Upper bounds on signed 2-independence number of graphs

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Abstract

A function f: $V \rightarrow \{-1,1\}$ defined on the vertices of a graph $G = (V, E)$ is a signed 2-independence function if the sum of its function values over any closed neighbourhood is at most one. That is, for every $v \in V$, $f(N[v]) \leq 1$, where $N[v]$ consists of v and every vertex adjacent to v. The weight of a signed 2-independence function is $f(V) = \sum f(v)$, over all vertices $v \in V$. The signed 2-independence number of a graph G, denoted $\alpha_s^2(G)$, is the maximum weight of a signed 2-independence function of G. In this article, we give some new upper bounds on $\alpha_s^2(G)$ of G, and establish a sharp upper bound on $\alpha_s^2(G)$ for an r-partite graph.

Key words: signed 2-independence function; signed domination; r-partite graph.

*Supported by NNSF of China (Grant No. 10101010).

†Supported by grant No.(R02-2000-00018) from the Basic Research Program of the KOSEF

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1 Introduction

We begin with the basic definitions, following the notation of [4]. Let G be a graph with vertex set V of order n and edge set E of size q, and let v be a vertex in V. The *open neighborhood* of v is $N(v) = \{u \in V | uv \in E(G)\}$, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a subset S of V, we set $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = S \cup N(S)$. If T is a subset of V disjoint from S, we let $e(S,T)$ denote the number of edges between S and T. G is r-*partite graph* with vertex classes $V_1, V_2, \ldots, V_r$ if $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$, $V_i \cap V_j = \emptyset$ whenever $1 \leq i < j \leq r$, and no edge joins two vertices in the same class. Moreover, for a subset $S \subseteq V$ and a vertex $v \in V$, we define $d(v, S)$ to be the number of vertices in S that are adjacent with v. In particular, let $d(v)$ instead of $d(v, V)$ denote the degree of v in G. The maximum (minimum) degree of the vertices in a graph G is denoted by $\Delta(G)$($\delta(G)$). If $d(v)$ is odd, the vertex v is called an odd vertex. Let f: $V \to \{-1, 1\}$ be a function which assigns an element of the set $\{-1, 1\}$ to each vertex of a graph $G = (V, E)$. The *weight* of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. For a vertex v in V, we denote $f(N[v])$ by $f[v]$ for notational convenience. The function f is said to be a ***signed dominating function*** of G if $f[v] \geq 1$ for every $v \in V$. The ***signed domination number***, denoted $\gamma_s(G)$, of G is the minimum weight of a signed dominating function on G. Signed domination has been studied in ([1]-[3], [5], [7], [9], [10]) and elsewhere. The function f is defined in [10] to be a ***signed 2-independence function***, denoted S2IF, on G if for every $v \in V$, $f[v] \leq 1$. The ***signed 2-independence number***, denoted $\alpha_s^2(G)$, of G is the maximum weight of an S2IF on G. Hence the signed 2-independence number is a certain dual to the signed domination number of a graph. In [6] Henning has established a good upper bounds for $\alpha_s^2(G)$ in terms of order and size of a graph.

Theorem 1 ([6]) *If G is a connected graph of order $n \geq 2$, then*

$$\alpha_s^2(G) \leq n + 2 - 2\sqrt{n+1}.$$

The paper is organized as follows: In section 2, we give some new upper bounds for $\alpha_s^2(G)$ in terms of order, size, number of odd vertices, maximum degree and minimum degree of a graph. In section 3, we give a sharp upper bound on $\alpha_s^2(G)$ for an r-partite graph.

2 Upper bounds

Theorem 2 *If G is a connected graph of order $n \geq 2$, size q, and n_0 is the number of odd vertices, then*

$$\alpha_s^2(G) \leq n + \frac{1}{2} - \sqrt{2q + n_0 + \frac{1}{4}}.$$

Proof. Let f be a S2IF on G satisfying $f(V) = \alpha_s^2(G)$ and we write

$$\begin{array}{rclrcl} P & = & \{v \in V | f(v) = 1\}, & M & = & \{v \in V | f(v) = -1\}, \\ P_o & = & \{v \in P |\ d(v) \text{ is odd}\}, & M_o & = & \{v \in M |\ d(v) \text{ is odd}\}. \end{array}$$

And let $|M| = m, |P| = p, P_e = P - P_o, M_e = M - M_o, |P_o| = p_o, |P_e| = p_e, |M_o| = m_o, |M_e| = m_e$. Since $f[v] \leq 1$ for each $v \in V$, it follows that

$$|N(v) \cap M| \geq \begin{cases} \frac{d(v)+1}{2} & \text{if } v \in P_o, \\ \frac{d(v)}{2} & \text{if } v \in P_e. \end{cases}$$

and

$$|N(v) \cap P| \leq \begin{cases} \frac{d(v)+1}{2} & \text{if } v \in M_o, \\ \frac{d(v)}{2} + 1 & \text{if } v \in M_e. \end{cases}$$

So we have

$$\frac{1}{2}\left(\sum_{v \in P} d(v) + p_o\right) = \sum_{v \in P_o} \frac{d(v)+1}{2} + \sum_{v \in P_e} \frac{d(v)}{2} \leq \sum_{v \in P} |N(v) \cap M| = e(P, M)$$

and

$$e(P,M) = \sum_{v\in M} |N(v)\cap P| \;\le\; \sum_{v\in M_o} \frac{d(v)+1}{2} + \sum_{v\in M_e} \left(\frac{d(v)}{2}+1\right)$$
$$\le\; \frac{1}{2}\sum_{v\in M} d(v) + \frac{1}{2}m_o + m_e.$$

Thus,

$$q + \frac{1}{2}n_0 \le \sum_{v\in M} d(v) + m.$$

Furthermore, we observe that for any vertex $v \in M$, $d(v) \le 2m-1$ if $d(v)$ is odd; $d(v) \le 2m$ if $d(v)$ is even. Hence, $q + \frac{1}{2}n_0 \le 2m^2 + m$. This implies that $m \ge \frac{-1+\sqrt{1+4(2q+n_0)}}{4}$. Therefore,

$$\alpha_s^2(G) = n - 2m \le n + \frac{1}{2} - \sqrt{2q + n_0 + \frac{1}{4}}.$$

□

Note that for a complete graph K_n of order $n = 2k+1$, we assign to only k vertices of K_n the value -1, then it produces an S2IF on K_n of weight $f(V(K_n)) = 1 = n + \frac{1}{2} - \sqrt{2q + n_0 + \frac{1}{4}}$. It is easily checked that this bound is better than that of Theorem 1 if $q \ge 2n - 3(\sqrt{n+1} - 1) - \frac{1}{2}n_0$. But if the edges of a graph are relatively sparse, then the bound in Theorem 1 is better.

Our first aim in this section is to establish a sharp upper bounds on $\alpha_s^2(G)$ in terms of order, size, number of odd vertices, minimum degree and maximum degree of a graph.

Theorem 3 *If G is a graph of order n and size q, n_0 is the number of odd vertices of G, then*

$$\alpha_s^2(G) \le \left\lfloor \min\left\{ n - \frac{2q+n_0}{\Delta(G)+1}, \frac{(1-\delta(G))n + 2q - n_0}{\delta(G)+1} \right\} \right\rfloor$$

Proof. Let f be an S2IF of G satisfying $f(V) = \alpha_s^2(G)$, and let P and M be defined as in Theorem 2. We let V_o and V_e denote the sets of odd and even vertices, respectively. Since $f[v] \leq 0$ for any $v \in V_o$ and $f[v] \leq 1$ for any $v \in V_e$, it implies that

$$\sum_{v\in V} f[v] = \sum_{v\in V_o} f[v] + \sum_{v\in V_e} f[v] \leq |V_e| = n - n_0.$$

On the other hand, we have

$$\begin{aligned}
\sum_{v\in V} f[v] &= \sum_{v\in V} f(v) + \sum_{v\in V}\sum_{u\in N(v)} f(u) \\
&= 2p - n + \sum_{v\in P} d(v) - \sum_{v\in M} d(v) \\
&= 2p - n + \sum_{v\in V} d(v) - 2\sum_{v\in M} d(v) \\
&= 2p - n + 2\sum_{v\in P} d(v) - \sum_{v\in V} d(v).
\end{aligned}$$

So

$$2p - n + 2q - 2(n-p)\Delta(G) \leq \sum_{v\in V} f[v] \leq n - n_0. \tag{1}$$

and

$$2p - n + 2p\delta(G) - 2q \leq \sum_{v\in V} f[v] \leq n - n_0. \tag{2}$$

Then

$$p \leq \frac{2n(\Delta(G)+1) - 2q - n_0}{2(\Delta(G)+1)}, \tag{3}$$

$$p \leq \frac{2n + 2q - n_0}{(\delta(G)+1)}. \tag{4}$$

By using (3) and (4), we have

$$\alpha_s^2(G) \leq \left\lfloor \min\left\{ n - \frac{2q+n_0}{\Delta(G)+1}, \frac{(1-\delta(G))n + 2q - n_0}{\delta(G)+1} \right\} \right\rfloor.$$

□

The following Figure 1 serves to illustrate that the bound in Theorem 4 is sharp.

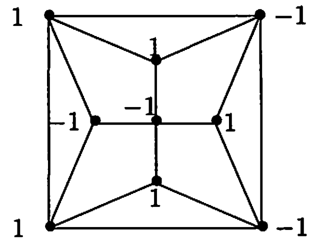

Figure 1: A graph with $\alpha_s^2(G) = 1$

As an immediate consequence of Theorem 3, we have the following result for a tree T.

Corollary 4 *If T is a tree of size $q \geq 1$ and n_0 is the number of odd vertices of T, then $\alpha_s^2(T) \leq q - \frac{1}{2}n_0$.*

The upper bound in Corollary 4 is sharp. For example, let $J_1, J_2, \ldots, J_k$ be k disjoint copies $K_{1,3}$. Now let T be the graph obtained from the union of $J_1, J_2, \ldots, J_k$ by joining the center of J_i and the center of J_{i+1}, $i = 1, 2, \ldots, k-1$. Then T is a tree of order $n = 4k$. Let f be a function on T by assigning to the center of each J_i the value -1 and to each vertex of degree 1 the value 1. It is easily seen that f is an S2IF on T and $\alpha_s^2(T) = 2k = q - \frac{1}{2}n_0$.

Corollary 5 *If G is a graph of order n and n_0 is the number of odd vertices of G, then*

$$\alpha_s^2(G) \leq \frac{n(\Delta(G) - \delta(G) + 2) - 2n_0}{\Delta(G) + \delta(G) + 2}.$$

Proof. Let f be an S2IF of G satisfying $f(V) = \alpha_s^2(G)$. By theorem 4, we have

$$2p(\Delta(G) + 1) \leq 2n(\Delta(G) + 1) - 2q - n_0. \tag{5}$$

$$2p(\delta(G) + 1) \leq 2n + 2q - n_0. \tag{6}$$

Adding (5) and (6), we have

$$p \leq \frac{n(\Delta(G)+2)-n_0}{\Delta(G)+\delta(G)+2}.$$

Therefore,

$$\alpha_s^2(G) = 2p - n \leq \frac{n(\Delta(G)-\delta(G)+2)-2n_0}{\Delta(G)+\delta(G)+2}.$$

□

As an immediate consequence of Corollary 5, we have the following result explicated by (Zelinka [10]).

Corollary 6 ([10]) *For any r-regular graph of order n,*

$$\alpha_s^2(G) \leq \begin{cases} n/(r+1) & \textit{for } r \textit{ even,} \\ 0 & \textit{for } r \textit{ odd.} \end{cases}$$

3 r-partite graphs

In this section we restrict our attention to r-partite graphs with order n. A sharp upper bound is established for $\alpha_s^2(G)$. We begin by stating an inequality explicated by Kang et al.[8].

Lemma 7 *For $r(r \geq 2)$ non-negative integers $m_1, m_2, \ldots, m_r$,*

$$\sqrt{\left(2+\frac{2}{r-1}\right)\sum_{i=1}^{r-1}\sum_{j=i+1}^{r} m_i m_j} \leq \sum_{i=1}^{r} m_i.$$

Theorem 8 *If $G = (V_1, V_2, \ldots, V_r; E)$ is an r-partite graph of order n, $r \geq 2$, then*

$$\alpha_s^2(G) \leq \frac{3r}{r-1} + n - \sqrt{\left(\frac{3r}{r-1}\right)^2 + \frac{4r}{r-1}n},$$

and this bound is sharp.

Proof. Let f be an S2IF on G satisfying $f(V) = \alpha_s^2(G)$, and let P and M be defined as in Theorem 2. Furthermore, we write $M_i = M \cap V_i$, $P_i = P \cap V_i$, and let $|M_i| = m_i$, $|P_i| = p_i$, for $i = 1, 2, \ldots, r$. Then

$$p + m = \sum_{i=1}^{r} p_i + \sum_{i=1}^{r} m_i = n. \tag{7}$$

Now, we calculate the value $e(P, M)$. Since $f[v] \leq 1$ for each vertex v of G, each vertex v of P is adjacent to at least a vertex of M, and so $|N(v) \cap M| = d(v, M) \geq 1$. On the other hand, each vertex v of M is adjacent to at most $d(v, M) + 2$ vertices of P, and so $d(v, P) \leq d(v, M) + 2$. Hence, we have

$$\begin{aligned}
\sum_{i=1}^{r} p_i \leq \sum_{v \in P} d(v, M) &= e(P, M) \\
&= \sum_{v \in M} d(v, P) \\
&= \sum_{i=1}^{r} \sum_{v \in M_i} d(v, P) \\
&\leq \sum_{i=1}^{r} \sum_{v \in M_i} (d(v, M) + 2) \\
&\leq \sum_{i=1}^{r} m_i(|M - M_i| + 2) \\
&= \sum_{i=1}^{r} m_i(\sum_{j=1, j \neq i}^{r} m_j + 2) \\
&= 2(\sum_{i=1}^{r-1} \sum_{j=i+1}^{r} m_i m_j + \sum_{i=1}^{r} m_i).
\end{aligned}$$

Using (1), we obtain

$$\frac{3}{2}p - n \leq \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} m_i m_j. \tag{8}$$

If $\frac{3}{2}p - n \leq 0$, then $p \leq \frac{2}{3}n$. Thus,

$$\alpha_s^2(G) = p - m = 2p - n \leq \frac{1}{3}n \leq \frac{3r}{r-1} + n - \sqrt{\left(\frac{3r}{r-1}\right)^2 + \frac{4r}{r-1}n},$$

the desired result follows. So we may assume $\frac{3}{2}p - n > 0$. By (7), (8) and Lemma 7, we obtain

$$p + \sqrt{\left(2 + \frac{2}{r-1}\right)\left(\frac{3}{2}p - n\right)} \leq n. \tag{9}$$

For notational convenience, we write $a = \sqrt{\frac{3}{2}p - n}$. Then, $p = \frac{2}{3}(a^2 + n)$, and so $\alpha_s^2(G) = f(V) = 2p - n = \frac{1}{3}(4a^2 + n)$. Now we define two functions as follows:

$$\begin{aligned} g(x) &= \frac{2}{3}(x^2 + n) + \sqrt{2 + \frac{2}{r-1}}\,x \ (x > 0), \\ h(x) &= \frac{1}{3}(4x^2 + n) \ (x > 0). \end{aligned}$$

Since

$$\frac{dg}{dx} = \frac{4}{3}x + \sqrt{2 + \frac{2}{r-1}} > 0 \ \text{ and } \ \frac{dh}{dx} = \frac{8}{3}x > 0.$$

This implies that $g(x)$ and $h(x)$ are monotonous increasing functions. By (9), we have

$$\begin{aligned} g(a) &= \frac{2}{3}(a^2 + n) + \sqrt{2 + \frac{2}{r-1}}\,a \\ &= p + \sqrt{\left(2 + \frac{2}{r-1}\right)\left(\frac{3}{2}p - n\right)} \\ &\leq n. \end{aligned}$$

Furthermore, we note that when

$$x_0 = \frac{-3\sqrt{2 + \frac{2}{r-1}} + \sqrt{9(2 + \frac{2}{r-1}) + 8n}}{4},$$

$g(x)$ takes the value n, i.e., $g(x_0) = n$. Hence $a \leq x_0$. Therefore

$$\begin{aligned} \alpha_s^2(G) = f(V) &= \frac{1}{3}(4a^2 + n) \\ &\leq \frac{1}{3}(4x_0^2 + n) \\ &= \frac{3r}{r-1} + n - \sqrt{\left(\frac{3r}{r-1}\right)^2 + \frac{4r}{r-1}n}. \end{aligned}$$

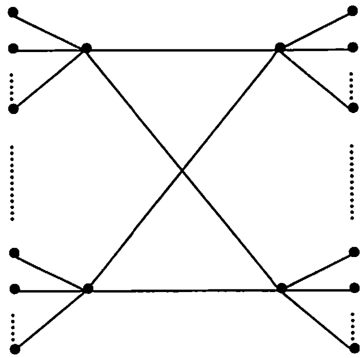

Figure 2: A bipartite graph G with $r = 2$ for which $\alpha_s^2(G) = 6 + n - 2\sqrt{9+2n}$

This establishes the desired upper bound for r-partite graphs. That the given upper bound is sharp, may be seen as follows. Let s be a positive integer, and let H is isomorphic to s disjoint copies of $K_{1,(r-1)s+2}$. Let $H_1, H_2, \ldots, H_r$ be r disjoint copies of H. Furthermore, let X_i and Y_i be the sets of vertices of degree 1 and $(r-1)s+2$, respectively, for $i = 1, 2, \ldots, r$. Now let G be the graph obtained from the disjoint union of $H_1, H_2, \ldots, H_r$ by joining every vertex of Y_i to every vertex of Y_j, for $1 \le i < j \le r$. Then, G is an r-partite graph of order $n = rs[(r-1)s+3]$ with partite sets $X_1 \cup Y_2, X_2 \cup Y_3, \ldots, X_{r-1} \cup Y_r, X_r \cup Y_1$. An example of a bipartite graph (r=2) is shown in Fig.2. Now we let f be a function on G and assign to each vertex of $\bigcup_{i=1}^r Y_i$ the value -1 and to each vertex of $\bigcup_{i=1}^r X_i$ the value 1. Then, it is easily checked that f is a S2IF on G and we have

$$\begin{aligned} w(f) = f(V) &= rs[(r-1)s+2] - rs \\ &= rs[(r-1)s+1] \\ &= \frac{3r}{r-1} + n - \sqrt{\left(\frac{3r}{r-1}\right)^2 + \frac{4r}{r-1}n}. \end{aligned}$$

Acknowledgements

The third author was supported by Changwon National University, which hosted her during the research was conducted. The authors wish to thank the anonymous refrees for their helpful suggestions.

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