomposable graphs and at least one of their complements is Hamiltonian decomposable, then the special product of G_1 and G_2 is Hamiltonian decomposable.

1 Introduction.

Unless otherwise specified, all graphs considered here are finite and simple. We start by introducing the definitions of the following six basic product graphs. Let G_1 and G_2 be two graphs. (1) The Cartesian product $G = G_1 \times G_2$ has vertex-set $V(G) = V(G_1) \times V(G_2)$ and edge-set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)\}$. (2) The direct product $G = G_1 \wedge G_2$ has vertex-set $V(G) = V(G_1) \times V(G_2)$ and edge-set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)\}$. (3) The semi-strong product $G = G_1 \bullet G_2$ has vertex-set $V(G) = V(G_1) \times V(G_2)$ and edge-set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)\}$. (4) The Strong product $G = G_1 \otimes G_2$ has vertex-set $V(G) = V(G_1) \times V(G_2)$ and edge-set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$ or $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)\}$. (5) The lexicographic product $G = G_1[G_2]$ has vertex-set $V(G) = V(G_1) \times V(G_2)$ and edge-set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_1v_1 \in E(G_1)\}$. (6) The special product $G = G_1 \oplus G_2$ has vertex-set $V(G) = V(G_1) \times V(G_2)$

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and edge-set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2)\}$.

From the above definitions, it is clear to see that the operations (1), (2), (4) and (6) are commutative and we also have the following relations:

$$G_1 \times G_2 \subset G_1 \otimes G_2 \subset G_1[G_2] \subset G_1 \obar G_2,$$
$$G_1 \wedge G_2 \subset G_1 \otimes G_2 \subset G_1[G_2] \subset G_1 \obar G_2,$$

and

$$|E(G_1 \obar G_2)| = |V(G_1)|^2|E(G_2)| + |V(G_2)|^2|E(G_1)| - 2|E(G_1)||E(G_2)|.$$

The question whether the lexicographic product of Hamiltonian decomposable graphs is Hamiltonian decomposable or not was studied by Baranyai and Szasz in [1], they proved the following.

Theorem 1.1 *(Baranyai and Szasz) The lexicographic product of two Hamiltonian decomposable graphs is Hamiltonian decomposable.*

Kotzing in [6] could prove that the Cartesian product of two cycles is Hamiltonian decomposable into two Hamiltonian cycles. The next result due to Stong is a major progress to answer the question for Cartesian product.

Theorem 1.2 *(Stong) Let G_1 and G_2 be two graphs that are decomposable into s and t Hamiltonian cycles, respectively, with $s \leq t$. Then $G = G_1 \times G_2$ is Hamiltonian decomposable if one of the following holds: (1) $s \leq 3t$, (2) $t \geq 3$, (3) The order of G_2 is even, or (4) The order of G_1 is at least $6\lceil \frac{s}{t} \rceil - 3$.*

The question about the direct product, semi-strong product and strong product were answered partially by Bosak and Zhou. In fact, they proved the following theorems:

Theorem 1.3 *(Bosak and Zhou) $C \wedge C^*$ is Hamiltonian decomposable if and only if at least one of them is of odd order.*

Theorem 1.4 *(Bosak and Zhou) The direct product of two Hamiltonian decomposable graphs is Hamiltonian decomposable if at least one of them is of odd order.*

Theorem 1.5 *(Zhou) $C \bullet C^*$ is Hamiltonian decomposable where C^* is a cycle.*

Theorem 1.6 *(Zhou) The semi-strong product and the strong product of two Hamiltonian decomposable graphs are Hamiltonian decomposable if at least one of the two graphs is of odd order.*

In 1998, Cong Fan and Jiuqiang Liu could give a complete answer for the strong product. In fact, they proved the following theorem.

Theorem 1.7 *(C. Fan, and J. Liu) The strong product of Hamiltonian decomposable graphs is Hamiltonian decomposable.*

The following facts are very important (see [2]).

Fact 1.1 *If $u_1u_2 \in E(C_1)$ and $v_1v_2 \in E(C_2)$ where C_1 and C_2 are two vertex-disjoint cycles, then $(C_1 \cup C_2 - \{u_1u_2, v_1v_2\}) \cup \{u_1v_1, u_2, v_2\}$ is a cycle.*

Fact 1.2 *Given a Hamiltonian cycle C, let u_1u_2 and v_1v_2 be two non-adjacent edges of C. If the order of the four end vertices appearing on C is u_1, u_2, v_1, v_2 along a given direction, then $(C - \{u_1u_2, v_1v_2\}) \cup \{u_1v_1, u_2v_2\}$ is still a Hamiltonian cycle.*

Throughout this paper, we shall use the notation $F_{t,i}$ which is used by Baranyai and Szasz [1] and a new notation $D_{j,t}$ as follows: Let $V_1 = \{u_1, u_2, \ldots, u_{|V(G_1)|}\}$ and $V_2 = \{v_1, v_2, \ldots, v_{|V(G_2)|}\}$. Set

$$F_{t,i} = \{(u_t, v_j)(u_{t+1}, v_{j+i}) | j = 0, 1, \ldots, |V(G_2)| - 1\} \text{ where } t = 0, 1, \ldots, |V(G_1)| - 1.$$

and

$$D_{j,t} = \{(u_i, v_t)(u_{i+j}, v_{t+1}) | i = 0, 1, \ldots, |V(G_1)| - 1\} \text{ where } t = 0, 1, \ldots, |V(G_2)| - 1.$$

In this paper, we prove that the special product of two cycles is Hamiltonian decomposable. We also show that if G_1 and G_2 are two Hamiltonian decomposable and at least one of their complements is Hamiltonian decomposable, then $G_1 \Phi G_2$ is Hamiltonian decomposable.

2 Main Results.

Throughout the paper, we set $V_1 = V(C_1)$, $V_2 = V(C_2)$, $m = |V(C_1)|$ and $n = |V(C_2)|$. Let $C_1 = u_0u_1 \ldots u_{m-1}u_0$ and $C_2 = v_0v_1 \ldots v_{n-1}v_0$ be two cycles. Note that the following sets are subsets of $C_1 \Phi C_2$

$$R = \bigcup_{j=0}^{n-1} R_j \text{ where } R_j = \{(u_i, v_j)(u_{i+1}, v_j) | i = 0, 1, \ldots, m-1\},$$

$$CL = \bigcup_{i=0}^{m-1} CL_j \text{ where } CL_i = \{(u_i, v_j)(u_i, v_{j+1}) | i = 0, 1, \ldots, n-1\},$$

$$RO_i = \begin{cases} (\bigcup_{t=0, t\ even}^{m-1} F_{t,i}) \bigcup (\bigcup_{t=1,t\ odd}^{m-2} F_{t,-i}), & \text{if } m \text{ is odd,} \\ (\bigcup_{t=0,\ even}^{m-2} F_{t,i}) \bigcup (\bigcup_{t=1,t\ odd}^{m-1} F_{t,-i}), & \text{if } m \text{ is even,} \end{cases}$$

and

$$CO_j = \begin{cases} (\bigcup_{t=0,t\ even}^{n-1} D_{j,t}) \bigcup (\bigcup_{t=1,t\ odd}^{n-2} D_{-j,t}), & \text{if } n \text{ is odd,} \\ (\bigcup_{t=0,t\ even}^{n-2} D_{j,t}) \bigcup (\bigcup_{t=1,t\ odd}^{n-1} D_{-j,t}), & \text{if } n \text{ is even.} \end{cases}$$

It is easy to see that $\{RO_i\}_{i=1}^{m}$ is a pairwise -edge disjoint set. Similarly, $\{CO_i\}_{i=1}^{n}$ is a pairwise-edge disjoint. Also, $RO_0 = R$ and $CO_0 = C$. Now, to simplify the work we introduce the following definition.

Definition 2.1 *Let $M_1, M_2, \ldots, M_k$ be an ordering of subgraphs of the graph G. We say that $\{M_i\}_{i=1}^{k}$ is a semi-pairwise disjoint set if $V(M_i) \cap V(M_j) = \phi$ for $1 < |i-j| < k-1$ and $|V(M_i) \cap V(M_j)| = 1$ if $|i-j| = 1$ or $k-1$.*

Lemma 2.1 *(see [1]) Let m be even. For each $0 \leq i \leq n-1$, let G_i be a graph whose vertex-set is $V = V_1 \times V_2$ and whose edge-set is*

$$E(G_i) = \{(RO_i - F_{m-1,-i}) \bigcup (F_{m-1,-i+1})\}.$$

Then G_i is a Hamiltonian cycle.

Lemma 2.2 *For each $2 \leq j \leq m-3$, let H_j be a graph whose vertex-set $V = V_1 \times V_2$ and whose edge-set*

$$E(H_j) = \{(CO_j - D_{-j,n-1}) \bigcup (D_{-j-1,n-1})\}.$$

Then H_j is a Hamiltonian cycle if n is even.

Proof. Take the following paths

$$P_{j,k} = (u_{j+k}, v_{n-1})(u_k, v_{n-2})(u_{j+k}, v_{n-3}) \ldots (u_k, v_0)(u_{j+k+1}, v_{n-1}).$$

Note that $\{P_{j,k}\}_{k=0}^{m-1}$ is a pairwise-edge disjoint set of subgraphs of H_j. Moreover, they are semi-pairwise disjoint. In fact, $V(P_{j,k}) \cap V(P_{j,k+1}) = (u_{j+k+1}, v_{n-1})$ and $V(P_{j,m-1}) \cap V(P_{j,0}) = (u_{j+1}, v_{n-1})$. Thus, H_j is a cycle. Also,

$$\begin{aligned} |E(H_j)| &= |E(CO_j)| - |E(D_{-j,n-1})| + |E(D_{-j-1,n-1})| \\ &= \sum_{t=0}^{n-1} |E(D_{j,t})| \\ &= \sum_{t=0}^{n-1} m \\ &= mn \\ &= |V(C_1 \oplus C_2)|. \end{aligned}$$

Thus, we get our results.

Lemma 2.3 *Let G_1^* be a graph whose vertex-set is $V = V_1 \times V_2$ and edge-set is*

$$E(G_1^*) = \{(CO_{m-2} - D_{-m+2,n-1}) \bigcup (\{(u_i, v_0)(u_{i+1}, v_0) | i = 1, 3, \ldots, m-1\})$$

$$\bigcup (\{(u_i, v_{n-1})(u_{n-1}, v_0) | i = 0, 2, \ldots, m-2\})\}.$$

If m is even, then G_1^* is a Hamiltonian cycle.

Proof. Note that

$$(CO_{m-2} - D_{-m+2,n-1}) = \bigcup_{l=0}^{m-1} P_l^*$$

where

$$P_l^* = (u_l, v_0)(u_{l+m-2}, v_1)(u_l, v_2)(u_{l+m-2}, v_3) \ldots (u_{l+n-2}, v_{n-1})$$

if n is even, and

$$P_l^* = (u_l, v_0)(u_{l+m-2}, v_1)(u_l, v_2)(u_{l+m-2}, v_3) \ldots (u_l, v_{n-1})$$

if n is odd. Clearly, $\{P_l^*\}_{l=0}^{m-1}$ is a family of pairwise-vertex disjoint paths. Now, construct the following semi-pairwise disjoint paths as follows:

$$P_l = P_l^* \bigcup (u_{l+m-2}, v_{n-1})(u_{l+m-1}, v_{n-1}) \text{ if } l = 0, 2, \ldots, m-2 \text{ and } n \text{ is}$$

even

$$P_l = P_l^* \bigcup (u_l, v_{n-1})(u_{l+1}, v_{n-1}) \text{ if } l = 0, 2, \ldots, m-2 \text{ and } n \text{ is odd,}$$

and

$$P_l = P_l^* \bigcup (u_l, v_0)(u_{l+1}, v_0) \text{ if } l = 1, 3, \ldots, m-1.$$

Note that, if n is even, then

$$V(P_l) \cap V(P_{l+1}) = \begin{cases} (u_{l+m-1}, v_{n-1}) & \text{If } l = 0, 2, \ldots, m-2, \\ (u_{l+1}, v_0) & \text{If } l = 1, 3, \ldots, m-1, \end{cases}$$

and if n is odd, then

$$V(P_l) \cap V(P_{l+1}) = \begin{cases} (u_{l+1}, v_{n-1}) & \text{If } l = 0, 2, \ldots, m-2, \\ (u_{l+1}, v_0) & \text{If } l = 1, 3, \ldots, m-1. \end{cases}$$

Thus,

$$\bigcup_{l=0}^{m-1} P_l$$

is a cycle. Now,

$$\begin{aligned} |E(G_1^*)| &= |E(CO_{|V(C_1)|-2}| - |D_{-m+2,n-1}| + \frac{m}{2} + \frac{m}{2} \\ &= mn. \end{aligned}$$

Thus G_1^* is a Hamiltonian cycle.

Lemma 2.4 *Let G_2^* be a graph whose vertex-set is $V = V_1 \times V_2$ and edge-set is*

$$E(G_2^*) = \{E(G_1) - \{(u_0, v_0)(u_{m-1}, v_0), (u_0, v_1)(u_{m-1}, v_1)\} \bigcup \{(u_0, v_0)(u_0, v_1), (u_{m-1}, v_0)(u_{m-1}, v_1)\}\}.$$

If m is even, then G_2^* is a Hamiltonian cycle.

Proof. Just apply Fact 2, and note that

$$\begin{aligned} |E(G_2^*)| &= |RO_1| - 2 + 2 \\ &= mn. \end{aligned}$$

Lemma 2.5 *Let G_3^* be a graph whose vertex-set is $V = V_1 \times V_2$ and edge-set is*

$$\begin{aligned} E(G_3^*) = \{E(CL) - \{\{(u_i, v_0)(u_i, v_{n-1})|i = 1, 2, \ldots, m-2\}, (u_0, v_0)(u_1, v_0), \\ (u_{m-1}, v_0)(u_{m-1}, v_1)\} \bigcup \{\{(u_i, v_0)(u_{i+1}, v_0)|i = 0, 2, \ldots, m-2\} \bigcup \\ \{(u_i, v_{n-1})(u_{i+1}, v_{n-1})|i = 1, 3, \ldots, m-3\} \bigcup \{(u_0, v_1)(u_{m-1}, v_1)\}\}\}. \end{aligned}$$

If m is even, then G_3^* is a Hamiltonian cycle.

Proof. Consider the following paths: For $l = 1, 2, \ldots, m-2$;

$$P_l^* = (u_l, v_0)(u_l, v_1)(u_l, v_2) \ldots (u_l, v_{n-1}),$$

$$P_0^* = (u_0, v_0)(u_0, v_{n-1})(u_0, v_{n-2}) \ldots (u_0, v_1),$$

and

$$P_{m-1}^* = (u_{m-1}, v_0)(u_{m-1}, v_{n-2}) \ldots (u_{m-1}, v_0).$$

Note that $\{P_l\}_{l=0}^{m-1}$ is a family of pairwise-vertex disjoint subgraphs of G_3^*. Now construct the following semi-pairwise disjoint paths as follows:

$$P_l = P_l^* \bigcup (u_l, v_0)(u_{l+1}, v_0) \text{ if } l = 0, 2, \ldots, m-2,$$

$$P_l = P_l^* \bigcup (u_l, v_{m-1})(u_{l+1}, v_{m-1}) \text{ if } l = 1, 3, \ldots, m-3,$$

and

$$P_{m-1} = P_{m-1}^* \bigcup (u_0, v_1)(u_{m-1}, v_1).$$

In fact,

$$V(P_l) \cap V(P_{l+1}) = \begin{cases} (u_{l+1}, v_0), & \text{if } l = 2, 4, \ldots, m-2, \\ (u_{l+1}, v_{n-1}), & \text{if } l = 1, 3, \ldots, m-3, \end{cases}$$

and

$$V(P_{m-1}) \cap V(P_0) = (u_0, v_1).$$

Thus $\cup_{l=0}^{m-1} P_l$ is a cycle and since

$$\begin{aligned} |E(\cup_{l=0}^{m-1} P_l)| &= \sum_{l=0}^{m-1} |E(P_l)| \\ &= \sum_{l=0}^{m-1} (|V(P_l^*)| + 1). \\ &= mn, \end{aligned}$$

G_3^* is a Hamiltonian cycle.

Lemma 2.6 *Let G_4^* be a graph whose vertex-set is $V = V_1 \times V_2$ and edge-set is*

$$E(G_4^*) = \{(E(G_0) - \{(u_i, v_j)(u_{i+1}, v_j) | i = 0, 1, 2, \ldots, m-2; j = 0, n-1\})$$
$$\bigcup (D_{-2,n-1}) \bigcup \{(u_i, v_0)(u_i, v_{n-1}) | i = 1, 2, \ldots, m-2\}\}.$$

If m is even, then G_4^* is a Hamiltonian cycle.

Proof. Define the following paths

$$P_1 = RO_0 - \{(u_i, v_j)(u_{i+1}, v_j) | i = 0, 1, \ldots, m-1\},$$

and

$$P_2 = D_{-2,n-1} \bigcup \{(u_i, v_0)(u_i, v_{n-1}) | i = 1, 2, \ldots, m-2\}$$
$$\bigcup \{(u_0, v_0)(u_{m-1}, v_{n-1})\}.$$

Since $V(P_1) \cap V(P_2) = \{(u_{m-1}, v_0), (u_0, v_{n-1})\}$ which are the end points of both of P_1 and P_2, $P_1 \cup P_2$ is a cycle. Now

$$\begin{aligned} |E(P_1)| + |E(P_2)| &= |E(RO_0)| - 2(m-1) + |E(D_{-2,n-1})| + m - 2. \\ &= mn - 2m + 2 + m + m - 2. \\ &= mn. \end{aligned}$$

Therefore, G_4^* is a Hamiltonian cycle.

Lemma 2.7 *Let G^* be a graph whose vertex-set is $V = V_1 \times V_2$ and edges-set is*

$$E(G^*) = \{E(G_0 \bigcup G_1) \bigcup C \bigcup (CO_{m-2} - D_{-m+2,n-1}) \bigcup D_{-2,n-1}\}.$$

If m and n are even, then G^* is decomposable into four Hamiltonian cycles.

Proof. Just note that $G^* = G_1^* \cup G_2^* \cup G_3^* \cup G_4^*$.

Since the proof of the following Lemma is not hard to see, we shall not prove it.

Lemma 2.8 $G_1 \Phi G_2 = (G_1 \times G_2) \cup (G_1 \wedge G_2) \cup (\bar{G}_1 \wedge G_2) \cup (G_1 \wedge \bar{G}_2)$ *and* $G_1[G_2] = (G_1 \times G_2) \cup (G_1 \wedge G_2) \cup (G_1 \wedge \bar{G}_2)$, *so* $G_1 \Phi G_2 = G_1[G_2] \cup (\bar{G}_1 \wedge G_2)$.

Lemma 2.9 *If $|V(C)|$ is odd, then $\bar{C}$ is a Hamiltonian decomposable.*

Theorem 2.1 *For each two cycles C_1 and C_2, $C_1 \Phi C_2$ is Hamiltonian decomposable into $m + n - 2$ Hamiltonian cycles.*

Proof. We need to consider three cases:
Case 1: If m is odd, then $\bar{C}_1$ is Hamiltonian decomposable, and by Theorem 1.4 and 1.1 $\bar{C}_1 \wedge C_2$ and $C_1[C_2]$ are Hamiltonian Decomposable. Thus, by Lemma 2.8 we get our result.
Case 2: If n is odd and m is even, then by noting that $C_1 \Phi C_2$ isomorphic to $C_2 \Phi C_1$ we have our result.
Case 3: If both m and n are even, then consider the following set of Hamiltonian cycles

$$\{G_i\}_{i=2}^{n-1} \bigcup \{H_i\}_{i=2}^{m-3} \bigcup G^*.$$

Clearly, they are edge-pairwise-disjoint and they are subgraphs of $C_1 \Phi C_2$. Now,

$$|E(\bigcup_{i=2}^{n-1} G_i)| + |E(\bigcup_{i=2}^{m-3} H_i)| + |E(G^*)| =$$
$$\sum_{i=2}^{n-1} |E(G_i)| + \sum_{i=2}^{m-3} |E(H_i)| + \sum_{i=1}^{4} |E(G_i^*)| =$$
$$\sum_{i=2}^{n-1} mn + \sum_{i=2}^{m-3} mn + 4mn =$$
$$(n-2)mn + (n-4)mn + 4mn =$$
$$m^2 n + n^2 m - 2mn =$$
$$|E(C_1 \Phi C_2)|.$$

Thus, $C_1 \Phi C_2$ is Hamiltonian decomposable into $m + n - 2$ Hamiltonian cycles.

Theorem 2.2 *Let G_1 and G_2 be two Hamiltonian decomposable graphs. Then $G_1 \oplus G_2$ is Hamiltonian decomposable into*

$$(\frac{d_{G_1}(u)|V(G_2)| + d_{G_2}(v)|V(G_1)| - d_{G_1}(u)d_{G_2}(v)}{2}),$$

if one of $\bar{G}_1$ and $\bar{G}_2$ is Hamiltonian decomposable where (u, v) is any vertex in $G_1 \oplus G_2$.

Proof. Just apply Lemma 2.8, Theorems 1.1 and 1.4 and the fact that if G and $\bar{G}$ are Hamiltonian decomposable, then G is of odd order.

Corollary 2.1 *If C is of odd order cycle and G is Hamiltonian decomposable, then $C \oplus G$ is Hamiltonian decomposable into*

$$|V(G)| - d_G(v) + \frac{d_G(v)|V(C)|}{2}.$$

Corollary 2.2 *For some integer n, $C_1 \oplus C_2 \oplus \ldots \oplus C_n$ is Hamiltonian decomposable, if all C_i are of odd length, except possibly one cycle is even or two consecutive cycles are even.*

Proof. Just apply Theorem 2.1 and Corollary 2.1.

Lemma 2.10 *If $G = \cup_{i=1}^{r} G_i$ where G_{i_0} is a Hamiltonian cycle for some i_0, then $H \oplus G = (H \oplus C_{i_0}) \cup (\bar{H} \wedge (\cup_{i=1 \text{ and } i \neq i_0}^{r} G_i)) \cup (N \times (\cup_{i=1 \text{ and } i \neq i_0}^{r} G_i))$ where N is a null graph whose vertex set is $V(H)$.*

Proof. Note that,

$$(H \oplus G_{i_0}) \bigcup (\bar{H} \wedge (\bigcup_{i=1 \text{ and } i \neq i_0}^{r} G_i)) \bigcup (N \times (\bigcup_{i=1 \text{ and } i \neq i_0}^{r} G_i) =$$

$$(H \times G_{i_0}) \bigcup (H \wedge G_{i_0}) \bigcup (\bar{H} \wedge G_{i_0}) \bigcup (H \wedge \bar{G}_{i_0})$$

$$\bigcup (N \times (\bigcup_{i=1 \text{ and } i \neq i_0}^{r} G_i)) \bigcup (H \wedge K_{|V(G)|}) \bigcup (\bar{H} \wedge G) =$$

$$(H \times G) \bigcup (H \wedge K_{|V(G_{i_0})|}) \bigcup (\bar{H} \wedge G) =$$

$$H \oplus G.$$

Theorem 2.3 *If C is of even order cycle and G is of odd order which can be decomposed into r Hamiltonian cycles, then $C \oplus G$ can be decomposed into $r|V(C)| + |V(G)| - 3r + 1$ Hamiltonian cycles, and $r - 1$ 2-factors.*

Proof. Sine $\bar{C} = (\bigcup_{i=2}^{\frac{|V(C)|}{2}-1} C_i') \cup M$ and Lemma 2.10, we have

$$(C \Phi G) = (C \Phi C_1) \bigcup$$

$$(((\bigcup_{i=2}^{\frac{|V(C)|}{2}-1} C_i') \bigcup M) \wedge (\bigcup_{j=2}^{r} C_j)) \bigcup (N \times (\bigcup_{j=2}^{r} C_j)) =$$

$$(C \Phi C_1) \bigcup ((\bigcup_{i=2}^{\frac{|V(C)|}{2}-1} C_i') \wedge (\bigcup_{j=2}^{r} C_j)) \bigcup (N \times (\bigcup_{j=2}^{r} C_j))$$

$$\bigcup (M \wedge (\bigcup_{j=2}^{r} C_j))$$

where M is a perfect matching, C_i' is a Hamiltonian cycle and N is a null graph whose vertex set is $V(C)$. Now,

$$((\bigcup_{i=2}^{\frac{|V(C)|}{2}-1} C_i') \wedge (\bigcup_{j=2}^{r} C_j)) \bigcup (N \times (\bigcup_{j=2}^{r} C_j)) =$$

$$((\bigcup_{i=3}^{\frac{|V(C)|}{2}-1} C_i' \wedge (\bigcup_{j=2}^{r} C_j)) \bigcup (C_3 \wedge (\bigcup_{j=2}^{r} C_j)) \bigcup (N \times (\bigcup_{j=2}^{r} C_j)) =$$

$$(\bigcup_{i=3}^{\frac{|V(C)|}{2}-1} \bigcup_{j=2}^{r} (C_i' \wedge C_j)) \bigcup (\bigcup_{j=2}^{r} (C_i' \bullet C_j)).$$

Now, by Theorems 2.1, 1.3 and 1.5, we have $(C \Phi C_1) \cup ((\bigcup_{i=2}^{\frac{|V(C)|}{2}-1} C_i') \wedge (\bigcup_{j=2}^{r} C_j)) \cup (N \times (\bigcup_{j=2}^{r} C_j))$ is a Hamiltonian decomposition. To this end, clearly $(M \wedge (G - C_1)$ is a $(m-1)$ 2-factors.
The following is an easy consequence of fact2.

Lemma 2.11 *Let* $G_4^{**} = G_4^* - \{(u_l, v_{j_1})(u_{l+1}, v_{j_1}), (u_{l+2}, v_{j_2})(u_{l+3}, v_{j_2})\} \cup \{(u_l, v_{j_1})(u_{l+2}, v_{j_2}), (u_{l+1}, v_{j_2})(u_{l+3}, v_{j_2})\}$ *or* $G_4^{**} = G_4^* - \{(u_l, v_{j_2})(u_{l+1}, v_{j_2}), (u_{l+2}, v_{j_1})(u_{l+3}, v_{j_1})\} \cup \{(u_l, v_{j_2})(u_{l+2}, v_{j_1}), (u_{l+1}, v_{j_2})(u_{l+3}, v_{j_1})\}$ *where* $0 < j_1 < j_2 < |V(C_2)| - 1$ *or* $0 > j_1 > j_2 > |V(C_2)| - 1$ *and for some* l*. Then* G_4^{**} *is a Hamiltonian cycle.*

Lemma 2.12 *Let* $G = C_1 \cup C_2$ *where* C_1 *and* C_2 *are even Hamiltonian cycles of* G*. If there are two edges* $v_{j_1}v_{j_4}$ *and* $v_{j_2}v_{j_3} \in E(G)$ *so that* $0 < j_1 < j_2 < j_3 < j_4 < |V(G)| - 1$ *or* $0 > j_1 > j_2 > j_3 > j_4 > |V(G)| - 1$*. Then* $C \Phi G$ *is Hamiltonian decomposable.*

Proof. By Corollary 2.1 we may assume that C is even. Since $C \oplus G = (C \oplus C_1) \cup (\bar{C} \wedge C_2) \cup (N \times C_2)$ and $(C \oplus C_1) - G_4^*$ is Hamiltonian decomposable, it suffices to show that $(\bar{C} \wedge C_2) \cup (N \times C_2) \cup G_4^*$ is Hamiltonian decomposable. Note that $\bar{C} \wedge C_2$ isomorphic to $\bar{C} \wedge C_1$, thus $\bar{C} \wedge C_2$ isomorphic to $\bigcup_{i=2}^{|V(G)|-2} CO_i$. Therefore

$$\bar{C} \wedge C_2 = (\bigcup_{j=2}^{|V(G)|-3} H_j') \bigcup (CO'_{|V(G)|-2} - D'_{|V(C)|-2,|V(G)|-1})$$

$$\bigcup (D'_{-2,|V(G)|-1})$$

where H_j' isomorphic to H_j, $CO'_{|V(G)|-2}$ isomorphic to $CO_{|V(G)|-2}$, and $D'_{i,t}$ isomorphic to $D_{i,t}$. Let

$$C_2 = v_{i_0} v_{i_1} \dots v_{i_{|V(C)|}}.$$

Then

$$(N \times C_2) \bigcup (CO'_{|V(C_1)|-2} - D'_{|V(C)|-2,|V(G)|-1}) \bigcup (D'_{-2,|V(G)|-1}) = \bigcup_{i=1}^{4} C_i^*$$

where

$$C_1^* = (u_0, v_{i_{|V(C)|-1}})(u_2, v_{i_{|V(C)|-2}})(u_0, v_{i_{|V(C)|-3}}) \dots (u_2, v_{i_0})(u_2, v_{i_{|V(C)|-1}}) (u_4, v_{i_{|V(C)|-2}}) \dots (u_4, v_{i_0})(u_4, v_{i_{|V(C)|-1}}) \dots (u_{|V(C)|-2}, v_{i_0})(u_{|V(C)|-2}, v_{i_{|V(C)|-1}})(u_0, v_{i_{|V(C)|-2}})(u_{|V(C)|-2}, v_{i_{|V(C)|-3}}) \dots (u_0, v_0)(u_0, v_{i_{|V(C)|-1}}),$$

$$C_2^* = (u_1, v_{i_{|V(C)|-1}})(u_3, v_{i_{|V(C)|-2}})(u_1, v_{i_{|V(C)|-3}}) \dots (u_3, v_{i_0})(u_3, v_{i_{|V(C)|-1}}) (u_5, v_{i_{|V(C)|-2}}) \dots (u_5, v_{i_0})(u_5, v_{i_{|V(C)|-1}}) \dots (u_{|V(C)|-1}, v_{i_0})(u_{|V(C)|-1}, v_{i_{|V(C)|-1}})(u_1, v_{i_{|V(C)|-2}})(u_{|V(C)|-1}, v_{i_{|V(C)|-3}}) \dots (u_1, v_{i_0})(u_1, v_{i_{|V(C)|-1}}),$$

$$C_3^* = (u_o, v_{i_{|V(C)|-1}})(u_0, v_{i_{|V(C)|-2}})(u_0, v_{i_{|V(C)|-3}}) \dots (u_0, v_{i_0})(u_2, v_{i_{|V(C)|-1}}) (u_2, v_{i_{|V(C)|-2}}) \dots (u_2, v_{i_0})(u_4, v_{i_{|V(C)|-1}}) \dots (u_{|V(C)|-2}, v_{i_0})(u_0, v_{i_{|V(C)|-1}}),$$

and

$$C_4^* = (u_1, v_{i_{|V(C)|-1}})(u_1, v_{i_{|V(C)|-2}})(u_1, v_{i_{|V(C)|-3}}) \dots (u_1, v_{i_0})(u_3, v_{i_{|V(C)|-1}}) (u_3, v_{i_{|V(C)|-2}}) \dots (u_3, v_{i_0})(u_5, v_{i_{|V(C)|-1}}) \dots (u_{|V(C)|-1}, v_{i_0})(u_1, v_{i_{|V(C)|-1}}).$$

Note that one of C_1^* and C_2^* contains at least one of $(u_l, v_{j_1})(u_{l+2}, v_{j_4})$ and $(u_{l+2}, v_{j_1})(u_l, v_{j_4})$, say the first, and so the other contains $(u_{l+1}, v_{j_1})(u_{l+3}, v_{j_4})$. G_4^* contains $(u_l, v_{j_1})(u_{l+1}, v_{j_1})$, $(u_l, v_{j_4})(u_l, v_{j_4})$, $(u_{l+2}, v_{j_4})(u_{l+3}, v_{j_4})$ and $(u_{l+2}, v_{j_1})(u_{l+3}, v_{j_1})$ for some l and they take the following forms:

$$\dots (u_l, v_{j_1})(u_{l+1}, v_{j_1}) \dots (u_{l+2}, v_{j_4})(u_{l+3}, v_{j_4}) \dots .$$

Thus by fact (1) and (2) and Lemma 2.11, we have that

$$\{C_1^* \bigcup C_2^* - \{(u_l, v_{j_1})(u_{l+2}, v_{j_1}), (u_{l+1}, v_{j_1})(u_{l+3}, v_{j_4})\}\} \bigcup \{(u_l, v_{j_1})$$

$$(u_{l+1}, v_{j_1}), (u_{l+2}, v_{j_4})(u_{l+3}, v_{j_4})\}$$

and

$$C_4^{**} = \{C_4^* - \{(u_l, v_{j_1})(u_{l+1}, v_{j_1}), (u_{l+2}, v_{j_4})(u_{l+3}, v_{j_4})\}\} \bigcup \{(u_l, v_{j_1})$$

$$(u_{l+2}, v_{j_4}), (u_{l+1}, v_{j_1})(u_{l+3}, v_{j_4})\}$$

are two Hamiltonian cycles. Similarly, one of C_3^* and C_4^* contains (u_k, v_{j_2}) (u_k, v_{j_3}) and the other contains $(u_{k+1}, v_{j_2})(u_{k+1}, v_{j_3})$, C_4^{**} contains (u_k, v_{j_2}) (u_{k+1}, v_{j_2}), $(u_k, v_{j_3})(u_{k+1}, v_{j_3})$, $(u_k, v_{j_3})(u_{k+1}, v_{j_3})$ and $(u_k, v_{j_2})(u_{k+1}, v_{j_2})$ which are not adjacent in C_4^{**} and take the form:

$$\ldots (u_k, v_{j_2})(u_{k+1}, v_{j_2}) \ldots (u_k, v_{j_3})(u_{k+1}, v_{j_3}) \ldots$$

Thus by using Facts (1) and (2) and Lemma 2.11 we get

$$\{C_3^* \bigcup C_4^* - \{(u_k, v_{j_2})(u_k, v_{j_3}), (u_{k+1}, v_{j_2})(u_{k+1}, v_{j_3})\}\} \bigcup \{(u_k, v_{j_2})$$

$$(u_{k+1}, v_{j_2}), (u_k, v_{j_3})(u_{k+1}, v_{j_3})\}$$

and

$$\{C_4^{**} - \{(u_k, v_{j_2})(u_{k+1}, v_{j_2}), (u_k, v_{j_3})(u_{k+1}, v_{j_3})\}\} \bigcup \{(u_k, v_{j_2})(u_k, v_{j_3}),$$

$$(u_{k+1}, v_{j_2})(u_{k+1}, v_{j_3})\}$$

are Hamiltonian cycles, and so $C \Phi G$ is Hamiltonian decomposable.

Theorem 2.4 *Let*

$$G = C^* \bigcup (\bigcup_{i=1}^{r} C_i)$$

where C^* and C_i are even Hamiltonian cycles for G. If there are $l_1, l_2, \ldots, l_{4r}$ such that $1 \leq l_1 < l_2 \ldots < l_{4r} \leq |V(G)| - 2$ or $1 \geq l_1 > l_2 \ldots > l_{4r} \geq |V(G)| - 2$ and C_i contains $v_{l_i} v_{l_{4r-i+1}}$ and $v_{l_{i+1}} v_{l_{4r-i}}$ for each $i = 1, 3, \ldots, 2r+1$, Then $C \Phi G$ has a Hamiltonian decomposition.

Proof. Note that $C \Phi G = (C \Phi C^*) \cup (\cup_{i=1}^{r} [(\bar{C} \wedge C_i) \cup (N \times C_i)])$, so apply Lemma 2.12 for $(\bar{C} \wedge C_i) \cup (N \times C_i)$ for each $i = 1, 2, \ldots, r$.

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