New Families of Sequential Graphs

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ABSTRACT. Let $G = (V(G), E(G))$ be a finite simple graph with p vertices and n edges. A labeling of G is an injection $f: V(G) \to Z_n$. A labeling of G is called 2-sequential if $f(V(G)) = \{r, r+1, \ldots, r+p-1\}$ $(0 \leq r < r+p-1 \leq n-1)$ and the induced edge labeling $f^*: E(G) \to \{0, 1, \ldots, n-1\}$ given by

$$f^*(u,v) = f(u) + f(v), \text{ for every edge } (u,v)$$

forms a sequence of distinct consecutive integers $\{k, k+1, \ldots, n+k-1\}$ for some k $(1 \leq k \leq n-2)$.

By utilizing the graphs having 2-sequential labeling, several new families of sequential graphs are presented.

1 Introduction

Harmonious labelings of graphs have been introduced in 1980 by Graham and Sloane [1]. Results on harmonious labelings can be found in [3]. Unless specified otherwise, we will assume that $G = (V(G), E(G))$ is a finite simple graph with p vertices and n edges. A *labeliag* of G is an injection $f: V(G) \to Z_n$. A labeling of G is called *harmonious* if the induced edge labeling $f^*: E(G) \to \{0, 1, \ldots, n-1\}$ given by

$$f^*(u,v) = f(u) + f(v) \pmod{n}, \text{ for every edge } (u,v)$$

is $1-1$. If G is a tree, then exactly one label may be used on two distinct vertices.

In 1983, Grace [2] defines a labeling f as a *sequential labeling* if the set of induced edge labelings given as

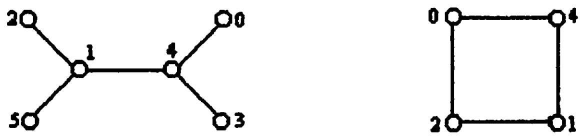
$$f^*(u,v) = f(u) + f(v), \text{ for all edges } (u,v)$$

forms a sequence of distinct consecutive integers, say $\{k, k+1, \ldots, n+k-1\}$ for some k $(1 \leq k \leq n-2)$. For example, the graphs shown in Figure 1 - (1) and (2) have a harmonious labeling and a sequential labeling, respectively. Clearly, a sequential labeling is a harmonious labeling by reducing the edge labelings modulo n. A graph G with a harmonious (sequential) labeling is known as a *harmonious* (*sequential*) *graph*.

(1) (2)

Figure 1

An *improperly labeling* of G is an injection $f\colon V(G) \to Z_{n+1}$. The definitions of an *improperly harmonious labeling* and an *improperly sequential labeling* of G are defined similarly. As an example, we can take the graphs depicted in Figure 2 - (1) and (2) together with an improperly harmonious labeling and an improperly sequential labeling. Note that the cycle C_4 in Figure 2 - (2) has no sequential labeling. Clearly, a harmonious (sequential) labeling is an improperly harmonious (improperly sequential) labeling as well.



(1) (2)

Figure 2

2 The construction

In this section, we will construct three families of sequential graphs by the graphs having sequential labelings or improperly sequential labelings. A (improperly) sequential labeling f of a graph G is called (*improperly*) *2-sequential* if $f(V(G)) = \{r, r+1, \ldots, r+p-1\}$,where $r \geq 0$ and $r+p-1 \leq n-1(n)$. It is obvious that each (improperly) sequential graph corresponds to at least a (improperly) 2-sequential graph. For example, the complete graph K_4, shown in Figure 3 - (1), has a sequential labeling and the graphs, depicted in Figures 3 - (2) and (3), have 2-sequential labelings. Remark that an improperly sequential tree itself is also an improperly 2-sequential tree. Figure 2 - (1) is an easy example.

A *caterpillar* is a tree with all vertices either on a single central path, or distance 1 away from it. The central path may be referred to be the longest path in the caterpillar, such that both end-vertices have degree 1.

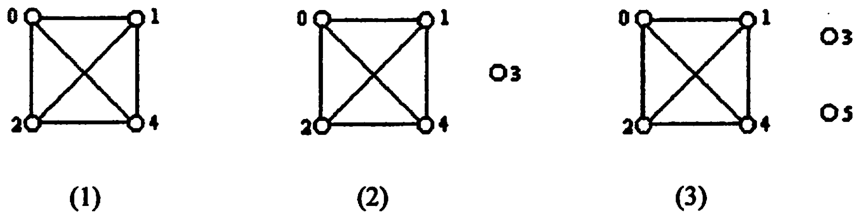


Figure 3

Construction I: *The join $G+H$ of disjoint graphs G and H is the graph obtained by joining each vertex of H to each vertex of G.*

Theorem 2.1. *Let G be an improperly 2-sequential graph. Then the graph $G+\overline{K_m}$ is sequential, $m \geq 1$.*

Proof: Let f be any improperly 2-sequential labeling of G and suppose that $V(G) = \{u_1, u_2, \ldots, u_p\}$ and $f(u_i) = r+i-1$ $(1 \leq i \leq p)$. For convenience, set $f^*(E(G)) = \{k, k+1, \ldots, k+n-1\}$ $(1 \leq k \leq n-2)$ and set $\overline{K_m} = \{v_1, v_2, \ldots, v_m\}$. Let us introduce a labeling g of $G+\overline{K_m}$ defined as

$$g(t) = \begin{cases} r+i-1, & \text{if } t = u_i,\ 1 \leq i \leq p, \\ n+k-r+(i-1)p, & \text{if } t = v_i,\ 1 \leq i \leq m, \end{cases}$$

where all vertices $t \in V(G+\overline{K_m})$.

By easy verification, it can be shown that g is a sequential labeling of $G+\overline{K_m}$. □

Remark. Theorem 2.1 generalizes the result of Proposition 4 - (1) of [2].

Lemma 2.2. *Each caterpillar has an improper 2-sequential labeling.*

Proof: Suppose that T is a caterpillar with $n+1$ vertices and n edges. Draw T as a bipartite graph in the plane with r (≥ 1) vertices on the left and $n+1-r$ vertices on the right, and with the first vertex of the central path on the left (see Figure 4). Let f be a labeling of T defined as

$$f(v_i) = i, \text{ for every vertex } v_i \in V(T).$$

It is clear that f is an improperly 2-sequential labeling of T. □

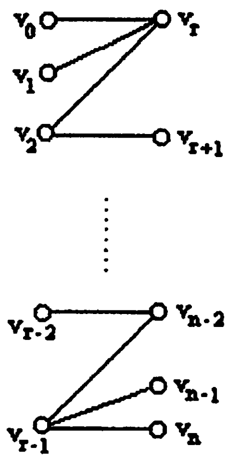


Figure 4

Combined Theorem 2.1 and Lemma 2.2, the following is readily given.

Corollary 2.3. *If T is a caterpillar, then $T + \overline{K_m}$ is sequential.*

Remark. Theorem 17 in [1] is a special case of Corollary 2.3 when T is a path and $m = 1$. Moreover, Chang, Hsu and Rogers [8] proved that the graph $S_m + K_1$ is harmonious, where S_m is the star with $m + 1$ vertices.

Lemma 2.4. *The cycle C_{2n+1} ($n \geq 1$) has a 2-sequential labeling.*

Proof: The cycle C_{2n+1} is the graph induced by the edges $\{(v_i, v_{i+1}), (v_0, v_{2n}) \mid i \in Z_{2n}\}$. Let f be a labeling of C_{2n+1} defined by

$$f(v_i) = \begin{cases} \frac{i}{2}, & \text{if } i = 0 \text{ or } i \text{ is even,} \\ n + \frac{i+1}{2}, & \text{if } i \text{ is odd,} \end{cases}$$

where each vertex v_i in C_{2n+1}.

It is not difficult to check that f is a 2-sequential labeling of C_{2n+1}. □

By utilizing Theorem 2.1 and Lemma 2.4, we have

Corollary 2.5. *The graph $C_{2n+1} + \overline{K_m}$ is sequential.*

Remark. Graham and Sloane [1] has proved that the graph $C_n + K_1$ is harmonious and Gnanajothi [4] has shown that the graph $C_n + \overline{K_2}$ is harmonious if n is odd and not harmonious if $n \equiv 2, 4, 6 \pmod 8$.

Construction II: The graph $G \odot \overline{K_p}$ is one obtained from G by attaching p (≥ 1) pendant edges at each vertex of G.

Let G_{2n+1}^{3n-k+1} denote the graph G with $2n+1$ vertices and $3n-k+1$ edges, where $1 \le k \le n$. A 2-sequential labeling f of G_{2n+1}^{3n-k+1} is said to be a *strongly 2-sequential labeling* if $f(V(G_{2n+1}^{3n-k+1})) = \{0, 1, \ldots, 2n\}$ and $f^*(E(G_{2n+1}^{3n-k+1})) = \{k, k+1, \ldots, 3n\}$. Notice that the cycle C_{2n+1} mentioned in Lemma 2.4 is the graph G_{2n+1}^{2n+1} having a strongly 2-sequential labeling.

Theorem 2.6. *Suppose that G_{2n+1}^{3n-k+1} has a strongly 2-sequential labeling. Then the graph $G_{2n+1}^{3n-k+1} \odot \overline{K_p}$ is 2-sequential, $p \ge 1$.*

Proof: Suppose that $V(G_{2n+1}^{3n-k+1}) = \{u_0, u_1, \ldots, u_{2n}\}$, and that $f(u_j) = j$, where f is any strongly 2-sequential labeling of G_{2n+1}^{3n-k+1} and $0 \le j \le 2n$. Let vertices $u_{j,1}, u_{j,2}, \ldots, u_{j,p}$ denote the pendant edges of the vertex u_j of $G_{2n+1}^{3n-k+1} \odot \overline{K_p}$, $0 \le j \le 2n$. Let g be a labeling of $G_{2n+1}^{3n-k+1} \odot \overline{K_p}$ defined as

$$g(v) = \begin{cases} j, & \text{if } v = u_j,\ 0 \le j \le 2n, \\ 3n+j+2, & \text{if } v = u_{j,1},\ 0 \le j \le n-1, \\ n+j+1, & \text{if } v = u_{j,1},\ n \le j \le 2n, \\ n+j+1+r(2n+1), & \text{if } v = u_{j,r},\ 0 \le j \le n-1 \text{ and } 2 \le r \le p, \\ j-n+r(2n+1), & \text{if } v = u_{j,r},\ n \le j \le 2n \text{ and } 2 \le r \le p, \end{cases}$$

where every vertex v in $G_{2n+1}^{3n-k+1} \odot \overline{K_p}$.

By easy verification, it can be shown that g is a sequential labeling of $G_{2n+1}^{3n-k+1} \odot \overline{K_p}$ and in fact, a 2-sequential labeling (The graph $C_5 \odot \overline{K_2}$, shown in Figure 5, is an example, where the cycle C_5 is depicted in Figure 1 - (2)). □

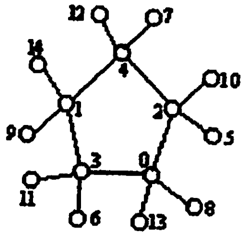


Figure 5

Combining Theorem 2.1 with Theorem 2.6 and Theorem 2.6 with Lemma 2.4, we have the following two results.

Corollary 2.7. *If G_{2n+1}^{3n-k+1} has a strongly 2-sequential labeling, then the graph $(G_{2n+1}^{3n-k+1} \odot \overline{K_p}) + \overline{K_m}$ is sequential.*

Corollary 2.8. *The graph $C_{2n+1} \odot \overline{K_p}$ is 2-sequential.*

Remark. Grace [2] and Liu, Zhang [5] proved that the graph $C_n \odot K_1$ is harmonious.

Construetion III: The product of graphs G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$, in which vertex u_1v_1 is adjacent to vertex u_2v_2 if and only if either $u_1 = u_2$ and $(v_1, v_2) \in E(H)$ or $v_1 = v_2$ and $(u_1, u_2) \in E(G)$.

Let f be a strongly 2-sequential labeling of G_{2n+1}^{3n-k+1}. A *dual labeling* f_d of f on G_{2n+1}^{3n-k+1} is defined as

$$f_d(u) = \begin{cases} n + f(u) + 1, & \text{if } 0 \le f(u) \le n - 1, \\ f(u) - n, & \text{if } n \le f(u) \le 2n, \end{cases}$$

where each vertex u in G_{2n+1}^{3n-k+1}.

As an example, consider the graph G_5^5 shown in Figure 6 - (1) and (2), where f is a strongly 2-sequential labeling of G_5^5.

(1) The graph G_5^5 with f (2) The graph G_5^5 with f_d

Figure 6

Theorem 2.9. *Suppose that G_{2n+1}^{3n-k+1} has a strongly 2-sequential labeling f and that $f_d^*(E(G_{2n+1}^{3n-k+1})) = \{n, n+1, \ldots, 4n-k\}$, where $n - 1 \le k \le n$. Then the graph $G_{2n+1}^{3n-k+1} \times P_m$ is sequential, $m \ge 2$. In particular, if $k = n$, then the graph $G_{2n+1}^{3n-k+1} \times P_m$ is 2-sequential.*

Proof: Let $V(G_{2n+1}^{3n-k+1}) = \{u_0, u_1, \ldots, u_{2n}\}$ and let $f(u_s) = s$, where f is any strongly 2-sequential labeling of G_{2n+1}^{3n-k+1} and $0 \le s \le 2n$. The graph $G_{2n+1}^{3n-k+1} \times P_m$ is shown in Figure 7.

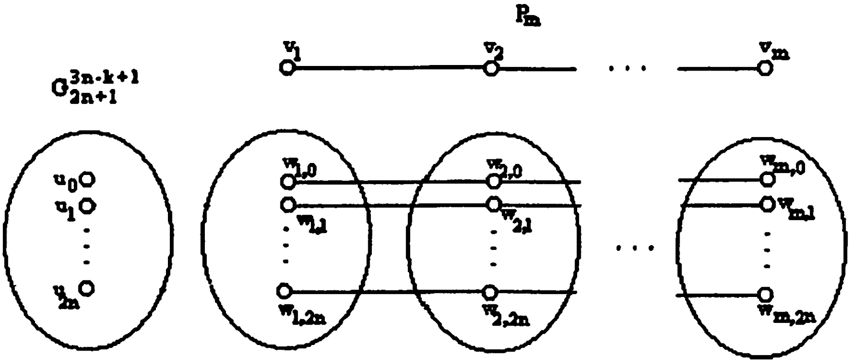


Figure 7. The graph $G_{2n+1}^{3n-k+1} \times P_m$

Case 1: $k = n$.

Subcase 1: m is odd.

Let us introduce a labeling g of $G_{2n+1}^{3n-k+1} \times P_m$ given by

$$g(t) = \begin{cases} f(u_s) + (r-1)(2n+1), \\ \quad \text{if } t \in w_{r,s}, \text{ for } r = 1, 3, \ldots, m \text{ and } 0 \le s \le 2n, \\ f_d(u_s) + (r-1)(2n+1), \\ \quad \text{if } t \in w_{r,s}, \text{ for } r = 2, 4, \ldots, m-1 \text{ and } 0 \le s \le 2n, \end{cases}$$

where each vertex u_s in G_{2n+1}^{3n-k+1} and each vertex t in $G_{2n+1}^{3n-k+1} \times P_m$.

Subcase 2: m is even.

Similar to Subcase 1 and omitted.

Case 2: $k = n - 1$.

Subcase 1: m is odd.

Let g be a labeling of $G_{2n+1}^{3n-k+1} \times P_m$ defined as

$$g(t) = \begin{cases} f(u_s) + (r-1)(2n+1) + \frac{r-1}{2}, \\ \quad \text{if } t \in w_{r,s}, \text{ for } r = 1, 3, \ldots, m \text{ and } 0 \le s \le 2n, \\ f_d(u_s) + (r-1)(2n+1) + \frac{r-2}{2}, \\ \quad \text{if } t \in w_{r,s}, \text{ for } r = 2, 4, \ldots, m-1 \text{ and } 0 \le s \le 2n, \end{cases}$$

where each vertex u_s in G_{2n+1}^{3n-k+1} and each vertex t in $G_{2n+1}^{3n-k+1} \times P_m$.

Subcase 2: m is even.

Similar to Subcase 1 and omitted.

By routine computation, it follows that g in each case is a sequential labeling of $G_{2n+1}^{3n-k+1} \times P_m$. (see Figure 8, where G_5^5 is the graph of Figure 6 - (1)). □

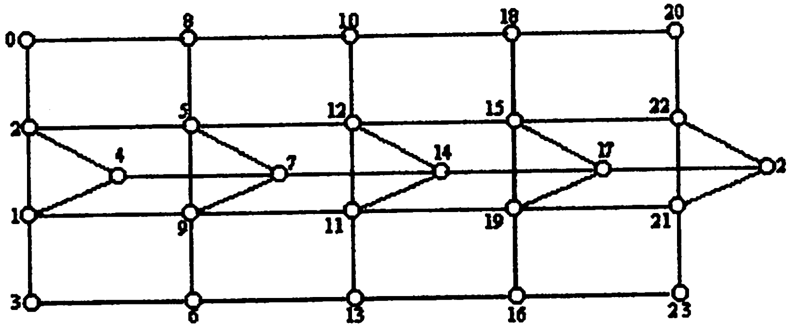


Figure 8. The graph $G_5^5 \times P_5$ with a 2-sequential labeling

Combining Theorem 2.1 with Theorem 2.9 and Lemma 2.4 with Theorem 2.9, the following result is given.

Corollary 2.10.

(1) *If* G_{2n+1}^{2n+1} *has a strongly 2-sequential labeling* f *and* $f_d^*(E(G_{2n+1}^{2n+1}))$ $= \{n, n+1, \ldots, 3n\}$, *then the graph* $(G_{2n+1}^{3n-k+1} \times P_m) + \overline{K_r}$ *is sequential.*

(2) *The prism* $C_{2n+1} \times P_m$ *is 2-sequential.*

Remark. The prism $C_n \times P_m$ is harmonious (1) if $m = 2$, $n \neq 4$ [6] (2) if m is odd [1] (3) if $n = 4$, $m \geq 3$ [7].

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