Average lower independence in trees and outerplanar graphs

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Abstract

For a vertex v of a graph $G = (V, E)$, the lower independence number $i_v(G)$ of G relative to v is the minimum cardinality of a maximal independent set in G that contains v. The average lower independence number of G is $i_{av}(G) = \frac{1}{|V|}\sum_{v\in V} i_v(G)$. In this paper, we show that if G is a tree of order n, then $i_{av}(G) \geq 2\sqrt{n} + O(1)$, while if G is an outerplanar graph of order n, then $i_{av}(G) \geq 2\sqrt{n/3} + O(1)$. Both bounds are asymptotically sharp.

Keywords: *average lower independence, outerplanar graphs, trees*

*Research supported in part by the University of Natal.
†Research supported in part by the University of Natal and the South African National Research Foundation.

1 Introduction

In this paper, we consider the concept of average independence in graphs, a concept closely related to the problem of finding large independent sets in graphs. The independent domination number $i(G)$ of a graph G can be viewed as a worst case bound on the performance of the 'naive' greedy-algorithm for approximating a maximum independent set of G: choose a vertex v, let $S = \{v\}$, and add vertices to S, one at a time, which are not adjacent to any vertex already in S. The algorithm stops when S is a maximal independent set. The class of those graphs for which this 'naive' greedy-algorithm always yields a maximum independent set is exactly the class of well-covered graphs. A graph is well-covered if every maximal independent set is also a maximum independent set. The study of well-covered graphs was proposed by Plummer [7].

The lower bound $i(G)$ on the cardinality of an independent set obtained by the 'naive' greedy-algorithm can be improved if one takes into account that the first vertex is chosen randomly. For a vertex v let $i_v(G)$ be the cardinality of a smallest maximal independent set containing v. Then $i_v(G)$ is a worst case bound on the cardinality of an independent set obtained by the 'naive' greedy-algorithm if we use v as a start vertex. We define the *lower average independence number* $i_{av}(G)$ by

$$i_{av}(G) = \frac{1}{n} \sum_{v \in V} i_v(G),$$

where $V = V(G)$ is the vertex set of the graph G and $n = |V|$. Then $i_{av}(G)$ is a lower bound on the expected value of the cardinality of the independent set obtained by the 'naive' greedy-algorithm if the first vertex is chosen randomly. The lower average independence number was first investigated in [6] where trees with equal lower average independence number and average domination number are characterized (the average domination number is defined in an analogous way as the lower average independence number).

Our aim in this paper is to determine lower bounds on the average lower independence number of a tree and an outerplanar graph in terms of their order. We show that if G is a tree of order n, then

$i_{av}(G) \geq 2\sqrt{n} + O(1)$, while if G is an outerplanar graph of order n, then $i_{av}(G) \geq 2\sqrt{n/3} + O(1)$. Both lower bounds can be achieved.

1.1 Notation

For notation and graph theory terminology we in general follow [4]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E of size q, and let v be a vertex in V. The open neighborhood of v is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* $N[S] = N(S) \cup S$. The subgraph of G induced by the vertices in S is denoted by $\langle S \rangle$. A *leaf* is a vertex of degree one and its neighbor is called a *support vertex.* We define a *branch vertex* as a vertex of degree at least 3. The set of branch vertices of a tree T is denoted by $B(T)$.

A set S is a *dominating set* of G if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to a vertex of S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. The *independence number* $\beta(G)$ of G is the maximum cardinality of an independent set in G, while the *lower independence number* (also called the *independent domination number*) $i(G)$ of G is the minimum cardinality of a maximal independent set of G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set, while a maximal independent set of cardinality $i(G)$ is called an $i(G)$-set.

For every graph G, $i(G) \leq i_v(G) \leq \beta(G)$, and so $i(G) \leq i_{av}(G) \leq \beta(G)$. Furthermore, $i_{av}(G) = \beta(G)$ if and only if the graph G is well-covered. Fricke, Haynes, Hedetniemi, Hedetniemi, and Laskar [3] defined a graph G to be *i-excellent* if $i_v(G) = i(G)$ for every vertex v of G. Hence, a graph G is *i-excellent* if $i(G) = i_{av}(G)$. A constructive characterization of i-excellent trees is given in [5].

2 Trees

Our aim in this section is to provide a lower bound on the average lower independence number of a tree in terms of its order. For this purpose we shall need the following lemma.

Lemma 1 *If T is a tree of order n satisfying $i(T) \leq 2\sqrt{n-1}$, then T has at least $n - 6\sqrt{n-1}$ leaves.*

Proof. Let I be an $i(T)$-set, and let n_1 denote the number of leaves in T. Then,

$$\begin{aligned} n_1 &= 2 + \textstyle\sum_{v \in B(T)} (\deg v - 2) \\ &\geq 2 + \textstyle\sum_{v \in I} (\deg v - 2) \\ &= 2 - 3|I| + \textstyle\sum_{v \in I} (\deg v + 1) \\ &\geq 2 - 3|I| + n \\ &\geq n - 6\sqrt{n-1} + 2. \end{aligned}$$

Hence, T has at least $n - 6\sqrt{n-1}$ leaves. □

Theorem 2 *If T is a tree of sufficiently large order n, then*

$$i_{av}(T) > 2\sqrt{n-1} - 12.$$

Proof. Let $T = (V, E)$. If $i(T) > 2\sqrt{n-1}$, then the result follows immediately since $i_{av}(G) \geq i(G)$ for every graph G. Hence, we may assume that $i(T) \leq 2\sqrt{n-1}$. Let L denote the set of all leaves in T and let $|L| = n_1$. By Lemma 1, $n_1 \geq n - 6\sqrt{n-1}$. Now let $S = \{v_1, v_2, \ldots, v_k\}$ be the set of support vertices. For $i = 1, 2, \ldots, k$, let L_i denote the set of leaves adjacent to v_i and let $|L_i| = \ell_i$. Then, $n_1 = \sum_{j=1}^{k} \ell_j$.

Let $u \in L_j$ where $1 \leq j \leq k$. Then any maximal independent set containing u must contain L_j and at least one vertex from each of the sets $L_i \cup \{v_i\}$ for $i \in \{1, 2, \ldots, k\} - \{j\}$. Hence, $i_u(T) \geq \ell_j + k - 1$. It follows that

$$\begin{aligned}
\sum_{v \in L} i_v(T) &\geq \sum_{j=1}^{k} \ell_j(\ell_j + k - 1) \\
&= (\sum_{j=1}^{k} \ell_j^2) + (k-1) \sum_{j=1}^{k} \ell_j \\
&\geq k \left(\frac{n_1}{k}\right)^2 + (k-1)n_1 \\
&= \frac{n_1^2}{k} + (k-1)n_1.
\end{aligned}$$

For each vertex $v \in V - L$, $i_v(T) \geq k$. Since T is a tree, $\langle V - L \rangle$ is a tree. For $j = 1, 2, \ldots, k$, let w_j be a neighbor of v_j in $\langle V - L \rangle$. Any maximal independent set containing w_j must contain L_j and at least one vertex from each of the sets $L_i \cup \{v_i\}$ for $i \in \{1, 2, \ldots, k\} - \{j\}$. Hence, $i_{w_j}(T) \geq \ell_j + k - 1$. It follows that the union of the $i_{w_j}(T)$-sets over all $j = 1, 2, \ldots, k$ contains the entire set L. Consequently,

$$\sum_{v \in V - L} i_v(T) \geq k\,|V - L| - k + |L| = k(n - n_1 - 1) + n_1.$$

Hence,

$$\sum_{v \in V} i_v(T) \geq \left(\frac{n_1^2}{k} + (k-1)n_1\right) + ((n - n_1 - 1)k + n_1) = \frac{n_1^2}{k} + k(n-1).$$

For constant n the expression $f(k) = k(n-1) + n_1^2/k$ is minimized when $k = n_1/\sqrt{n-1}$. Thus, by Lemma 1,

$$
\begin{aligned}
\sum_{v \in V} i_v(T) &\geq 2n_1\sqrt{n-1} \\
&\geq 2\left(n - 6\sqrt{n-1}\right)\sqrt{n-1} \\
&= 2n\sqrt{n-1} - 12(n-1).
\end{aligned}
$$

Consequently,

$$
i_{av}(T) = \frac{1}{n}\sum_{v \in V} i_v(T) \geq 2\sqrt{n-1} - 12 + \frac{12}{n} > 2\sqrt{n-1} - 12,
$$

as desired. □

It remains an open question to determine a sharp lower bound on the average lower independence number of a tree. We believe the lower bound can be improved by about 9.

Conjecture 1 *If T is a tree of order $n \geq 1$, then*

$$
i_{av}(T) \geq 2\sqrt{n} - 3 + \frac{2}{\sqrt{n}}.
$$

If Conjecture 1 is true, then the bound is sharp. To see this, let $k \geq 2$ be an integer. Let $T = (V, E)$ be the tree obtained from the disjoint union of k stars $K_{1,k-1}$ by adding $k - 1$ edges between the leaves of different stars. Then, T is a tree of order $n = k^2$. If v is the center of one of the k original stars, then $i_v(T) = k$, while if v is a leaf of one of the k original stars, then $i_v(T) = 2(k-1)$. Consequently,

$$
\sum_{v \in V} i_v(T) = k(k) + 2(k-1)(k^2 - k) = 2k^3 - 3k^2 + 2k.
$$

Hence,

$$
i_{av}(T) = \frac{1}{k^2}\left(\sum_{v \in V} i_v(T)\right) = 2k - 3 + \frac{2}{k} = 2\sqrt{n} - 3 + \frac{2}{\sqrt{n}}.
$$

3 Outerplanar Graphs

An *outerplanar graph* is a graph that can be embedded in the plane such that all vertices are on the outer face boundary. Our aim in this section is to provide a lower bound on the average lower independence number of an outerplanar graph in terms of its order.

If e is an edge of a graph G, the graph derived from G by deleting e and identifying each pair of ends of e is said to be obtained by *contracting* e. A *minor* of G is any graph that can be obtained from G by a sequence of vertex deletions, edge deletions and edge contractions. Any subgraph of a graph is a minor of the graph. We shall need the following characterization of outerplanar graphs in terms of minors.

Theorem 3 (Thomassen [8]) *A graph G is outerplanar if and only if it contains no minor isomorphic to K_4 or $K_{2,3}$.*

Before proceeding further, we introduce some additional notation. Assume that G is an outerplanar graph of order n. Then G is a spanning subgraph of a maximal outerplanar graph H. Let $C = a_0, a_1, \ldots, a_{n-1}, a_0$ be the unique hamilton cycle of H. For vertices a_i, a_j we define $[a_i, a_j]$ to be the subset $\{a_i, a_{i+1}, a_{i+2}, \ldots, a_{j-1}, a_j\} \subseteq V(G)$ (indices modulo n). Such a set is called a *segment* of G. The vertices a_i and a_j are called *marginal vertices* of the segment.

Lemma 4 *Let $G = (V, E)$ be an outerplanar graph and let $S = \{v_1, v_2, \ldots, v_k\}$ be a (not necessarily minimal) dominating set of G. Let $W_1, W_2, \ldots, W_k$ be disjoint sets in $V - S$ such that $\sum_{i=1}^{k} W_i = V - S$ and $W_i \subseteq N(v_i)$ for $i = 1, 2, \ldots, k$ (possibly $W_i = \emptyset$). Let t_i denote the minimum number of segments of G into which W_i can be partitioned. Then*

$$\sum_{i=1}^{k} t_i \leq 3k - 2.$$

Proof. The proof is by induction on k. If $k = 1$, then $S = \{v_1\}$ and v_1 is adjacent to every other vertex of G. Hence $W_1 = V - \{v_1\}$ and $t_1 = 1$.

Now let $k \geq 2$. If each W_i consists of at most two segments, then $\sum t_i \leq 2k \leq 3k - 2$. Hence we may assume that at least one W_i, say, W_1 has at least three segments. Let $w_1 \in W_1$ belong to a segment that neither precedes nor follows v_1 on C. Let G' and G''b e the outerplanar graphs induced in G by $[v_1, w_1]$ and $[w_1, v_1]$, respectively. The segments of G' (G'') are defined by restricting the hamilton cycle C to $V(G')$ ($V(G'')$) and adding the edge v_1w_1.

Since G is outerplanar and v_1w_1 is an edge of G, the sets $S' = S \cap V(G')$ and $S'' = S \cap V(G'')$ are dominating sets of G' and G'' which have only v_1 in common. Moreover, $W_i \subset V(G'')$ if and only if $v_i \in V(G'')$. Hence, for $i \neq 1$ and $v_i \in V(G')$ ($v_i \in V(G'')$) the number of segments of W_i in G' (G'') equals t_i. Denote the number of segments of $W_1 \cap V(G')$ in G' by t_1' and the number of segments of $W_1 \cap V(G'')$ in G'' by t_1''. Then $t_1' + t_1'' = t_1 + 1$ since the two segments of G' and G'' containing w_1 form a single segment in G. Since $|S'|, |S''| < k$ we can apply the induction hypothesis to G' and G''. Thus,

$$t_1' + \sum_{v_i \in S' - \{v_1\}} t_i \leq 3|S'| - 2$$

and

$$t_1'' + \sum_{v_i \in S'' - \{v_1\}} t_i \leq 3|S''| - 2.$$

Adding these two inequalities yields

$$1 + \sum_{v_i \in S} t_i \leq (3|S'| - 2) + (3|S''| - 2) = 3(|S| + 1) - 4 = 3k - 1,$$

which implies the statement of the lemma. □

We shall also need the following property of minimum dominating sets in graphs established by Bollobás and Cockayne [1].

Proposition 5 (Bollobás, Cockayne [1]) *If G is a graph with no isolated vertex, then there exists a minimum dominating set S of vertices of G such that for every vertex $v \in S$, there exists a vertex $w \in V(G) - S$ such that $N(w) \cap S = \{v\}$.*

We are now in a position to establish a lower bound on the average lower independence number of an outerplanar graph in terms of its order. Recall that a *linear forest* is a forest of which each component is a path.

Theorem 6 *Let $G = (V, E)$ be a connected outerplanar graph of order n. Then*

$$i_{av}(G) \geq 2\sqrt{\frac{n}{3}} + O(1),$$

and this bound is best possible.

Proof. For a vertex $v \in V$ we define $\gamma'_v(G)$ to be the minimum cardinality of a dominating set of G that contains v, but no neighbor of v. From this definition it follows directly that, for each vertex v, $i_v(G) \geq \gamma'_v(G)$. Hence it suffices to show that

$$\frac{1}{n}\sum_{v \in V} \gamma'_v(G) \geq 2\sqrt{\frac{n}{3}} + O(1). \tag{1}$$

Let $S = \{v_1, v_2, \ldots, v_\gamma\}$ be a $\gamma(G)$-set that satisfies the statement of Proposition 5. We can assume that $\gamma \leq \sqrt{4n/3}$, since otherwise for each vertex v of G the inequality $\gamma'_v(G) \geq \gamma$ implies the desired result immediately. Clearly, for every $v_i \in S$,

$$\gamma'_{v_i}(G) \geq \gamma. \tag{2}$$

By our choice of S, we can partition $V - S$ into sets $W_1, W_2, \ldots, W_\gamma$ such that $W_i \subseteq N(v_i)$ for $i = 1, 2, \ldots, \gamma$. Let t_i denote the minimum number of segments of G into which W_i can be partitioned.

We now find a lower bound on $\gamma'_w(G)$, for $w \in W_i$. Let S_w be a smallest dominating set of G that contains w but no neighbor of w.

Then $v_i \notin S_w$. Let $\tilde{W}_i$ be the set of vertices of W_i which are not marginal in the maximal segments of W_i. By Theorem 3, a vertex in $\tilde{W}_i$ has, apart from v_i, only neighbors in W_i. If $G[W_i]$ contains a vertex u of degree at least 3, then u and v_i have three common neighbors, and so G contains $K_{2,3}$ as a subgraph, contradicting Theorem 3. Hence, $G[W_i]$ is a linear forest, and so no vertex in W_i can dominate more than three vertices of $\tilde{W}_i$. Thus

$$|S_w \cap W_i| \geq \frac{1}{3}|\tilde{W}_i| \geq \frac{1}{3}(|W_i| - 2t_i). \tag{3}$$

Since $(S_w - \tilde{W}_i) \cup \{v_i\}$ is a dominating set of G, we have $|(S_w - \tilde{W}_i) \cup \{v_i\}| \geq \gamma$ and thus

$$|S_w - W_i| \geq |S_w - \tilde{W}_i| - 2t_i \geq \gamma - 1 - 2t_i. \tag{4}$$

Adding (3) and (4) yields

$$\gamma'_w(G) = |S_w| \geq \gamma - 1 + \frac{1}{3}|W_i| - \frac{8}{3}t_i. \tag{5}$$

Adding (2) over all i and (5) over all $w \in W_i$ and then over all i yields

$$\begin{aligned}
\sum_{v \in V} \gamma'_v(G) &= \sum_{i=1}^{\gamma} \gamma'_{v_i}(G) + \sum_{i=1}^{\gamma} \sum_{w \in W_i} \gamma'_w(G) \\
&\geq \gamma^2 + \sum_{i=1}^{\gamma} |W_i| \Big(\gamma - 1 + \frac{1}{3}|W_i| - \frac{8}{3}t_i\Big) \\
&= \gamma^2 + (\gamma - 1)(n - \gamma) + \frac{1}{3} \sum_{i=1}^{\gamma} \Big(|W_i|^2 - 8|W_i|t_i\Big). \quad (6)
\end{aligned}$$

In order to bound the last expression we shall minimize the sum $\sum_i |W_i|^2 - 8|W_i|t_i$ subject to the constraints $\sum_i |W_i| = n - \gamma$ and, by Lemma 4, $\sum_i t_i \leq 3\gamma - 2$. Without loss of generality we can assume that $|W_1| \geq |W_i|$ for all $i = 2, \ldots, \gamma$. In order to minimize the above sum we can assume that $t_1 = 3\gamma - 2$, $t_i = 0$ for $i = 2, \ldots, \gamma$, and $|W_2| = |W_3| = \cdots = |W_\gamma|$. If we denote $|W_1|$ by a, then a lower bound on the sum is

$$f(a) := a^2 - 8(3\gamma - 2)a + (\gamma - 1)\Big(\frac{n - \gamma - a}{\gamma - 1}\Big)^2.$$

A simple calculation shows that $f(a)$ achieves its minimum at $a = (4(3\gamma - 2)(\gamma - 1) + n - \gamma)/\gamma$. Substituting this value for a and some further calculations yield

$$\begin{aligned} f(a) \geq\ & \Big(\frac{4(3\gamma-2)(\gamma-1)+n-\gamma}{\gamma}\Big)^2 \\ & -8(3\gamma-2)\left(\frac{4(3\gamma-2)(\gamma-1)+n-\gamma}{\gamma}\right) \\ & +(\gamma-1)\left(\frac{n-\gamma-\frac{4(3\gamma-2)(\gamma-1)+n-\gamma}{\gamma}}{\gamma-1}\right)^2 \\ =\ & \frac{(n-\gamma)^2}{\gamma} + \frac{8(3\gamma-2)(n-\gamma)}{\gamma^2} - \frac{16(3\gamma-2)^2(\gamma-1)}{\gamma}. \end{aligned}$$

In conjunction with (6) we obtain

$$\begin{aligned} \frac{1}{n}\sum_{v\in V}\gamma'_v(G) \geq\ & \frac{\gamma^2}{n} + \frac{(\gamma-1)(n-\gamma)}{n} + \frac{(n-\gamma)^2}{3\gamma n} \\ & + \frac{8(3\gamma-2)(n-\gamma)}{3\gamma^2 n} - \frac{16(3\gamma-2)^2(\gamma-1)}{3\gamma n} \\ >\ & \frac{(\gamma-1)(n-\gamma)}{n} + \frac{(n-\gamma)^2}{3\gamma n} - \frac{16(3\gamma-2)^2(\gamma-1)}{3\gamma n}. \end{aligned}$$

By our assumption $\gamma \leq \sqrt{4n/3}$ we have $\dfrac{16(3\gamma-2)^2(\gamma-1)}{3\gamma n} = O(1)$. Hence

$$\frac{1}{n}\sum_{v\in V}\gamma'_v(G) \geq \frac{(\gamma-1)(n-\gamma)}{n} + \frac{(n-\gamma)^2}{3\gamma n} + O(1)$$
$$\geq \gamma + \frac{n}{3\gamma} + O(1).$$

It is easy to verify that the term $\gamma + \frac{n}{3\gamma}$ is minimal if $\gamma = \sqrt{n/3}$. Hence,

$$\frac{1}{n}\sum_{v\in V}\gamma'_v(G) \geq 2\sqrt{\frac{n}{3}} + O(1),$$

as desired.

That this bound is best possible may be seen as follows. For $k \geq 2$ an integer, let $G = (V,E)$ be the graph obtained from a path $P_{k(3k-1)}$: $u_1, u_2, \ldots, u_{k(3k-1)}$ by adding k new vertices $v_1, v_2, \ldots, v_k$ and for each $i = 1,2,\ldots,k$, adding the edges $v_i u_{(i-1)(3k-1)+j}$ where $1 \leq j \leq 3k-1$. Then, G is a connected outerplanar graph of order $n = 3k^2$. If $v = v_i$ where $1 \leq i \leq k$, then $i_v(G) = k$. If $v = u_{(i-1)(3k-1)+j}$ where $1 \leq i \leq k$ and $j \equiv 0 \pmod 3$, then $i_v(G) = 2k$. All remaining vertices v of G satisfy $i_v(G) = 2k-1$. Consequently,

$$\sum_{v\in V} i_v(G) = k(k) + k(k-1)(2k) + 2k^2(2k-1) = 3k^2(2k-1).$$

Hence,

$$i_{av}(G) = \frac{1}{3k^2}\left(\sum_{v\in V} i_v(G)\right) = 2k-1 = 2\sqrt{\frac{n}{3}} - 1.$$

This completes the proof of the theorem. □

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