A Note on the Decomposition Dimension of Complete Graphs

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Abstract. A decomposition $\mathcal{F} = \{ F_1, F_2, \ldots, F_r \}$ of the edge set of a graph G is called a resolving r-decomposition if for any pair of edges e_1 and e_2, there exists an index i such that $d(e_1, F_i) \neq d(e_2, F_i)$, where $d(e, F)$ denotes the distance from e to F. The decomposition dimension $\text{dec}(G)$ of a graph G is the least integer r such that there exists a resolving r-decomposition. Let K_n be the complete graph with n vertices. It is proved that $\text{dec}(K_n) \leq (1/2)(\log_2 n)^2(1 + o(1))$.

1 Introduction

Let G be a finite undirected graph without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. In this paper, we always assume that G is a connected graph. Let e_1 and e_2 be edges of G. The distance from e_1 to e_2, denoted by $d_G(e_1, e_2)$ or simply $d(e_1, e_2)$, is defined as the number of vertices contained in the shortest path in G from e_1 to e_2. Note that $d(e, e) = 0$ for any edge e. For an edge e and an edge set $F \subset E(G)$, we define $d(e, F)$ as the minimum $d(e, f)$ over $f \in F$.

The notion of the decomposition dimension of a graph G was introduced in [1]. Suppose that a decomposition $\mathcal{F} = \{ F_1, F_2, \ldots, F_r \}$ of $E(G) = F_1 \cup F_2 \cup \ldots \cup F_r$ is given. We call $\mathcal{F}$ a *resolving r-decomposition* if for any pair of edges e_1 and e_2, there exists some index i such that $d(e_1, F_i) \neq d(e_2, F_i)$. The *decomposition dimension* $\text{dec}(G)$ of a graph G is the least integer r for which there exists a resolving r-decomposition of G. The decomposition dimension of a tree is discussed in [2].

In this paper, we focus on complete graphs K_n. In [1], it is proved that $\text{dec}(K_n) \leq \lfloor (2n + 5)/3 \rfloor$ for $n \geq 3$. This bound is improved.

Theorem 1. $2(\log_2 n)(1 + o(1)) \leq \text{dec}(K_n) \leq (1/2)(\log_2 n)^2(1 + o(1))$.

2 Proof of Theorem 1

Proof of lower bound. Let $n \geq 4$ and $r = \text{dec}(K_n)$. We employ the following lemma [2]. (It was essentially proved in [1].) The *edge-diameter* of a graph G is defined as the largest distance of two edges e_1 and e_2 over $E(G)$.

Lemma 2. *Let G be a graph with edge-diameter d and decomposition dimension r. Then $|E(G)| \leq r\, d^{r-1}$.* □

Because the edge-diameter of K_n is 2 and $|E(K_n)| = n(n-1)/2$, Lemma 2 implies that $n(n-1)/2 \leq r\, 2^{r-1}$. Therefore, we have $r \geq 2(\log_2 n)(1+o(1))$, as required.

Proof of upper bound. Let t be a sufficiently large integer. Let X be an underlying set with $X = \{x_1, x_2, \dots, x_{2t}\}$. First we prove the theorem when the order n of a complete graph is exactly $\binom{2t}{t} + \binom{2t}{2}$. Set $g(t) = \binom{2t}{t} + \binom{2t}{2}$. We define a complete graph K_n and a partition of the edge set as follows.

$$\begin{aligned} V(K_n) &= \{A \subset X : |A| = t \text{ or } |A| = 2\}, \\ E(K_n) &= F_0 \cup \bigcup_{1 \leq k < l \leq d} F_{k,l}, \\ & \quad \text{where } F_{k,l} = \{(A, \{x_k, x_l\}) : |A| = t \text{ and } A \supset \{x_k, x_l\}\}, \\ & \quad \text{and } F_0 = E(K_n) \setminus \bigcup_{1 \leq k < l \leq d} F_{k,l}. \end{aligned}$$

Let us denote this partition by $\mathcal{F}$. We want to show that $\mathcal{F}$ is a resolving decomposition. In the following, we denote two endvertices of an edge e by $A_+(e)$ and $A_-(e)$. These sets $A_+(e)$ and $A_-(e)$ actually represent subsets of X. Suppose that there exists a pair of edges e_1 and e_2 in a common edge class such that $d(e_1, F) = d(e_2, F)$ for any $F \in \mathcal{F}$.

Case 1. $\{e_1, e_2\} \subset F_{k,l}$ for some k and l.

We may assume $A_+(e_i) \supset A_-(e_i)$ and $A_-(e_i) = \{x_k, x_l\}$ for $i = 1, 2$. Since $e_1 \neq e_2$, we have $A_+(e_1) \neq A_+(e_2)$. It follows that there exists an element $x_i \in X$ such that $x_i \in A_+(e_1) \setminus A_+(e_2)$. This implies that $d(e_1, F_{i,k}) = 1$ and $d(e_2, F_{i,k}) = 2$, a contradiction.

Case 2. $\{e_1, e_2\} \subset F_0$.

Since $e_1 \in F_0$, we have $A_+(e_1) \not\supset A_-(e_1)$ and $A_-(e_1) \not\supset A_+(e_1)$. Choose two elements x_k and x_l with $x_k \in A_+(e_1) \setminus A_-(e_1)$ and $x_l \in A_-(e_1) \setminus A_+(e_1)$.

We may assume $x_k \in A_+(e_2) \cup A_-(e_2)$. Indeed, if $x_k \notin A_+(e_2) \cup A_-(e_2)$, take an element x_i with $x_i \in A_+(e_1) \setminus \{x_k\}$. Then we have $\{x_i, x_k\} \subset A_+(e_1)$, $\{x_i, x_k\} \not\subset A_+(e_2)$, and $\{x_i, x_k\} \not\subset A_-(e_2)$. It follows that $d(e_1, F_{i,k}) = 1$ and $d(e_2, F_{i,k}) = 2$, a contradiction. Therefore, we have $\{x_k, x_l\} \subset A_+(e_2) \cup A_-(e_2)$.

If $\{x_k, x_l\} \subset A_+(e_2)$ or $\{x_k, x_l\} \subset A_-(e_2)$, then we have $d(e_1, F_{k,l}) = 2$ and $d(e_2, F_{k,l}) = 1$, a contradiction. Hence, we may assume $x_k \in A_+(e_2) \setminus A_-(e_2)$ and $x_l \in A_-(e_2) \setminus A_+(e_2)$.

Since $e_1 \neq e_2$ holds, we may assume there exists an element $x_i \in A_+(e_1) \setminus A_+(e_2)$. Then we have $\{x_i, x_k\} \subset A_+(e_1)$, $\{x_i, x_k\} \not\subset A_+(e_2)$ and $\{x_i, x_k\} \not\subset A_-(e_2)$. It follows that $d(e_1, F_{i,k}) = 1$ and $d(e_2, F_{i,k}) = 2$, a contradiction.

Hence, $\mathcal{F}$ is a resolving decomposition, as required.

We shall calculate the number n of vertices and the number r of partitions. By using Stirling's formula, we have $n = g(t) \sim 2^{2t}/\sqrt{\pi t}$. Hence, we have $t = (1/2)(\log_2 n)(1 + o(1))$. On the other hand, $r = 1 + \binom{2t}{2} \leq 2t^2$. Therefore, we have $\mathrm{dec}(K_n) \leq 2t^2 = (1/2)(\log_2 n)^2(1 + o(1))$.

It is left for us to show the theorem when n is not strictly $g(t)$. Suppose that $g(t-1) < n < n' = g(t)$. Let X be a set of $2t$ elements as before. We can choose K_n such that $V(K_n) = V_n \cup \binom{X}{2}$, where $V_n \subset \binom{X}{t}$ and V_n contains at least one t-set A with $A \supset B$ for each 2-set B. Then the same argument works as in the proof for $K_{n'}$. It follows that $\mathrm{dec}(K_n) \leq (1/2)(\log_2 n')^2(1 + o(1)) = (1/2)(\log_2 n)^2(1 + o(1))$, as required. □

Remark. Recently, Kündgen and West proved that $\mathrm{dec}(K_n) \leq 3.2(\log_2 n)(1+o(1))$ [3]. Their proof is based on the probabilistic method.

References

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