Algebraic connectivity of the line graph, the middle graph and the total graph of a regular graph

Masakazu Nihei
Joso Gakuin High School
Tuchiura, Ibaraki, 300-0849, Japan
e-mail mznihei@anet.ne.jp

Abstract

Let G be a graph on p vertices and denote by $L(G) = D(G) - A(G)$ the difference between the diagonal matrix of vertex degrees and the adjacency matrix. It is not difficult to see that $L(G)$ is positive semidefinite symmetric and its second smallest eigenvalue, $a(G) > 0$, if and only if G is connected. This observation led M.Fiedler to call $a(G)$ the algebraic connectivity of G.

The algebraic connectivity of the line graph, the middle graph and the total graph of a regular graph are given.

1. Introduction and Preliminaries

We begin with a few definitions and some notations. We consider finite undirected graphs without loops or multiple edges. We let $V(G) = \{v_1, \cdots, v_p\}$ and $E(G) = \{e_1, \cdots, e_q\}$ be the set of vertices and set of edges of a graph, respectively.

Let $A = A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise, be the adjacency matrix of G.

Let $L(G) = D(G) - A(G) = D - A$, where $D = \mathrm{diag}(d_1, \cdots, d_p)$ and $d_i = d(v_i)$ is the degree of vertex v_i ($i = 1, \cdots, p$). Following [4] we will refer to $L(G)$ as a Laplacian matrix.

Let $p \geqq 2$ and $0 = \lambda_1 \leqq \lambda_2 = a(G) \leqq \lambda_3 \leqq \cdots \leqq \lambda_p$ be the eigenvalues of the matrix $L(G)$. It is well known that the second smallest eigenvalue $a(G)$ is zero if and only if G is not connected ([4]). This observation led M.Fiedler to think of $a(G)$ as a quantitative measure of connectivity ([4]). Following him, we call it the algebraic connectivity of the graph G.

For example, let K_p and C_p be the complete graph ($p \geqq 1$) and the circuit graph of order p ($p \geqq 3$), respectively. Then $a(K_p) = p$ and $a(C_p) = 2(1 - \cos(2\pi/p))$ ([4]).

Furthermore, it is also known ([4]) that $a(G) \leqq \kappa(G) \leqq \kappa_1(G)$ for any non-complete graph G, where $\kappa(G)$ and $\kappa_1(G)$ are the vertex-connectivity and the edge-connectivity of a graph G, respectively.

The line graph G_L of G is the graph on $E(G)$ in which $e, f \in E(G)$ are adjacent as vertices if and only if they are adjacent as edges in G.

ARS COMBINATORIA 69(2003), pp. 215-221

The middle graph $M(G)$ is the graph obtain from G by inseting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G.

The total graph $T(G)$ of a graph G is the graph whose vertex set can be put in one-to-one correspondence with the set $V(G) \cup E(G)$ such that two vertices of $T(G)$ are adjacent if and only if the corresponding element of G are adjacent or incident.

The characteristic polynomial of $A(G)$ will be denoted by $\Phi(G; \lambda)$, and referred to as the characteristic polynomial of G. The characteristic polynomial of $L(G)$ will be denoted by $\Phi(L(G); \lambda)$. We refer to a regular graph G of degree k as k-regular, and we denote a graph with p vertices and q edges by (p, q) graph.

Theorem A([2], p61). If G is a k-regular (p, q) graph, then

$$\Phi(G_L; \lambda) = (\lambda + 2)^{q-p} \Phi(G; \lambda - k + 2),$$

where $q = kp/2$.

Theorem B([3], p18). If λ is an eigenvalue of a line graph G_L, then $\lambda \geqq -2$.

Theorem C([2], p64). If G is a k-regular (p, q) graph, then

$$\Phi(T(G); \lambda) = (\lambda + 2)^{q-p} \prod_{i=1}^{p} (\lambda^2 - (2\lambda_i + k - 2)\lambda + \lambda_i^2 + (k-3)\lambda_i - k),$$

where λ_i $(i = 1, \cdots, p)$ being the eigenvalues of $A(G)$ and $q = kp/2$.

Let us denote the largest degree of G by Δ, and the diameter of G by $\mathrm{diam}(G)$, respectively. Then the following theorem holds.

Theorem D([5], B.Mohar). Let G be a graph of order p, then

$$\mathrm{diam}(G) \leqq 2 \left\lceil (\Delta + a(G)) \log(p-1) / 4a(G) \right\rceil,$$

where $\log x$ is the natural logarithm and $\lceil x \rceil$ is the smallest integer not less than x.

The main purpose of this article is to give the algebraic connectivity of the line graph, the middle graph and the total graph of a regular graph.

Terms not defined here can be found in [1].

2. Results

Let us denote the identity matrix of order p by I_p and the determinant of a square matrix A by detA. Furthermore, in what follows, we will assume that a graph G is connected since the algebraic connectivity of the line graph G_L, the middle graph $M(G)$ and the total graph $T(G)$ are all zero when G is disconnected.

Lemma 1. Let G be a k-regular graph of order p, then

$$\Phi(L(G); \lambda) = (-1)^p \Phi(G; -\lambda + k).$$

Proof. $\Phi(L(G); \lambda) = \det(\lambda I_p - (kI_p - A(G)) = (-1)^p \det((-\lambda + k)I_p - A(G))$.

This completes the proof. □

From Lemma 1 and the definition of a(G), we can easily obtain the following result.

Lemma 2. Let G be a k-regular graph of order p with at least two vertices and $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_{p-1} \leqq \lambda_p$ be the eigenvalues of G, then

$$a(G) = k - \lambda_{p-1}$$

We first will give the algebraic connectivity of the line graph and the middle graph of a regular graph.

Theorem 1. Let G be a k-regular graph with at least three vertices, then

$$a(G_L) = a(G),$$

where G_L denotes the line graph of G.

proof. Let G be a k-regular (p, q) graph and $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_p$ be the eigenvalues of G, then we have

$$\Phi(G; \lambda) = \prod_{i=1}^{p}(\lambda - \lambda_i).$$

Since G is k-regular, by Theorem A, we have

$$\Phi(G_L; \lambda) = (\lambda + 2)^{q-p} \Phi(G; \lambda - k + 2)$$

$$= (\lambda + 2)^{q-p} \prod_{i=1}^{p}(\lambda - (\lambda_i + k - 2)).$$

Here, since $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_p$, we obtain

$$\lambda_1 + k - 2 \leqq \lambda_2 + k - 2 \leqq \cdots \leqq \lambda_p + k - 2.$$

On the other hand, from Theorem B we easily see that $\lambda_i + k - 2 \geqq -2$ ($i = 1, \cdots, p$).

This implies that the second largest eigenvalue of G_L is $\lambda_{p-1} + k - 2$. Since G_L is $(2k - 2)$-regular graph, from Lemma 2 we obtain

$$a(G_L) = 2k - 2 - (\lambda_{p-1} + k - 2) = k - \lambda_{p-1}.$$

By the way, Since G is k-regular, from Lemma 2 we see that $a(G) = k - \lambda_{p-1}$.

This completes the proof. □

For example, the line graph of K_p is sometimes called the triangle graph and denoted by Δ_p.

Then, from Theorem 1, we see that $a(\Delta_p) = a(K_p) = p$ ($p \geqq 3$).

By the way, if a graph G is not regular, Theorem 1 does not hold. In fact, the line graph of the star graph $K(1, p)$ is isomorphic to the complete graph K_p. But the algebraic connectivity of $K(1, p)$ is always one although $a(K_p)$ is p.

From Theorem D and Theorem 1, we also have the following result.

Corollary 1. Let G be a k-regular graph of order p with at least three vertices, then

$$\operatorname{diam}(G_L) \leqq 2 \lceil (2k + a(G) - 2)\log(kp/2 - 1)/4a(G) \rceil .$$

Lemma 3. If G is a k-regular (p, q) graph with at least two vertices, then

$$\Phi(L(M(G)); \lambda) = (-1)^q (\lambda - k)^{p-q} (\lambda - k - 1)^q \Phi(G_L; (\lambda^2 - 3k\lambda + 2k^2 - 2)/(k + 1 - \lambda)).$$

proof. Since G is a k-regular (p, q) graph, without loss of generality we may write

$$A(M(G)) = \begin{bmatrix} O & B \\ B' & A_L \end{bmatrix} \quad \text{and} \quad D(M(G)) = \begin{bmatrix} kI_p & O \\ O & 2k\, I_q \end{bmatrix}$$

where B and A_L are the incidence matrix of G and adjacency matrix of the line graph of G, respectively.

Hence, we obtain

$$\Phi(L(M(G)); \lambda) = \det(\lambda I_{p+q} - L(M(G)))$$

$$= \begin{vmatrix} (\lambda - k)I_p & B \\ B' & (\lambda - 2k)I_q + A_L \end{vmatrix}$$

$$= \det((\lambda - k)I_p)\det\{(\lambda - 2k)I_q + A_L - B'((\lambda - k)I_p)^{-1}B\}.$$

Here, noticing that $B'B = A_L + 2I_q$ and $((\lambda - k)I_p)^{-1} = (\lambda - k)^{-1} I_p$, we have

$$\Phi(L(M(G)); \lambda) = (-1)^q (\lambda - k)^{p-q} (\lambda - k - 1)^q \det\{((\lambda^2 - 3k\lambda + 2k^2 - 2)/(k + 1 - \lambda))I_q - A_L\} = (-1)^q (\lambda - k)^{p-q} (\lambda - k - 1)^q \Phi(G_L; (\lambda^2 - 3k\lambda + 2k^2 - 2)/(k + 1 - \lambda)).$$

This completes the proof. □

Theorem 2. Let G be a k-regular graph with at least two vertices, then

$$a(M(G)) = (k + a + 2 - \sqrt{(k - a)^2 + 4(k + 1)})/2,$$

where $a = a(G)$.

proof. For the sake of brevity, let us set $k + 1 - \lambda = b$ and $\lambda^2 - 3k\lambda + 2k^2 - 2 = c$. Let G be a (p, q) graph. Then, from Lemma 3 and $\lambda - k = 1 - b$, we obtain

$$\Phi(L(M(G)); \lambda) = (-1)^q (1 - b)^{p-q} (-b)^q \Phi(G_L; c/b). \qquad ①$$

On the other hand, from Theorem A, we have

$$\Phi(G_L; c/b) = (c/b + 2)^{q-p} \Phi(G; c/b - k + 2). \qquad ②$$

Hence, from ① and ② we have

$$\Phi(L(M(G));\lambda) = (1-b)^{p-q} b^{p} (c+2b)^{q-p} \Phi(G;(c-bk+2b)/b).$$

Here let us set $\Phi(G;\lambda) = \prod_{i=1}^{p}(\lambda - \lambda_i)(\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_p)$.

Then from $1-b = \lambda - k$, $c+2b = (\lambda-k)(\lambda-2k-2)$ and $c - bk + 2b = \lambda^2 - 2(k+1)\lambda + k^2 + k$, we can obtain

$$\Phi(L(M(G));\lambda) = (\lambda-2k-2)^{q-p}\prod_{i=1}^{p}(\lambda^2-(2k+2-\lambda_i)\lambda+k^2+(1-\lambda_i)k-\lambda_i).$$

Now, it is easy to see that

$\lambda^2-(2k+2-\lambda_i)\lambda+k^2+(1-\lambda_i)k-\lambda_i$

$= \{\lambda-(2k+2-\lambda_i+\sqrt{\lambda_i^2+4k+4})/2\}\{\lambda-(2k+2-\lambda_i-\sqrt{\lambda_i^2+4k+4})/2\}$.

Moreover, we can easily check that the function $f(x) = (c - x - \sqrt{x^2+2c})/2$ $(c > 0)$ is decreasing on the closed interval $[-k, k]$.

From these it is easy to see that

$a(M(G)) = (2k+2-\lambda_{p-1} - \sqrt{\lambda_{p-1}^2+4k+4})/2$.

By the way, since $a(G) = k - \lambda_{p-1}$, we can obtain

$a(M(G)) = (k+a+2-\sqrt{(k-a)^2+4(k+1)})/2$.

This completes the proof. □

Corollary 2. Let G be a regular graph with at least two vertices, then

$a(M(G)) \leqq a(G)$.

proof. Let G be a k-regular graph, then from Theorem 2 and $(k-a+2)^2 \leqq (k-a)^2+4(k+1)$, we have

$a(M(G)) \leqq \{(k+a+2)-(k-a+2)\}/2 = a$,

where $a = a(G)$. This completes the proof. □

Lastly we will give the algebraic connectivity of the total graph $T(G)$ of a regular graph G.

If G is a k-regular (p, q) graph, $T(G)$ becomes 2k-regular graph of the order $p+q$. Then, from Theorem C, we see that the distinct eigenvalues of $T(G)$ are

-2 and $(2\lambda_i+k-2\pm\sqrt{4\lambda_i+k^2+4})/2$ $(i = 1, \cdots, p)$.

Here, notice that $-k \leqq \lambda_i \leqq k$ $(i = 1, \cdots, p)$ and the function

$g(x) = (2x+k-2+\sqrt{4x+k^2+4})/2$

is increasing on the closed interval $[-k, k]$.

These imply that the second largest eigenvalue of $T(G)$ is

$(2\lambda_{p-1}+k-2+\sqrt{4\lambda_{p-1}+k^2+4})/2$,

where λ_{p-1} is the second largest eigenvalue of G.

Since $T(G)$ is 2k-regular graph, by Lemma 2, we have

$a(T(G)) = 3k/2 - \lambda_{p-1} + 1 - \sqrt{4\lambda_{p-1}+k^2+4}/2$.

On the other hand, since $\lambda_{p-1} = k - a(G)$, we can obtain the following theorem.

Theorem 3. Let G be a k-regular graph with at least two vertices, then

$$a(T(G)) = a + k/2 + 1 - \sqrt{(k+2)^2 - 4a}\,/\,2,$$

where $a = a(G)$.

From this theorem, we immediately have

Corollary 3. Let G be a regular graph with at least two vertices, then

$$a(G) \leqq a(T(G)).$$

Furthermore, since the function $h(x) = (x + 2k)/4x$ $(k > 0)$ is decreasing on the open interval $(0, \infty)$, we also obtain the following result.

Corollary 4. Let G be a k-regular graph of order $p\,(p \geqq 2)$, then

$$\mathrm{diam}(T(G)) \leqq 2 \lceil (2k + a(G)) \log(k\,p/2 + p - 1)/4a(G) \rceil .$$

3. Conclusion

If G is a regular graph with at least three vertices, combining Theorem 1 with Corollary 2 and 3, we have

$$a(M(G)) \leqq a(G) = a(G_L) \leqq a(T(G)).$$

Here, we give the definition of the subdivision graph of a graph.

The subdivision graph $S(G)$ of a graph G is obtained from G by inserting an additional vertex each edge of G. Obviously, $S(G)$ is a spanning subgraph of $M(G)$ and $T(G)$.

If G is a k-regular graph, using the same argument as in Lemma 3 and Theorem 2, we have

$$a(S(G)) = (k + 2 - \sqrt{(k+2)^2 - 4a}\;)/2.$$

Noticing that $a(S(G)) \leqq a(M(G))$, we also have

$$a(S(G)) \leqq a(M(G)) \leqq a(G) = a(G_L) \leqq a(T(G)),$$

where G is a regular graph with at least three vertices.

Acknowlegement : The author would like to thank the referee for useful suggestions.

References

[1] G.Chartrand and L.Lesniak, Graphs and Digraphs, Wadsworth and Brooks/Cole Advanced Books and Software, Monterey, 1986.

[2] D.M.Cvetković, M.Doob and H.Sachs, Spectra of Graphs, Academic Press, New York, 1979.

[3] N.Biggs, Algebraic Graph Theory, Cambrige University Press, 1974.

[4] M.Fiedler, Algebraic connectivity of graphs, Czech.Math.J. 23 (1973), 298-305.

[5] B.Mohar, Eigenvalues, diameter, and mean distance in graphs, Graphs and Combinatorics 7 (1991) 53-64.
[6] T.Hamada and I.Yoshimura, Traversability and connectivity of the middle graph of a graph, Discrete Math.14(1976), 247-255.