Circulant Distant Two Labeling and Circular Chromatic Number

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Abstract

Let G be a graph and d, d' be positive integers, $d' \geq d$. An m-(d, d')-circular distance two labeling is a function f from $V(G)$ to $\{0, 1, 2, \cdots, m-1\}$ such that $|f(u) - f(v)|_m \geq d$ if u and v are adjacent; and $|f(u) - f(v)|_m \geq d'$ if u and v are distance two apart, where $|x|_m := \min\{|x|, m - |x|\}$. The minimum m such that there exists an m-(d, d')-circular labeling for G is called the $\sigma_{d,d'}$-number of G and denoted by $\sigma_{d,d'}(G)$. The $\sigma_{d,d'}$-numbers for trees can be obtained by a first-fit algorithm. In this article, we completely determine the $\sigma_{d,1}$-numbers for cycles. In addition, we show connections between generalized circular distance labeling and circular chromatic number.

Keywords. Vertex-labeling, distance two labeling, circular chromatic number.

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1 Introduction

For a graph G, the distance between any two vertices u and v is denoted by $d_G(u,v)$. Given G and positive integers d, d' with $d \geq d'$, an $L(d,d')$-labeling (distance two labeling) of G is a function $f : V(G) \to \{0,1,2,\cdots\}$ such that $|f(u)-f(v)| \geq d$ if $uv \in E(G)$; and $|f(u)-f(v)| \geq d'$ if $d_G(u,v) = 2$. The *span* of f is defined as $\min_{u,v\in V(G)} |f(u)-f(v)|$. The minimum span of an $L(d,d')$-labeling for G is denoted by $\lambda_{d,d'}(G)$. The $L(d,d')$-labelings, for different values of d and d', have been studied by several authors in the past decade. (See [1, 2, 3, 4, 8, 10, 12].)

The circular distance two labeling is a variation of $L(d,d')$-labeling by using a different measurement. Given G and positive integers d and d' with $d \geq d'$, an m-$(d,d')_c$- labeling is a function, $f : V(G) \to \{0,1,2,\cdots,m-1\}$, such that $|f(u)-f(v)|_m \geq d$ if $uv \in E(G)$, and $|f(u)-f(v)|_m \geq d'$ if $d_G(u,v) = 2$, where $|x|_m := \min\{|x|, m-|x|\}$. For any graph G and any $d \geq d'$, the *$\sigma_{d,d'}$-number* of G, $\sigma_{d,d'}(G)$, is the minimum m such that there exists an m-$(d,d')_c$-labeling for G.

In Section 2, we give exact values of $\sigma_{d,d'}$-number for trees and $\sigma_{d,1}$-number for cycles. Georges and Mauro [3] proved that $\lambda_{ad,ad'}(G) = a\lambda_{d,d'}(G)$ holds for any graph G and any positive integers a, d, d' with $d \geq d'$. The results for cycles shown in this article indicate that a similar result of Georges and Mauro does not hold for circular distance two labelings. That is, there exist graphs such that $\sigma_{ad,ad'}(G) \neq a\sigma_{d,d'}(G)$ for some positive integer a.

In Section 3, we investigate close relations between generalized circular distance labeling and circular chromatic number.

2 The σ-number for trees and cycles

In this section, we give the exact values of the $\sigma_{d,d'}$-numbers for trees for all $d \geq d'$, and $\sigma_{d,1}$-numbers for cycles.

Theorem 2.1 *Let T be a tree with maximum degree Δ. Then for any $d \geq d'$, $\sigma_{d,d'}(T) = 2d + (\Delta-1)d'$.*

Proof. Observe that for any star $G = K_{1,n}$, $\sigma_{d,d'}(G) = 2d + (n-1)d'$. Therefore, $\sigma_{d,d'}(T) \geq 2d + (\Delta-1)d'$, since T contains a $K_{1,\Delta}$. So it suffices to find a $(2d+(\Delta-1)d')$-$(d,d')_c$-labeling for T. This can be done by a first-fit algorithm. First, fix a root of T of degree Δ, label it by 0 and its neighbors by $d, d+d', d+2d', \cdots, d+(\Delta-1)d'$; then label the neighbors of these labeled vertices one by one using the labels from the set $\{0,1,2,\cdots,2d+(\Delta-1)d'-1\}$; and continue this process until all vertices are labeled. At each step, if a vertex v is labeled by x, then there are exactly Δ labels, $\{x+d, x+d+d', x+d+2d', \cdots, x+d+(\Delta-1)d'\}$ (mod

$2d+(\Delta-1)d')$, that can be used by the neighbors of v, including the father of v. Thus, this process produces a valid $(2d+(\Delta-1)d')$-$(d,d')_c$-labeling for T. Q.E.D.

Theorem 2.2 *For any $d \geq d'$, $\sigma_{d,d'}(C_n) \geq 2d+d'$. Moreover, $\sigma_{d,d'}(C_n) = 2d+d'$ if and only if $n \equiv 0 \pmod{\frac{2d+d'}{r}}$, where $r = \gcd(2d+d', d)$.*

Proof. Observe that $\sigma_{d,d'}(C_n) \geq 2d+d'$, since $\sigma_{d,d'}(K_{1,2}) = 2d+d'$. Suppose $\sigma_{d,d'}(C_n) = 2d+d'$ and let f be a $(2d+d')$-$(d,d')_c$-labeling for C_n. For any vertex v of C_n, suppose $f(v) = x$ for some $0 \leq x \leq 2d+d'-1$, then there are exact two values, $x-d$, $x+d \pmod{2d+d'-1}$, that can be assigned to the two neighbors of x. Let the vertices of C_n, in clockwise order, be $v_1, v_2, \cdots, v_n$. Without loss of generality, assume $f(v_1) = 0$ and $f(v_2) = d$. Then for every $i \geq 3$, the value of $f(v_{i+1})$ is fixed, $f(v_{i+1}) = (f(v_i)+d) \bmod (2d+d')$. Because $f(v_n)$ must be $d+d'$, f is well-defined if and only if $n \equiv 0 \pmod{\frac{2d+d'}{r}}$, where $r = \gcd(2d+d', d)$. Q.E.D.

Corollary 2.3 *Suppose $d \geq d'$, then $\sigma_{kd,kd'}(C_n) = k\sigma_{d,d'}(C_n) = k(2d+d')$ if $n \equiv 0 \pmod{\frac{2d+d'}{r}}$, where $r = \gcd(2d+d', d)$.*

It is known [7] and not hard to observe the following inequalities:

$$\lambda_{d,d'}(G)+1 \leq \sigma_{d,d'}(G) \leq \lambda_{d,d'}(G)+d, \quad \text{for any graph } G. \qquad (*)$$

To derive the values of $\sigma_{d,1}(C_n)$, we first quote the result of Georges and Mauro [3] on $\lambda_{d,1}(C_n)$.

Theorem 2.4 *([3]) Let C_n be a cycle, then*

$$\lambda_{d,1}(C_n) = \begin{cases} d+2, & n \equiv 0 \pmod 4; \\ d+3, & n \equiv 2 \pmod 4; \\ 2d, & n \equiv 1 \pmod 2). \end{cases}$$

Combining (*) with Theorems 2.2 and 2.4. we have

$$2d+1 \leq \sigma_{d,1}(C_n) \leq \begin{cases} 2d+2, & n \equiv 0 \pmod 4; \\ 2d+3, & n \equiv 2 \pmod 4; \\ 3d, & n \equiv 1 \pmod 2). \end{cases}$$

Theorem 2.5 *Let C_n be a cycle and let $d \geq 2$. Then*

$$\sigma_{d,1}(C_n) = \begin{cases} 2d+1, & n \equiv 0 \ (\text{mod } 2d+1); \\ 2d+2, & n \text{ is even, and } n \not\equiv 0 \ (\text{mod } 2d+1); \\ 2d+2, & n \text{ is odd, } n > 2d, \text{ and } n \not\equiv 0 \ (\text{mod } 2d+1); \\ 2d+\lceil \frac{d}{k} \rceil, & n = 2k+1,\ d < n \leq 2d, \text{ and } n \not\equiv 0 \ (\text{mod } 2d+1). \end{cases}$$

Proof. Because gcd $(2d+1, d) = 1$, by Theorem 2.2, $\sigma_{d,1}(C_n) \geq 2d+1$, and $\sigma_{d,1}(C_n) = 2d+1$ if and only if $n \equiv 0 \pmod{2d+1}$. This proves the first line in the formula.

For the next two lines in the formula, it suffices to find a $(2d+2)$-$(d,1)_c$-labeling for C_n. Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ in clockwise order. It is enough to find n integers z_i where $z_i \in \{d, d+1, d+2\}$, $i = 1, 2, \cdots, n$ such that:

(1) $z_1 + z_2 \cdots + z_n$ is a multiple of $(2d+2)$, and

(2) if $z_i + z_j = 2d+2$ then i and j are not circular consecutive.

Then a mapping f defined by $f(v_1) = z_1$ and $f(v_{i+1}) = f(v_i) + z_{i+1} \pmod{2d+2}$ for $1 \leq i \leq n-1$, will satisfy what we are looking for.

Now we claim the following cases:

<u>Case 1</u> n is even and $n \not\equiv 0 \pmod{2d+1}$. Let $n = 2k$. The integers z_i, $i = 1, 2, \cdots 2k$ of the following order will satisfy (1) and (2) in the above.

$$\underbrace{d,\ d,\ d, \cdots,\ d,}_{(k-1)\text{ terms}}\ d+1,\ \underbrace{d+2,\ d+2, \cdots,\ d+2,}_{(k-1)\text{ terms}}\ d+1.$$

<u>Case 2</u> n is odd, $n = 2k+1$, $n \geq 2$, $n > 2d$, and $n \not\equiv 0 \pmod{2d+1}$. By considering the parity of d, we separate this case into two subcases:

<u>Sub case 2.1</u> d is even. Let $d = 2m$ and $x = k - m \geq 1$.

If $x = 1$ and $x \geq 2$, define the numbers z_i, $i = 1, 2, \cdots n$, respectively, by the following:

$$d+1,\ d,\ d+1,\ \underbrace{d,\ d,\ d, \cdots,\ d}_{(n-3)\text{ terms}}\ ;$$

$$d+1,\ \underbrace{d,\ d,\ d, \cdots,\ d,}_{(n-x-1)\text{ terms}}\ \ d+1,\ \underbrace{d+2,\ d+2, \cdots,\ d+2}_{(x-1)\text{ terms}}.$$

It is easy to verify that each of the sets of numbers z_i, $i = 1, 2, \cdots, n$, in the above satisfy (1) and (2).

<u>Sub case 2.2</u> d is odd. Let $d = 2m+1$. Then $k > 2m$. Let $x = k - m > m \geq 1$. So $x \geq 2$.

If $x = 2$ and $x \geq 3$, define the numbers z_i, $i = 1, 2, \cdots n$, respectively, by the following:

$$d+1,\ d,\ d+1,\ \underbrace{d,\ d,\ d, \cdots,\ d,}_{(n-5)\text{ terms}}\ d+1,\ d;$$

$$d+1,\ \underbrace{d,\ d,\ d, \cdots,\ d,}_{(n-x-2)\text{ terms}}\ \ d+1,\ \underbrace{d+2,\ d+2, \cdots,\ d+2,}_{(x-2)\text{ terms}}\ d+1,\ d.$$

It is easy to verify that each of the sets of numbers z_i, $i = 1, 2, \cdots, n$, in the above satisfy (1) and (2).

Case 3 n is odd, $n = 2k+1$, $d < n \leq 2d$, and $n \not\equiv 0 \pmod{2d+1}$. Let $\lceil \frac{d}{k} \rceil = q$. Suppose $\sigma_{d,1}(C_{2k+1}) = l \leq 2d+q-1$. Let f be an l-$(d,1)_c$-labeling for C_{2k+1}. Let the vertices of C_{2k+1}, in clockwise order, be v_0, a_1, $a_2, \cdots, a_k, b_k, b_{k-1}, \cdots, b_1$. Without loss of generality, let $f(v_0) = 0$. By definition of circular labeling, we have

$$\begin{aligned} & |f(a_1) - f(b_1)|_l \leq l - 2d \leq q - 1 \text{ (since } l \leq 2d+q-1) \\ \Rightarrow\ & |f(a_2) - f(b_2)|_l \leq 2(q-1) \\ \Rightarrow\ & |f(a_3) - f(b_3)|_l \leq 3(q-1) \\ \Rightarrow\ & \cdots\cdots\cdots \\ \Rightarrow\ & |f(a_k) - f(b_k)|_l \leq k(q-1) < d. \end{aligned}$$

(The above inequalities follow from the fact that for any vertex v, if $f(v) = x$ then the labels of the two neighbors of v must be within the range from $x-d$ to $x+d$, i.e., $[x-d, x+d]_l$, here we view the labels $\{0, 1, 2, \cdots, l-1\}$ in clockwise order modular l, for instance $[2,4]_7 = \{2,3,4\}$ and $[4,1]_6 = \{4,5,0,1\}$. The last inequality is true since $\lceil \frac{d}{k} \rceil = q$.) This contradicts the fact that a_k and b_k are adjacent.

To complete the proof, it is enough to find a $(2d+q)$-$(d,1)_c$-labeling for C_{2k+1}. Now we let the vertices of C_n, in clockwise order, be $v_1, v_2, \cdots, v_{2k+1}$ and let $d = xk + r$ for some x and $1 \leq r \leq k$. Then $q = x+1$. Define the coloring by $f(v_1) = 0$; and

$$f(v_{i+1}) = \begin{cases} f(v_i) + d + 1, & \text{if } 1 \leq i < 2(k-r) - 1 \text{ and } i \text{ is odd;} \\ f(v_i) + d, & \text{otherwise.} \end{cases}$$

(The above coloring function is taken under modular $(2d+q)$.) Then $f(v_{2k+1}) = d + q$, and f is a $(2d+q)$-$(d,1)_c$-labeling for C_n. Q.E.D.

The result above has also been proved, lately and independently, by Wu and Yeh [13].

3 Generalized Circular Distance Labeling and Its Relations to Circular Chromatic Number

The circular distance two labeling is a special case of circular distance p labeling, $p \geq 1$, introduced and studied by ven den Heuvel at el [7]. Given a graph G and integers $d_1 \geq d_2 \geq \cdots \geq d_p \geq 1$, an m-labeling f of G, $f : V(G) \to \{0, 1, 2, \cdots, m-1\}$, satisfies the constraints $(d_1, d_2, \cdots, d_p)$ if $|f(u) - f(v)|_m \geq d_i$, for all $i \in \{1, 2, \cdots, p\}$ and $d_G(u,v) = i$. The minimum

m of such a labeling for G is denoted by $\sigma(G; d_1, d_2, \cdots, d_p)$; and for the case $d_1 = d_2 = \cdots d_p$, it is denoted by $\sigma(G; (d_1)^p)$.

The σ-number of a graph G is closely related to the circular chromatic number of G. Let a and b be positive integers such that $a \geq 2b$. An (a, b)-coloring of a graph $G = (V, E)$ is a mapping c from V to $\{0, 1, \cdots, a-1\}$ such that $|c(x) - c(y)|_a \geq b$ for any edge xy in G. The *circular chromatic number* $\chi_c(G)$ of G is the infimum of a/b for which there exists an (a, b)-coloring of G. The circular chromatic number is also known as the *star-chromatic number* in the literature [11].

Theorem 3.1 *For any graph G and positive integer d, $\sigma(G; d) = \lceil d\chi_c(G) \rceil$.*

Proof. By definition, $\chi_c(G) \leq \frac{\sigma(G;d)}{d}$. Since $\sigma(G; d)$ is an integer, we have $\sigma(G; d) \geq \lceil d\chi_c(G) \rceil$.

Let $\chi_c(G) = p/q$, $\gcd(p, q) = 1$, and $m = \lceil pd/q \rceil$. To complete the proof, it suffices to find an m-labeling f, $f : V(G) \to \{0, 1, 2, \cdots, m-1\}$ such that $|f(u) - f(v)|_m \geq d$ for all $uv \in E(G)$. This is equivalent to finding an (m, d)–coloring for G. Since $\chi_c(G) = p/q$, there is a (p, q)-coloring for G, implying the existence of an (m, d)–coloring for G. Q.E.D.

Given a graph G and a positive integer k, the *k-th power of G*, denoted by G^k, is defined by $V(G^k) = V(G)$ and $uv \in E(G^k)$ if and only if $d_G(u, v) \leq k$. It is easy to see that $\sigma(G; (d)^k) = \sigma(G^k; d)$, so we have:

Corollary 3.2 *For any graph G and positive integers k and d, $\sigma(G; (d)^k) = \lceil d\chi_c(G^k) \rceil$.*

It is known [9] that $\chi_c(C_n^k) = \frac{n}{\lfloor \frac{n}{k+1} \rfloor}$, thus we have:

Corollary 3.3 *For any positive integers $n \geq 3$, k and d, $\sigma(C_n; (d)^k) = \lceil \frac{nd}{\lfloor \frac{n}{k+1} \rfloor} \rceil$.*

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