# A RUNLENGTH OPERATOR ON PARTITIONS OF INTEGERS, APPLIED TO INVENTORY CHAINS

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**Abstract.** The inventory of a $2 \times m$ array $A = A(i,j)$ consisting of $n$ not necessarily distinct positive integers $\mathbb{I}(2,j)$ is the $2 \times n$ array $\mathbb{I}(A) = \mathbb{I}(i,j)$, where $\mathbb{I}(1,j)$ is the number of occurrences of $\mathbb{I}(2,j)$ in $A$. Define $\mathbb{I}^q(A) = \mathbb{I}(\mathbb{I}^{q-1}(A))$ for $q \geq 1$, with $\mathbb{I}^0(A) = A$. For every $A$, the chain $\{\mathbb{I}^q(A)\}$ of inventories is eventually periodic, with period 1, 2, or 3. The proof depends on runlengths of partitions of integers. A final section is devoted to an open question about cumulative inventory chains.

## 1 INTRODUCTION

We begin with an example. Let $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If asked what $A$ contains, one can reply "two ones" and write this inventory as $\mathbb{I}(A) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The inventory of $\mathbb{I}(A)$ can be read as "one one and one two" and recorded as $\mathbb{I}(\mathbb{I}(A)) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Continuing,

$$\mathbb{I}(\mathbb{I}(\mathbb{I}(A))) = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbb{I}^4(A) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix},$$

and so on. We are interested in the long-range structure of inventory chains such as $(A, \mathbb{I}(A), \mathbb{I}(\mathbb{I}(A)), \ldots)$.

In this paper, an *array* is a matrix,

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{pmatrix}, \tag{1}$$

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where $m \geq 1$ and $a_i$ and $b_i$ are positive integers satisfying $b_1 < b_2 < \cdots < b_m$. The *inventory* of $A$ is the array given by

$$\mathbb{I}(A) = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ d_1 & d_2 & \cdots & d_n \end{pmatrix},$$

where $n$ is the number of distinct terms $d_i$ in $A$, $c_i$ is the number of occurrences of $d_i$ in $A$, and $d_1 < d_2 < \cdots < d_n$. Suppose for given array $A$ that there exist distinct $u$ and $v$ such that $\mathbb{I}^v(A) = \mathbb{I}^u(A)$. Then $A$ has *eventually periodic inventory*, or is *eventually periodic*. If $v > u$ and $v - u$ is minimal, then $v - u$ is the *period* of $A$.

**Example 1.** Here we use the notation $A \to A'$ for $\mathbb{I}(A) = A'$. Let $A = \begin{pmatrix} m \\ 1 \end{pmatrix}$, where $m \geq 5$. Then

$$A \to \begin{pmatrix} 1 & 1 \\ 1 & m \end{pmatrix} \to \begin{pmatrix} 3 & 1 \\ 1 & m \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & m \end{pmatrix} \to \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 2 & 3 & m \end{pmatrix}$$

$$\to \begin{pmatrix} 4 & 1 & 2 & 1 \\ 1 & 2 & 3 & m \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & m \end{pmatrix} \to \begin{pmatrix} 4 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 4 & m \end{pmatrix}$$

$$\to \begin{pmatrix} 3 & 3 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 & m \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 4 & m \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 4 & m \end{pmatrix} \to \cdot$$

Note that $\mathbb{I}^{10}(A) = \mathbb{I}^9(A)$, so that $A$ is eventually periodic, with period 1, beginning at $\mathbb{I}^9(A)$. It is easy to check that if $1 \leq m \leq 4$, then $A$ is eventually periodic, with period 1, beginning at $\begin{pmatrix} 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

In Example 1 and in general, $\mathbb{I}(A)$ is invariant of the order in which the top numbers $a_1, a_2, \ldots, a_m$ occur. That is, $\mathbb{I}$ is a many-to-one mapping determined by the top numbers as a multiset. In Example 1, $\mathbb{I}^5(A)$ has top numbers $4, 1, 2, 1$: two of one number, one of another, and one of another; i.e., the partition $1^2 2$ of the integer 4. The chain in Example 1 beginning at $\mathbb{I}^5(A)$ is thus matched to a chain of partitions:

$$1^2 2 \to 1^2 3 \to 12^2 \to 12^2 \to 12^2 \to 12^2.$$

The match between arrays and partitions motivates the introduction of an operator $\mathcal{R}$ on partitions, in Section 2. In Section 3, we shall apply the developments of Section 2 to prove that every array is eventually periodic. In Section 4, we define and consider cumulative inventory arrays.

## 2 PARTITIONS AND THE RUNLENGTH OPERATION

For array $A$ as in (1), the multiset $\{a_1, a_2, \ldots, a_m\}$ corresponds to a partition of the integer $h := a_1 + a_2 + \cdots + a_m$ as follows:

i. assume that the $a_i$ are in nondecreasing order as a word, $w$;

ii. let $e_1, e_2, \ldots, e_k$ be the sequence of runlengths in $w$;

iii. write the partition of $h$ as

$$\underbrace{a_1 + \cdots + a_1}+\underbrace{a_{e_2} + \cdots + a_{e_2}}+\cdots+\underbrace{a_{e_k} + \cdots + a_{e_k}},$$

where $a_1$ is written $e_1$ times, $a_{e_2}$ is written $e_2$ times, and so on, so that

$$h = e_1a_1 + e_2a_{e_2} + \cdots + e_ka_{e_k};$$

iv. change to multiplicative notation: $a_1^{e_1}a_2^{e_2}\ldots a_k^{e_k}$.

Given a partition $p = a_1^{e_1}a_2^{e_2}\ldots a_k^{e_k}$, we shall call each of the expressions $a_i^{e_i}$ a *term* of $p$. Next, we define the titular runlength operator, $\mathcal{R}$: suppose $p = a_1^{e_1}a_2^{e_2}\ldots a_k^{e_k}$ is a partition of $h = e_1a_1 + e_2a_2 + \cdots + e_ka_k \geq 1$; then

$$\mathcal{R}(p) := \begin{cases} 1h & \text{if} \quad p = h \\ h & \text{if} \quad p = 1^h \\ e_1e_2\ldots e_k(h - e_1 - e_2 - \cdots - e_k) & \text{otherwise.} \end{cases}$$

**Example 2.** $p = 123^2$ denotes the partition $1 + 2 + 3 + 3$ of 9, and $\mathcal{R}(p) = 1^225$.

**Example 3.** $\mathcal{R}$-chains (i.e., $p$, $\mathcal{R}(p), \mathcal{R}(\mathcal{R}(p)), \ldots$) for the partitions of 2 are

$$2 \to 12 \to 1^3 \to 3 \to 13 \to 1^22 \to 1^22 \to \cdots;\ 1^2 \to 2 \to \cdots.$$

$\mathcal{R}$-chains for the paritions of 3 are

$$3 \to 13 \to 1^22 \to 1^22 \to \cdots;\ 12 \to 1^3 \to 3 \to \cdots;\ 1^3 \to 3 \to \cdots.$$

Suppose for given partition $p$ that there exist distinct $u$ and $v$ such that $\mathcal{R}^v(p) = \mathcal{R}^u(p)$. Then $p$ is *eventually periodic.* If $v > u$ and $v - u$ is minimal, then $v - u$ is the *period* of $p$. Example 3 shows that all partitions of 2 and 3 are eventually periodic with period 1.

**Example 4.** Every partition of 4 and 5 is eventually periodic with period 1 or 2. Every partition of 6 is eventually periodic with period 2 or

3. The number 7 has one partition of period 1, namely $1^2 23$; two of period 2, namely 7 and $1^7$; the remaining twelve partitions of 7 have period 3.

**Lemma 1.** *Suppose $h \geq 8$ and $p$ is a partition of $h$. Then there exists $j$ such that $R^j(p)$ consists of 1, 2, or 3 terms.*

*Proof.* If not, let $h$ be least for which $\mathcal{R}^j(p)$ has at least 4 terms for every $j$. Let $k$ be the least number of terms that occurs, and let $j'$ be such that $\mathcal{R}^{j'}(p)$ has $k$ terms. Write $q$ for $\mathcal{R}^{j'}(p)$, so that $q$ has the form $a_1^{e_1} a_2^{e_2} \ldots a_k^{e_k}$, and

$$\mathcal{R}(q) = e_1 e_2 \ldots e_k e_{k+1}, \tag{2}$$

where $e_{k+1} := h - \sum_{i=1}^{k} e_i$. By hypothesis, when $\mathcal{R}(q)$ is written in the multiplicative form $c_1^{f_1} c_2^{f_2} \ldots c_m^{f_m}$, we have $m \geq k$. Equation (2) shows that $m$ must be $k$ or $k+1$.

**Case 1:** $m = k$. In this case, the number of distinct elements in the multiset $\{e_1, e_2, \ldots, e_k, e_{k+1}\}$ is $k$; we may assume $e_1 = e_2$. Let $f = h - k - 1$.

**Case 1.11:** $m = k$, $e_1 = e_2$, $f \neq e_i$ for $1 \leq i \leq k+1$. We have from (2),

$$\mathcal{R}(q) \to 1^k 2(h-k-2) \to \begin{cases} 1^2 k(h-k-2) & \text{if} \quad h-k \notin \{3,4\} \\ 1^{k+1}2 & \text{if} \quad h-k=3 \\ 1^k 2 & \text{if} \quad h-k=4 \end{cases},$$

so that $\mathcal{R}^3(q)$ has fewer than 4 terms, a contradiction.

**Case 1.12:** $m = k$, $e_1 = e_2$, $f = e_1$. In this case, $\mathcal{R}(q) \to 1^{k-3}3(h-k)$, so that $\mathcal{R}^2(q)$ has fewer than 4 terms, a contradiction.

**Case 1.13:** $m = k$, $e_1 = e_2$, $f = e_i$ for some $i \geq 3$. In this case, $\mathcal{R}(q) \to 2^2 1^{k-1}(h-k-1)$, so that $\mathcal{R}^2(q)$ has fewer than 4 terms, a contradiction.

**Case 2:** $m = k+1$. In this case the $e_i$ in (2) are distinct, so that $\mathcal{R}^2(q) = 1^{k-1}w$, again a contradiction. □

**Theorem 1.** *Every partition of every $h \geq 8$ is eventually periodic with period 1 or 2.*

*Proof.* Let $h \geq 8$, and let $q$ be a partition of $h$. By Lemma 1, $\mathcal{R}^j(q)$, for some $j$, has fewer than 4 terms. Write $p$ for $\mathcal{R}^j(q)$. We consider three cases:

**Case 1:** $p$ consists of a single term, $a^e$. If $h \neq 2e$, then

$$a^e \to ew \to 1^2w' \to 12w'' \to 1^2w',$$

where $w = h - e$ and $w' = h - 2$, so that $p$, and hence $q$, has period 2. If $h = 2e$, then

$$a^e \to e^2 \to 2w' \to 1^2w' \to 12(w' - 1),$$

and here, too, $q$ has period 2.

**Case 2.1:** $p = a_1^{e_1}a_2^{e_2}$, where $e_1 \neq e_2$. The $\mathcal{R}$-chain in case the number $f := h - e_1 - e_2$ satisfies $f \neq e_1$ and $f \neq e_2$ begins with

$$a_1^{e_1}a_2^{e_2} \to e_1e_2f \to 1^3(w' - 1) \to 13(w' - 2) \to 1^2w' \to 12(w' - 1),$$

indicating a period of 2. (The hypothesis $h \geq 8$ applies, for example, to ensure the link $1^3(w' - 1) \to 13(w' - 2)$.) If $f = e_1$, we have instead

$$a_1^{e_1}a_2^{e_2} \to e_2f^2 \to 12(w' - 1) \to 1^2w' \to 12(w' - 1),$$

and similarly if $f = e_2$.

**Case 2.2:** $p = a_1^{e_1}a_2^{e_2}$, where $e_1 = e_2$. Let $g = h - 2e_1$. If $g \neq e_1$, then

$$a_1^{e_1}a_2^{e_1} \to e_1^2g \to 12(w' - 1) \to 1^2w' \to 12(w' - 1),$$

whereas if $g = e_1$, the chain includes $a_1^{e_1}a_2^{e_1} \to e_1^3$, and, as above, $q$ has period 2.

**Case 3.1:** $p = a_1^{e_1}a_2^{e_2}a_3^{e_3}$, where the $e_i$ are distinct. Let $e_4 = h - e_1 - e_2 - e_3$. If $e_4$ is not one of $e_1, e_2, e_3$, then

$$a_1^{e_1}a_2^{e_2}a_3^{e_3} \to e_1e_2e_3e_4 \to 1^4(h-4) \to 14(h-5) \to 1^3(h-3) \to 13(h-4) \to 1^3(h-3),$$

indicating a period of 2. If $e_4$ equals one of $e_1, e_2, e_3$, then $\mathcal{R}^2(p) = 1^22(h-4)$. In this case, $\mathcal{R}^3(p) = \mathcal{R}^2(p)$, indicating a period of 1.

**Case 3.2:** $p = a_1^{e_1}a_2^{e_2}a_3^{e_3}$, where the multiset $\{e_1, e_2, e_3\}$ contains 2 distinct elements. Without loss, assume $e_1 \neq e_2 = e_3$. Let $f = h - e_1 - 2e_2$. If $f \notin \{e_1, e_2\}$, then $\mathcal{R}(p) = e_1e_2^2f$ and $\mathcal{R}^2(p) = 1^22(h - 4)$, so that $q$ has period 1. If $f = e_1$, then

$$p \to e_2^2f^2 \to 2^2(h - 4) \to 12(h - 3) \to 1^3(h - 3) \to 13(h - 4) \to 1^3(h - 3),$$

indicating a period of 2. If $f = e_2$, then

$$p \to e_1 f^3 \to 13(h-4) \to 1^3(h-3),$$

indicating a period of 2.

**Case 3.3:** $a_1^e a_2^e a_3^e$, Let $f = h - 3e$. If $f \neq e$, then $\mathcal{R}^2(p) = 13(h-4)$, indicating a period of 2. If $f = e$, then

$$p \to e^4 \to 4(h-4) \to 1^2(h-2) \to 12(h-3),$$

indicating a period of 2. □

To summarize, for every $h$ except 6 and 7, every partition of $h$ is eventually periodic with period 1 or 2. As found in the proof of Theorem 1, every $\mathcal{R}$-chain of a partition of period 1 eventually reaches the repeating partition $1^2 2(h-4)$, and every $\mathcal{R}$-chain of a partition of period 2 eventually reaches the repeating link $1^3(h-3) \to 13(h-4)$.

In Section 3, these periodicities will be applied to arrays in the manner exemplified in Section 1. Here, we continue with a consideration of initial $\mathcal{R}$-segments from a given partition $p$ down to a repeating partition. These are shown for the partitions of 6 in Table 1. The final two columns show the period $r(p)$ of $p$ and the length $l(p)$ of the initial segment from $p$ to, but not including, the first repeating partition.

TABLE 1. $\mathcal{R}$-CHAINS FOR THE PARTITIONS OF 6

| partition, $p$ | $\mathcal{R}(p)$ | $\mathcal{R}^2(p)$ | $\mathcal{R}^3(p)$ | $\mathcal{R}^4(p)$ | $\mathcal{R}^5(p)$ | $\mathcal{R}^6(p)$ | $r(p)$ | $l(p)$ |
|---|---|---|---|---|---|---|---|---|
| 6 | 16 | $1^2 5$ | **124** | **$1^3 4$** | **$13^2$** | 124 | 3 | 3 |
| 15 | $1^2 4$ | **123** | **$1^3 3$** | 123 | $1^3 3$ | 123 | 2 | 2 |
| 24 | $1^2 4$ | **123** | **$1^3 3$** | 123 | $1^3 3$ | 123 | 2 | 2 |
| $3^2$ | 24 | $1^2 4$ | **123** | **$1^3 3$** | 123 | $1^3 3$ | 2 | 3 |
| $1^2 4$ | **123** | **$1^3 3$** | 123 | $1^3 3$ | 123 | $1^3 3$ | 2 | 1 |
| **123** | **$1^3 3$** | 123 | $1^3 3$ | 123 | $1^3 3$ | 123 | 2 | 0 |
| $2^3$ | $3^2$ | 24 | $1^2 4$ | **123** | **$1^3 3$** | 123 | 2 | 4 |
| **$1^3 3$** | **123** | $1^3 3$ | 123 | $1^3 3$ | 123 | $1^3 3$ | 2 | 0 |
| $1^2 2^2$ | $2^3$ | $3^2$ | 24 | $1^2 4$ | **123** | **$1^3 3$** | 2 | 5 |
| $1^4 2$ | $1^2 4$ | **123** | **$1^3 3$** | 123 | $1^3 3$ | 123 | 2 | 2 |
| $1^6$ | 6 | 16 | $1^2 5$ | **124** | **$1^3 4$** | **$13^2$** | 3 | 4 |

In Table 1, consecutive bold-face partitions in a row indicate the period; e.g., for $p = 6$, the partitions $124, 1^4 3$, $13^2$ indicate that $\mathcal{R}^{3+k}(p) = \mathcal{R}^k(p)$

for all $k \geq 3$. It would be of interest to know the maximal length of the initial $\mathcal{R}$-segment as a function of $h$.

Also of interest is that fact that, for $h \geq 8$, according to Theorem 1, the set of partitions of $h$ is partitioned into two parts: those of period 1 and those of period 2. In Table 2, $p_h$ is the number of partitions of $h$, and $p_h^{(i)}$ is the number of those having period $i$.

TABLE 2. TWO KINDS OF PARTITIONS

| $h$ | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|
| $p_h$ | 22 | 30 | 42 | 56 | 77 | 101 |
| $p_h^{(1)}$ | 2 | 7 | 8 | 18 | 25 | 41 |
| $p_h^{(2)}$ | 20 | 23 | 34 | 38 | 52 | 60 |

It would be of interest to know asymptotic growth rates for $p_h^{(1)}$ and $p_h^{(2)}$.

# 3 EVENTUALLY PERIODIC INVENTORIES

Let $A$ be an array as in (1). The $j$th array in the $\mathbb{I}$-chain of $A$ will be denoted by

$$\mathbb{I}^j(A) = \begin{pmatrix} a_1^{(j)} & a_2^{(j)} & \cdots & a_m^{(j)} \\ b_1^{(j)} & b_2^{(j)} & \cdots & b_m^{(j)} \end{pmatrix} \tag{3}$$

for $j = 0, 1, \ldots$, beginning with $\mathbb{I}^0(A) := A$. Let

$$a* = \max\{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m\}.$$

**Lemma 2.** *Let*

$$j_0 = \begin{cases} b_m & \text{if } \max\{a_1, a_2, \ldots, a_m\} \leq b_m \text{ and } a_i \neq b_m \text{ for some } i \\ \max\{m+1, b_m\} & \text{if } a_i = a_1 \text{ for } 1 \leq i \leq m \text{ and } b_i = a_1 \text{ for some } i \\ a^* & \text{if } \max\{a_1, a_2, \ldots, a_m\} > \max\{b_1, b_2, \ldots, b_m\} \end{cases}$$

*Then* $m(j) = m(j_0)$ *for all* $j \geq j_0$.

*Proof.* We consider three cases:

**Case 1:** max $\{a_i\} \leq b_m$ and $a_i \neq b_m$ for some $i$. Here, no term of $A$ occurs more than $m$ times, so that $\mathbb{I}(A)$ has the form (3) with $j = 1$ and $b_{m(1)}^{(1)} = b_m$. The condition defining case 1 thus holds for array $\mathbb{I}(A)$; that is, $\max\{a_i^{(1)}\} \leq b_m$ and $a_i^{(1)} \neq b_m$ for some $i$. Consequently, by induction, $m(j) = m(b_m)$ for all $j \geq b_m$.

**Case 2:** $a_i = a_1$ for $1 \leq i \leq m$ and $a_i = b_i$ for some $i$. In this case, $a_1$ occurs $m+1$ times in $A$, so that $m+1$ is one of the numbers $b_j^{(1)}$. Consequently, $b_{m(1)}^{(1)} = \max\{m+1, b_m\}$. If $m = 1$ and $b_1 = 2$, then $m(j) = 1$ for all $j \geq j_0$, as required. Otherwise, case 1 now applies to $\mathbb{I}(A)$, so that $m(j) = m(b_{m(1)}^{(1)})$ for all $j \geq b_{m(1)}^{(1)}$.

**Case 3:** $\max\{a_i\} \geq \max\{b_i\}$. Here, we have $b_{m(1)}^{(1)} = \max\{a_i\} = a^*$, and case 1 applies to $\mathbb{I}(A)$. Hence, $m(j) = m(a^*)$ for all $j \geq a^*$. □

**Lemma 3.** *If* $A \neq \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, *there exists* $j \geq 0$ *such that* $I^j(A)$ *contains* 1 *as a term in row 2; i.e.,* $b_1^{(j)} = 1$.

*Proof.* If not, then for $j = 0$ in equation (1), every $b_i$ is also an $a_i$; otherwise, if some $b_{i'}$ is not one of the $a_i$, then $\mathbb{I}(A)$ contains exactly one $b_{i'}$, contrary to the hypothesis. Since the $b_i$ are distinct, the $a_i$ must be distinct, so that

$$\mathbb{I}(A) = \begin{pmatrix} 2 & 2 & \cdots & 2 \\ b_1 & b_2 & \cdots & b_m \end{pmatrix}.$$

If $m \neq 1$, then some $b_{i'}$ is not 2, so that $\mathbb{I}^2(A)$ contains exactly one $b_i$, contrary to the hypothesis. If $m = 1$ and $b_1 \neq 2$, then $\mathbb{I}^2(A)$ contains exactly one $b_1$, again a contradiction.

**Lemma 4.** *If* $A \neq \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ *then there exist positive integers* $J$ *and* $M$ *such that for all* $j \geq J$, *the array* $I^j(A)$ *has the form*

$$\begin{pmatrix} a_1^{(j)} & a_2^{(j)} & a_3^{(j)} & a_4^{(j)} & \cdots & a_M^{(j)} & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & 4 & \cdots & M & b_{M+1} & b_{M+2} & \cdots & b_{m(J)} \end{pmatrix}$$

*where* $a_i^{(j)} \leq M$ *for* $1 \leq i \leq M$. (If $b_{m(J)} = M$, then the final column is $\begin{pmatrix} a_M^{(j)} \\ M \end{pmatrix}$.)

*Proof.* For $j \geq 0$, let $B_j = (b_1^{(j)}, b_2^{(j)}, \ldots, b_{m(j)}^{(j)})$; that is, $B_j$ is the bottom row of array $\mathbb{I}^j(A)$ in (3). By Lemma 3, there exists $j'$ such that $b_1^{(j)} = 1$ for all $j \geq j'$. For each $j \geq j'$, let $n(j)$ be the greatest index $n$ such that $b_n^{(j)} = n$. By Lemma 2, the set $\{n(j) : j \geq j'\}$ is bounded above, so that, by the well-ordering principle, this set contains a greatest number, $M$. Let $J'$ be the least $j$ for which $m(j) = M$.

In case $a_i^{(j)} > M$ for some $j \geq J'$ and some $i$ satisfying $1 \leq i \leq M$, let $j_0$ be as in Lemma 2, and let $J'' = \max\{J', j_0\}$. For all $j \geq J''$, we have $a_i^{(j)} \leq M$. Let $J = J'' + 1$. Since $b_i > M$ for $M + 1 \leq i \leq m(J'')$, we have $a_i = 1$ for all $i$ satisfying $M + 1 \leq i \leq m(J)$, for all $j \geq J$. □

**Theorem 2.** *Every array is eventually periodic, with period* 1, 2, *or* 3.

*Proof.* Let $A$ be an array. If $A = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then $A$ is purely periodic with period 1. Otherwise, let $J$ and $M$ be as in Lemma 4. Then the multiset

$$\{a_1^{(J)}, a_2^{(J)}, \ldots, a_M^{(J)}\}$$

matches a partition $p$ of the integer

$$h := a_1^{(J)} + a_2^{(J)} + \cdots + a_M^{(J)}$$

as described in Section 2.

Likewise, the inventory chain $A, \mathbb{I}(A), \mathbb{I}(\mathbb{I}(A)), \ldots$ of arrays matches the $\mathcal{R}$-chain $p, \mathcal{R}(p), \mathcal{R}(\mathcal{R}(p)), \ldots$ of partitions of $h$. By Examples 3 and 4 and Theorem 1, this $\mathcal{R}$-chain is eventually periodic with period 1 or 2 if $2 \leq h \leq 5$ or h$\geq 8$, with period 2 or 3 if $h = 6$, and with period 1, 2, or 3 if $h = 7$.

Referring to the array in (4) and $j \geq J$, write $A_j := (a_1^{(j)}, a_2^{(j)}, \ldots, a_M^{(j)})$ Since $A_{j+1}$ counts runlengths of numbers in $A_j$ when those numbers are arranged in nondecreasing order, and since the runlength sequence is periodic, the sequence $A_j$ is, by the pidgeonhole principle, also periodic, with the same period. □

Although the period of the inventory chain of an array $A$ equals the period of the matching $\mathcal{R}$-chain, the initial inventory segment may contain more links than the initial $\mathcal{R}$-segment does. This is illustrated by Example 1, for which the least $u$ for which $\mathbb{I}^{u+1}(A) = \mathbb{I}^u(A)$ is 9, whereas the least $u$ for which $\mathcal{R}^{u+1}(p) = \mathcal{R}^u(p)$ is 7.

**Example 5.** Suppose $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ 1 & 2 & \cdots & m \end{pmatrix}$, where $(a_1, a_2, \ldots, a_m)$ is any permutation of $(1, 2, \ldots, m)$. Then for $j \geq 1$, the inventory $\mathbb{I}^j(A)$ is invariant of the particular permutation. For $m \geq 7$, the inventory chain has period 2, and the first repeating array is

$$\mathbb{I}^8(A) = \begin{pmatrix} m-1 & 4 & 1 & 1 & \cdots & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 4 & \cdots & m-2 & m-1 & m & m+1 \end{pmatrix}.$$

For $1 \le m \le 7$, the first repeating arrays are these:

$$\mathbb{I}^{14}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \text{ period } 1;$$

$$\mathbb{I}^{9}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \text{ period } 1;$$

$$\mathbb{I}^{10}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \text{ period } 1;$$

$$\mathbb{I}^{10}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \text{ period } 1;$$

$$\mathbb{I}^{7}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}, \text{ period } 2;$$

$$\mathbb{I}^{8}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \text{ period } 3.$$

# 4 CUMULATIVE INVENTORY CHAINS

Consider the following initial multiset $\{1\}$ and resulting chain of arrays:

$$\{1\} \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} \rightarrow$$
$$\begin{pmatrix} 6 & 2 & 1 \\ 1 & 3 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 8 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \rightarrow \cdots$$

For example, the array $\begin{pmatrix} 6 & 2 & 1 \\ 1 & 3 & 4 \end{pmatrix}$ inventories all preceding arrays and initial multiset: six $1's$, two $3's$, and one 4. A question posed in [1] is whether every positive integer enters the chain. It appears that the question remains open. Here, we introduced the term "cumulative inventory problem" for the much more general conjecture that the cumulative inventory chain beginning with an arbitrary multiset of positive integers contains every positive integer.

Let $m$ be a positive integer and

$$\mathcal{I}^j(m) := \begin{cases} \text{the multiset } \{m\} & \text{if } j = 0 \\ j\text{th array in the cumulative inventory chain of } \{m\} & \text{if } j \ge 1, \end{cases}$$

$L(m, j) :=$ least $j'$ such that $j$ is a term in $\mathcal{I}^{j'}(m)$,
$C(m, j) :=$ number of columns in $\mathcal{I}^j(m)$,
$G(m, j) :=$ greatest term in $\mathcal{I}^j(m)$.

Additional open questions about cumulative inventories, such as asymptotic growth rates, are suggested by Table 3:

TABLE 3. SEQUENCES $L(m,j)$, $C(m,j)$, $G(m,j)$

| $m$ | $L(m,j)$, $1 \le j \le 11$ | $C(m,j)$, $1 \le j \le 10$ | $G(m,j)$, $1 \le j \le 10$ |
|---|---|---|---|
| 1 | 1,4,2,3,6,4,8,5,9,8,6 | 1,1,2,3,5,6,8,9,12,15 | 1,3,4,6,8,11,13,16,18,22 |
| 2 | 1,1,3,3,4,5,5,6,6,7,10 | 1,2,2,4,5,7,9,11,13,15 | 2,2,4,5,7,9,12,15,18,21 |
| 3 | 1,2,1,5,4,4,5,6,6,7,8 | 1,2,3,3,5,7,9,11,13,15 | 3,3,3,6,7,9,12,15,18,21 |
| 4 | 1,2,3,1,4,5,6,6,7,9,10 | 1,2,3,4,5,6,8,9,11,12 | 4,4,4,5,6,8,9,12,14,17 |
| 5 | 1,2,3,4,1,6,10,6,7,7,8 | 1,2,3,4,5,6,7,9,10,12 | 5,5,5,5,6,8,10,11,13,15 |

## REFERENCE

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