TO DETERMINE 1-EXTENDABLE GRAPHS AND ITS ALGORITHM

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Abstract. Let G be a graph with a perfect matching M_0. It is proved that G is 1-extendable if and only if for any pair of vertices x and y with an M_0-alternating xy-path P_0 of length three which starts with an edge that belongs to M_0, there exists an M_0-alternating path P connecting x and y, of which the starting and the ending edges do not belong to M_0. With this theorem, we develop a polynomial algorithm that determines whether the input graph G is 1-extendable, the tine complexity of the algorithm is $O(|E|^2)$.

1. Introduction

. Let k be a positive integer. A graph G is said to be k-**extendable** if $1 \leq k \leq (|V(G)| - 2)/2$ and every matching of size k can be extended to a perfect matching of G. For any graph G with a perfect matching, the extendability of G, $ext(G)$, is defined to be the maximum integer k such that G is k-extendable. For current results in the subject area of matching extension in graphs, readers are referred to the survey papers [9] and [10] by M. D. Plummer.

An important fundamental question about extendability is to find polynomial algorithms to determine the extendability of a given graph G. This problem remains unsolved.

In this paper, we give a necessary and sufficient condition of 1-extendability of a graph G, which will help to develop a polynomial algorithm to determine whether a graph G is 1-extendable. So for $k = 1$, the answer to the extendability problem is positive, and we will give the polynomial algorithm together with the analysis of its time complexity, which is $O(|E|^2)$. Some other results about 1-extendable graphs will be presented in Section 4.

A graph G is said to be **1-extendable**, if for each edge $e \in E(G)$, there is a perfect matching M of G, such that $e \in M$. An M-alternating path is a path of which the edges appear alternately in M and $E(G) - M$.

Many results have been found for 1-extendable graphs, the following are some necessary and sufficient conditions obtained for 1-extendable graphs.

Little, Grant and Holton[6] gave a necessary and sufficient condition for graphs to be 1-extendable.

Thoereml: Let G be a graph of even order. Then G is 1-extendable if and only if
1.$o(G - S) \leq |S|$, for all $S \subseteq V(G)$, and
2. $o(G - S) = |S|$ implies that S is independent.

ARS COMBINATORIA 69(2003), pp. 223-228

Lovász and Plummer[7] gave a characterization of 1-extendable bipartite graphs in terms of Bipartite Ear Decomposition, which is referred to as the Ear Structure Theorem. A Bipartite Ear Decomposition is a decomposition of a bipartite graph G. Let e be an edge, P_1 is an odd length path connecting the endpoints of e, then let $G_1 = e + P_1$, proceed inductively to build a sequence of bipartite graphs as follows: If $G_{r-1} = e + P_1 + \cdots + P_{r-1}$ has been constructed, add an r^{th} ear P_r by joining any two vertices in different partitions of G_{r-1} , here P_r is an odd path having no other vertices in common with G_{r-1} . The decomposition $G_r = e + P_1 + \cdots + P_r$ is called a Bipartite Ear Decomposition of G_r . The following theorem can be used to find all 1-extendable bipartite graphs.

Thoerem2: A graph G is 1-extendable and bipartite if and only if G has a Bipartite Ear Decomposition.

Lovász and Plummer[8] extended the above result by pointing out the ear structure of 1-extendable general graphs. A subgraph G' of a graph G is called nice if $G - V(G')$ has a perfect matching. An ear of G relative to G' is any odd path in G having both endpoints—but no interior vertices— in G' . An ear decomposition of G starting with G' is a representation of G in the form $G = G' + P_1 + \cdots + P_k$,where P_1 is an ear of $G' + P_1$ relative to G' and P_i is an ear of $G' + P_1 + \cdots + P_i$ relative to $G' + P_1 + \cdots + P_{i-1}$ for $2 \leq i \leq k$.

Thoerem3: Let G be 1-extendable and G' a subgraph of G. Then G has an ear decomposition starting with G' if and only if G' is a nice subgraph of G.

In the following sections, we shall give a new necessary and sufficient condition of 1-extendability of a graph in terms of M-alternating paths. A graph G is 1-extendable if and only if for every pair of vertices x and y, such that x and y are connected by an M-alternating path P_0 of length three, where M is a perfect matching of G, and P_0 starts with an edge in M, there is an M-alternating path P connecting x and y , which starts and ends with edges not in M. According to this necessary and sufficient condition, we develop a polynomial algorithm to determine 1-extendability of a graph. The algorithm has high efficiency since it suffices to find only one perfect matching to determine 1-extendability.

2. A necessary and sufficient condition of 1-extendable graphs

. In this section, we show a necessary and sufficient condition of 1-extendable graphs using M-alternating paths.

Thoerem4: Let G be a graph with a perfect matching M_0, then the following are equivalent:

(a) G is 1-extendable;

(b) For any pair of vertices x and y with an M_0-alternating xy-path P_0 of length three which starts with an edge that belongs to M_0, there exists an M_0-alternating path P connecting x and y, of which the starting and the ending edges are not in M_0.

Proof.

(a) implies (b):

Let $e = ab$ be the edge in $E(P_0) - M_0$. Since G is 1-extendable, e can be extended to a new perfect matching M_0'. Let $H_0 = M_0 \Delta M_0'$, then H_0 contains only even cycles with edges alternately appearing in the two perfect mathcings M_0 and M_0'. Let $P_0 = xaby$ and $e = ab$, where $xa, by \in M_0$. Since $e \notin M_0$ and $P_0 \subseteq H_0$, let C_0 be the cycle in which P_0 lies, then $C_0 - \{a, b\}$ is an M_0-alternating xy-path, which starts and ends with edges that do not belong to M_0.

(b) implies (a):

Let e be any edge of G. If e is in M_0, then e can be extended to M_0. If e is not in M_0, then we can construct a new perfect matching containing e in the following way. Let $e = x_0y_0$, $xx_0 \in M_0$, $yy_0 \in M_0$. Then $P_0 = xx_0y_0y$ is an M_0-alternating path of length three which starts with an edge in M_0. By the assumption, there is an M_0-alternating xy-path P that starts and ends with edges not in M_0. Let $C = P_0 \cup P$, C is an M_0-alternating cycle. Let $M = E(C)\Delta M_0$, then M is a new perfect matching of G that contains e. It follows that G is 1-extendable. □

By the theorem above, we can determine 1-extendability of a graph by checking whether the condition in (b) holds for any given perfect matching. We only need to find one perfect matching of the input graph during our algorithm, which reduces the time complexity of it.

3. Algorithm to determine the 1-extendable graphs

. The algorithm is similar to that of finding out a maximum matching in a general graph. Firstly, we find a perfect matching M in the graph G. Then for each edge that does not belong to M, we extend the edge to an M-alternating path of length three, and construct an M-alternating tree from the origin of the path to the terminal. If there is such a pseudo-M-augmenting path, i.e. an M-alternating path which starts and ends with edges not in M, between each pair of such vertices, then according to Theorem 4 the graph G is 1-extendable. The first polynomial algorithm to find a maximum matching in a general graph was found by Edmonds [2]. Later on, Even and Kariv [3] and Bartnik [1] independently obtained an algorithm to find a maximum matching with the time complexity of $O(|V|^{2.5})$. Kameda and Munro [4] introduced an $O(|E|\,|V|)$ algorithm for maximum matching of graphs.

Now we give the following algorithm to determine the 1-extendable graphs:

Algorithm:

1. Use the algorithm of [4] to find a perfect matching M in G; // O(|V||E|)
2. IF M does not exist THEN RETURN(false); (G is not 1-extendable) // O(1)

3. ELSE
FOR each edge $e = x_0y_0 \in E(G)$ DO // $O(|E|^2)$
BEGIN
IF $e \notin M$ THEN // $O(1)$
BEGIN
Find two vertices x and y such that $x_0x, y_0y \in M$; // $O(1)$
Use Procedure 1 to find an M-alternating path P from x to y such that P starts and ends with edges in $E(G)\backslash M$; // $O(|E|)$
END;
IF the path P does not exist THEN RETURN(false); (G is not 1-extendable) // $O(1)$
END;
4. RETURN(true); (G is 1-extendable) // $O(1)$

Procedure 1 uses the idea of [4] for finding an (x, y) M-augmenting path by building an M-alternating tree from x. It only takes $O(|E|)$ time. The loop in Step 3 repeats $O(|E|)$ times, so Step 3 totally takes $O(|E|^2)$ time. It is easy to see the whole algorithm takes $O(|E|^2)$ time.

If we use the definition of 1-extendable graphs to design an algorithm, then we delete every edge $e = xy$ with its end vertices x and y and find a perfect matching in $G-\{x, y\}$. In this case, the algorithm needs $O(|E|^2|V|)$ time. So our algorithm has higher efficiency.

4. Other results about 1-extendable graphs

. In this section, we give two results which shows that there is an M-alternating path between each pair of vertices x and y for any perfect matching M in a 1-extendable graph.

Theorem 5: Let G be a connected 1-extendable graph. For any pair of distinct vertices x and y, and any perfect matching M of G, there exists an M-alternating xy-path $P : x = a_0, a_1, \cdots, a_t = y$, such that $a_0a_1 \notin M$.

Proof. We proceed induction on the distance $d_G(x, y)$.

If $d_G(x, y) = 1$, then $xy \in E(G)$. If $xy \notin M$, then xy is the required M-alternating xy-path. If $xy \in M$, then $\deg_G(x) \geq 2$, since G is connected and 1-extendable, G must be 2-connected. Let $x' \in N(x) - y$, and extend xx' to a perfect matching M_0. Let $H = M \Delta M_0$, H contains even cycles of which edges appear alternately in M and M_0. Since $xy \in M - M_0$ and $xx' \in M_0 - M$, $x'xy$ lies in a cycle $C = y \cdots x'xy$ in H, and $P = y \cdots x'x$ is the required M-alternating xy-path.

If $d_G(x, y) \geq 2$, let $y' \in V(G)$ be a vertex with $d_G(x, y') = d_G(x, y) - 1$ and $yy' \in E(G)$. By the induction hyperthesis, there is an M-alternating xy'-path P:$x = a_0, a_1, \cdots, a_t = y'$ such that $a_0a_1 \notin M$.

If $y \in V(P)$, then the segment of P from x to y is the required M-alternating xy-path, we denote it as $P(x, y)$.

Otherwise, if $y \notin V(P)$ let $y''y' \in E(P)$. If $y''y' \in M$, then $yy' \notin M$ and $P + y$ is the required M-alternating xy-path. If $y''y' \notin M$ and $yy' \in M$, then $P + y$ is the required M-alternating xy-path.

The remaning case is that $y''y' \notin M$ and $yy' \notin M$. Since G is 1-extendable, let M_0 be the perfect matching that contains yy'. Since $yy' \in M_0 - M$, yy' is in $H = M \Delta M_0$. Let C be the M-M_0-alternating cycle in H which yy' lies in. We denote $C(x_1, x_2)$ the path from x_1 to x_2 in C which starts with an edge not in M and $C'(x_1, x_2)$ the path from x_1 to x_2 in C which starts with an edge in M. If $V(C) \cap P(x, y'') = \emptyset$, then $P(x, y') + C'(y', y)$ is the required M-alternating xy-path. If $V(C) \cap P(x, y'') \neq \emptyset$, choose $a_k \in V(C) \cap P(x, y'')$ such that k is as small as possible. If $x = a_k$,then $x, y \in V(C)$, then $C(x, y)$ is the required M-alternating xy-path. If $x \neq a_k$, then $P(x, a_k)$ must be of odd length, i.e. $a_k a_{k+1} \in M$, since k is chosen as small as possible. Since $a_k, y \in V(C)$, then $P(x, a_k) + C'(a_k, y)$ is the required M-alternating xy-path. □

Theorem 6: Let G be a graph and $x, y \in V(G)$, $x \neq y$. Let M_0 and M be any two different perfect matchings of G. If G has an M_0-alternating xy-path P_0, then G has an M-alternating xy-path P.

Proof. Let $K = (V(G), E(P_0) \Delta M_0 \Delta M)$, $d_K(v)$ be the degree of vertex v in K, and $N_K(v)$ be the neighbour set of vertex v in K. Firstly, we show that

$$d_K(v) = 0 \text{ or } 2 \text{ if } v \notin \{x, y\} \qquad (1)$$

and if $d_K(v) = 2$, one of the two edges incident with v is in M; and

$$d_K(v) = 1 \text{ or } 3 \text{ if } v \in \{x, y\} \qquad (2)$$

Case1: $v \in V(G) - V(P_0)$

Since M_0 and M are perfect matchings, there are v_1, v_2 such that $vv_1 \in M_0$, $vv_2 \in M$. If $v_1 = v_2$, then $d_K(v) = 0$, else $N_K(v) = \{v_1, v_2\}$ and $vv_2 \in M$.

Case2: $v \in V(P_0) - \{x, y\}$

Let $vv_1 \in M_0$, $vv_2 \in M$, $vv_3 \in E(P_0) - M_0$. If $v_2 = v_1$, then $N_K(v) = \{v_2, v_3\}$ and $vv_2 \in M$. If $v_2 = v_3$, then $d_K(v)$= 0. If $v_2 \neq v_1$ and $v_2 \neq v_3$, then $N_K(v) = \{v_2, v_3\}$ and $vv_2 \in M$.

Case3: $v = x$

Let $xx_1 \in M_0$, $xx_2 \in M$, $xx_3 \in E(P_0)$. Firstly, suppose that $x_1 \neq x_3$. If $x_1 = x_2$, then $N_K(v) = \{x_3\}$ and $xx_3 \notin M$. If $x_3 = x_2$, then $N_K(v) = \{x_1\}$ and $xx_1 \notin M$. If $x_1 \neq x_2 \neq x_3$, then $N_K(v) = \{x_1, x_2, x_3\}$.

If $x_1 = x_3$ and $x_1 \neq x_2$, then $N_K(v) = \{x_2\}$ and $xx_2 \in M$. If $x_1 = x_3$ and $x_1 = x_2$, then $N_K(v) = \{x_1\}$ and $xx_1 \in M$.

Case4: $v = y$

Similar to the case $v = x$.

By (1) and (2), the graph K has only two vertices of odd degrees, namely x and y, and so x and y must be in the same component of K. Therefore, there is a path joining x to y and it is easy to see that this path is an M-alternating xy-path. □

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