Φ-Strong(weak) domination in a graph

V. Swaminathan
Department of Mathematics, Saraswathi Narayanan College
Madurai 625022, Tamil Nadu, India
P. Thangaraju *
School of Mathematics, Madurai Kamaraj University
Madurai 625021, Tamil Nadu, India

Abstract

Let $G = (V, E)$ be a graph. Let $\Phi : V \rightarrow \mathbf{R}$, where $\mathbf{R}$ is the set of all reals ($\mathbf{R}$ can be replaced by any chain). We say that u Φ-strongly dominates v and v Φ-weakly dominates u if $uv \in E$ and $\Phi(u) \geq \Phi(v)$. When Φ is a constant function, we have the usual domination and when Φ is the degree function of the graph, we have the strong(weak) domination studied by Sampathkumar et al. In this paper, we extend the results of O.Ore regarding minimal dominating sets of a graph. We also extend the concept of fully domination balance introduced by Sampathkumar et al and obtain a lower bound for strong domination number of a graph.

1 Introduction

All graphs will be finite undirected graphs without loops and multiple edges. For general terminology we refer to Harary [2].

Let $G = (V, E)$ be a graph. Let $\Phi : V \rightarrow \mathbf{R}$, where $\mathbf{R}$ is the set of all reals($\mathbf{R}$ can be replaced by any chain). If $uv \in E$ and $\Phi(u) \geq \Phi(v)$, we say that u Φ-strongly(Φs) dominates v and v Φ-weakly(Φw) dominates u. If $uv \in E$ and $\Phi(u) > \Phi(v)$, we say that u strictly Φ-strong dominates v and v strictly Φ-weak dominates u. If $\Phi(v) = c$ (a constant) for every $v \in V$, we have the usual domination. If $\Phi(v) = deg\ v$ for every $v \in V$, we have strong(weak) domination studied by Sampathkumar et al [4].

A subset D of V is a Φ-strong(weak) dominating set of G if every vertex $v \in V - D$ is $\Phi s(\Phi w)$-dominated by some $u \in D$. We use the abbreviations

*Research supported by UGC(No.F.1-075 MINOR/SRO)

Φsd-set(Φwd-set) for Φ-strong(weak) dominating sets. The Φ-strong(weak) domination number $\Phi_s(G)$ ($\Phi_w(G)$) of G is the size of its smallest Φsd-set (Φwd-set). The domination, strong(weak) domination numbers of a graph G are denoted by $\gamma(G), \gamma_s(G)(\gamma_w(G))$. When no ambiguity exists as to the graph in question the reference to G as (G) will be omitted. We observe that for a regular graph G, $\gamma_s = \gamma_w$ but Φ_s need not be equal to Φ_w.

A Φsd-set(Φwd-set) D of G is minimal if $D - \{u\}$ is not a Φsd-set(Φwd-set) for any $u \in D$. A Φsd-set(Φwd-set) D of G is minimum if $| D |= \Phi_s$ ($| D |= \Phi_w$). Every minimum Φsd-set(Φwd-set) is a minimal Φsd-set(Φwd-set). The converse of this result is not true.

The Φ-strong neighbourhood of a vertex $u \in V$ denoted by $N_{\Phi s}(u)$ is defined by $N_{\Phi s}(u) = \{v \in V : uv \in E \text{ and } \Phi(u) \leq \Phi(v)\}$. Similarly we define the Φ-weak neighbourhood $N_{\Phi w}(u)$ of $u \in V$. The open neighbourhood, strong neighbourhood and weak neighbourhood of u are respectively denoted by $N(u)$, $N_s(u)$ and $N_w(u)$.

A graph G is fully Φ domination balanced if there exists a partition of the vertex set $V = S_1 \cup S_2$ of G such that S_1 is a Φsd-set with $| S_1 |= \Phi_s$ and S_2 is a Φwd-set with $| S_2 |= \Phi_w$.

In Section 2 of this paper, we extend some of the results in [3] regarding the relation between minimal dominating sets, its complement and neighbourhoods of vertices in it, and characterise fully Φ domination balanced graphs. In Section 3, a lower bound for strong domination number of a graph is obtained.

2 Ore's theorems in Φ-strong(weak) domination

Proposition 2.1 (O.Ore [3]) Let $G = (V, E)$ be a graph without isolated vertices. If D is a minimal dominating set of G, then $V - D$ is a dominating set of G.

We find that if D is a minimal Φsd-set(Φwd-set) of a graph without isolated vertices, then $V - D$ need not be a Φwd-set(Φsd-set). For, in graph G_1, $D = \{v_1, v_3, v_5, v_6\}$ is a minimal Φsd-set. But $V - D = \{v_2, v_4\}$ is not a Φwd-set.

G_1

In the following proposition we identify the vertices of a Φsd-set D which have to be included in $V - D$ to get a Φwd-set.

Proposition 2.2 *Let $G = (V, E)$ be a graph without isolated vertices. If D is a minimal Φsd-set, then $(V - D) \cup D_1 \cup D_2$ is a Φwd-set, where $D_1 = \{u \in D : u$ is strictly Φs-dominated by some vertex v of D and v does not Φs-dominate any vertex of $V - D\}$ and $D_2 = \{u \in D : u$ is not adjacent to any vertex of D and u does not Φs-dominate any vertex of $V - D\}$.*

Proof. By definition of D_2, no vertex of V Φw-dominates any vertex of D_2. So it is enough to prove that every vertex $u \in D - (D_1 \cup D_2)$ is Φw-dominated by some vertex $v \in (V - D) \cup D_1$. Assume the contrary. Then either u is not adjacent to any $v \in (V - D) \cup D_1$ or whenever u is adjacent to some $v \in (V - D) \cup D_1$ then $\Phi(u) < \Phi(v)$. Since D is a Φsd-set and since u does not Φs-dominate any vertex of $V - D$, every vertex of $V - D$ is Φs-dominated by some vertex of $D - \{u\}$. Now we have the following three cases.

Case(i) u is not adjacent to any vertex of $(V - D) \cup D_1$.

Since no vertex of D_2 is adjacent to any vertex of D and since u is not an isolated vertex, there is a vertex $v \in D - (D_1 \cup D_2)$ adjacent to u. Since $v \notin D_1, \Phi(u) \leq \Phi(v)$. Hence $D - \{u\}$ is a Φsd-set.

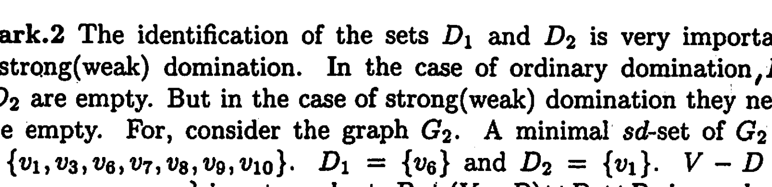
Case(ii) u is adjacent to some $v \in V - D$ such that $\Phi(u) < \Phi(v)$ and u is not adjacent to any $v \in D_1$.

Since $u \notin D_2$, there is a vertex $w \in D - (D_1 \cup D_2)$ which is adjacent to u. As $w \notin D_1$, $\Phi(u) \leq \Phi(w)$. Hence $D - \{u\}$ is a Φsd-set.

Case(iii) u is adjacent to some $v \in D_1$ such that $\Phi(u) < \Phi(v)$. Then $D - \{u\}$ is a Φsd-set.

Thus we get a contradiction to the minimality of D, which proves the proposition. □

Remark.1 If Φ is a constant function, then D_1 and D_2 are empty, and Ore's theorem follows.



G_2

Remark.2 The identification of the sets D_1 and D_2 is very important in Φ-strong(weak) domination. In the case of ordinary domination, D_1 and D_2 are empty. But in the case of strong(weak) domination they need not be empty. For, consider the graph G_2. A minimal *sd*-set of G_2 is $D = \{v_1, v_3, v_6, v_7, v_8, v_9, v_{10}\}$. $D_1 = \{v_6\}$ and $D_2 = \{v_1\}$. $V - D = \{v_2, v_4, v_5, v_{11}, v_{12}, v_{13}\}$ is not a *wd*-set. But $(V - D) \cup D_1 \cup D_2$ is a *wd*-set. Sampathkumar et al [4] pointed out that the complement of a sd-set need not be a wd-set but did not probe further. In the above proposition this task has been successfully completed in a more general set up than that considered by Sampathkumar et al.

The weak version of proposition 2.2 is given in the following:

Proposition 2.3 *Let $G = (V, E)$ be a graph without isolated vertices. If D is a minimal Φwd-set, then $(V - D) \cup D_3 \cup D_4$ is a Φsd-set, where $D_3 = \{u \in D : u$ is strictly Φw-dominated by some v of D and v does not Φw-dominate any vertex of $V - D\}$ and $D_4 = \{u \in D : u$ is not adjacent to any vertex of D and u does not Φw-dominate any vertex of $V - D\}$.*

Proposition 2.4 *Let G be a graph of order n without isolated vertices. Then $\Phi_s + \Phi_w \leq n +$ min $\{\mid D_1 \mid + \mid D_2 \mid, \mid D_3 \mid + \mid D_4 \mid\}$ where D_1, D_2, D_3 and D_4 are defined in propositions 2.2 and 2.3.*

Let $G = (V, E)$ be a graph and $D \subset V$. We say that D is full if every $u \in D$ is adjacent to some $v \in V - D$. Similarly we define strong full(s-full), weak full(w-full), Φ-strong full(Φs-full) and Φ-weak full(Φw-full) sets of a graph.

The following proposition is obvious.

Proposition 2.5 *For any subset D of V (i) D is Φs-full if and only if D_1 and D_2 are empty (ii) D is Φw-full if and only if D_3 and D_4 are empty.*

The full number $f(G)$ of a graph G is the maximum number of vertices in the full set of G. Similarly we define the strong full number $(f_s(G))$, the weak full number$(f_w(G))$, the Φ-strong full number $(f_{\Phi s}(G))$ and Φ-weak full number$(f_{\Phi w}(G))$ of G.

Proposition 2.6 (Sampahkumar et al [4]) If G is a graph of order n, then (i) $f(G) + \gamma(G) = n$ (ii) $f_s(G) + \gamma_w(G) = n$ and (iii) $f_w(G) + \gamma_s(G) = n$.

Proposition 2.7 *If G is a graph of order n, then (i) $f_{\Phi s}(G) + \Phi_w(G) = n$ and (ii) $f_{\Phi w}(G) + \Phi_s(G) = n$.*

Proof. (i) Let S be a Φwd-set with $\mid S \mid = \Phi_w$. Then $V - S$ is Φs-full. Hence $f_{\Phi s}(G) \geq n - \Phi_w(G)$. Let T be a Φs-full set of maximum cardinality. Then $V - T$ is a Φwd-set. Hence $\Phi_w(G) \leq n - f_{\Phi s}(G)$. Proof of (ii) is similar to proof of (i). □

Let $\Delta_\Phi(\delta_\Phi)$ be the maximum(minimum) of the degrees of the vertices having maximum(minimum) Φ-value. Then $\Delta_\Phi \leq \Delta$ and $\delta_\Phi \geq \delta$. If Φ is a constant function or the degree function of the graph, then $\delta_\Phi = \delta$ and $\Delta_\Phi = \Delta$.

The following proposition is obvious.

Proposition 2.8 *If G is a graph of order n, then $\gamma \leq \Phi_s \leq n - \Delta_\Phi$ and $\gamma \leq \Phi_w \leq n - \delta_\Phi$.*

Proposition 2.9 *If G is a graph of order n, then $\delta_\Phi \leq f_{\Phi s}(G) \leq f(G)$ and $\Delta_\Phi \leq f_{\Phi w}(G) \leq f(G)$.*

Proof. Proof follows from the following equations:

$$\begin{array}{lll} \gamma(G) & \leq \Phi_s(G) & \leq n - \Delta_\Phi \\ \gamma(G) & \leq \Phi_w(G) & \leq n - \delta_\Phi \\ f(G) & + \gamma(G) & = n \\ f_{\Phi s}(G) & + \Phi_w(G) & = n \\ f_{\Phi w}(G) & + \Phi_s(G) & = n. \end{array}$$

□

A graph G is fully domination balanced(fd-balanced) if there exists a partition of the vertex set G into S_1 and S_2 such that S_1 is a sd-set with $|S_1| = \gamma_s$ and S_2 is a wd-set with $|S_2| = \gamma_w$. Similarly fully Φ domination balanced($f\Phi d$-balanced) graphs are defined.

Proposition 2.10 (Sampathkumar et al [4]) A graph of order n is fd-balanced if and only if (i) $f_s + f_w = n$ and (ii) there exists a sd-set(wd-set) with cardinality $\gamma_s(\gamma_w)$ which is s-full(w-full).

Proposition 2.11 *A graph of order n is $f\Phi d$-balanced if and only if (i) $f_{\Phi s} + f_{\Phi w} = n$ and (ii) there exists a Φsd-set (Φwd-set) with cardinality Φ_s (Φ_w) which is Φs-full(Φw-full).*

Proof. Suppose that G is $f\Phi d$-balanced. Then by definition, (i) holds. Also there exists a partition of V into S_1 and S_2 such that S_1 is a Φsd-set with cardinality Φ_s and S_2 is a Φwd-set with cardinality Φ_w. Then S_1 is Φs-full and S_2 is Φw-full. Conversely, suppose (i) and (ii) hold. Let S_1 be a Φsd-set with cardinality Φ_s which is Φs-full. Then $S_2 = V - S_1$ is Φwd-set. Since $f_{\Phi s} + f_{\Phi w} = n$ and $\Phi_w + f_{\Phi s} = n$, we get $|S_2| = \Phi_w$. This proves that G is $f\Phi d$-balanced. □

Proposition 2.12 (O.Ore [3]) Let $G = (V, E)$ be a graph. A dominating set D of G is minimal if and only if for every $u \in D$ one of the following conditions holds. (i) $N(u) \cap D$ is empty and
(ii) There is a vertex $v \in V - D$ such that $N(u) \cap D = \{u\}$.

The following result can be proved by considering Φ-strong neighbourhoods of vertices of G.

Proposition 2.13 *Let $G = (V, E)$ be a graph. A Φsd-set D of G is minimal if and only if for each $u \in D$ one of the following conditions holds.*
(i) $N_{\Phi s}(u) \cap D$ is empty and
(ii) There is a vertex $v \in V - D$ such that $N_{\Phi s}(v) \cap D = \{u\}$.

In a similar way, we can characterize minimal Φwd-sets of a graph.

Corollary 2.14 *Let $G = (V, E)$ be a graph. A sd-set D of G is minimal if and only if for every $u \in D$ one of the following conditions holds.*
(i) $N_s(u) \cap D$ is empty and
(ii) There is a vertex $v \in V - D$ such that $N_s(u) \cap D = \{u\}$.

Minimal *wd*-sets of a graph can be similarly characterised.
Remark. In [4] only the necessary part of corollary 2.14 has been proved.

3 Lower bound for strong domination number of a graph

In [7] Walikar et al have given a lower bound for domination number of a graph in the following form: For a graph of order n, $\gamma(G) \geq \lceil \frac{n}{1+\Delta(G)} \rceil$, where $\Delta(G)$ is the maximum degree of a vertex in G and $\lceil x \rceil$ denotes the smallest integer not less than x. A γ-set is a minimum dominating set of cardinality γ. In proposition 3.2, we give an improved lower bound for domination number of a graph.

Let G be a non-regular graph of order n with degree sequence $\Pi = (d_1^{n_1}, d_2^{n_2}, \cdots, d_k^{n_k})$, where $d_1 > d_2 \cdots > d_k$.

Lemma 3.1 *Let D be a γ-set of G and $m_i (1 \leq i \leq k)$ be the number of vertices of degree d_i in D. Let $n_1' = \lceil \frac{n}{d_1+1} \rceil$, $p_1 = min\ \{n_1, n_1'\}$. For $2 \leq i \leq k$, let $n_i' = \left\lceil \frac{n - \sum_{j=1}^{i-1} p_j(d_j+1)}{d_i+1} \right\rceil$ if $n - \sum_{j=1}^{i-1} p_j(d_j+1) > 0$ and zero otherwise, and $p_i = min\ \{n_i, n_i'\}$. If $m_i > p_i$, then $p_i = n_i'$ and $p_{i+1} = 0$ $(1 \leq i \leq k-1)$. Also there exists a positive integer $r \leq k$ such that $n - \sum_{j=1}^{r} p_j(d_j+1) \leq 0$.*

Proof. If $m_i \leq n_i'$ then as $m_i \leq n_i$, we get $m_i \leq \min\ \{n_i, n_i'\} = p_i$. Since $m_i > p_i$ we get $m_i > n_i'$. Therefore $p_i = n_i' = \left\lceil \frac{n - \sum_{j=1}^{i-1} p_j(d_j+1)}{d_i+1} \right\rceil \geq \frac{n - \sum_{j=1}^{i-1} p_j(d_j+1)}{d_i+1}$. Hence $\sum_{j=1}^{i} p_j(d_j+1) \geq n$. That is $n_{i+1}' = 0$. Therefore $p_{i+1} = 0$.

Now $n - \sum_{j=1}^{k} n_j(d_j+1) \leq 0$. Hence there exists a least positive integer r such that $n - \sum_{j=1}^{r} n_j(d_j+1) \leq 0$ for some $r \leq k$. That is $n - \sum_{j=1}^{i} n_j(d_j+1) > 0$

for $i = 1$ to $r-1$. From this we get that $p_j = n_j$ for $j = 1$ to $r-1$. $n - \sum_{j=1}^{r} n_j(d_j+1) \leq 0$ implies that $n - \sum_{j=1}^{r-1} n_j(d_j+1) \leq n_r(d_r+1)$.

Hence $n_r^{'} \leq n_r$ which means $p_r = n_r^{'}$. Since $n - \sum_{j=1}^{r-1} n_j(d_j+1) \leq n_r^{'}(d_r+1)$,

$n - \sum_{j=1}^{r} p_j(d_j+1) = n - \sum_{j=1}^{r} n_j(d_j+1) + (n_r - n_r^{'})(d_r+1) \leq 0$. Hence the lemma. □

Proposition 3.2 $\gamma(G) \geq \sum_{i=1}^{k} p_i$, *where* p_i*'s are defined in lemma 3.1.*

Proof. Let r be the minimum suffix $i-1 (\geq 1)$ such that $n - \sum_{j=1}^{i-1} p_j(d_j+1) \leq 0$.

Case(i) $r = 1$. Then $n - p_1(d_1+1) \leq 0$. Therefore $p_i = 0$ for $i \geq 2$. Now $\sum_{i=1}^{k} m_i(d_1+1) \geq \sum_{i=1}^{k} m_i(d_i+1) \geq n$. Then $\gamma(G) = \sum_{i=1}^{k} m_i \geq \lceil \frac{n}{d_1+1} \rceil = n_1' = p_1$.

Case(ii) $2 \leq r \leq k$. Then $p_1, p_2, \cdots, p_r$ are > 0 and $n - \sum_{i=1}^{r} p_i(d_i+1) \leq 0$.

Hence $p_{r+1} \cdots$ are zero. Now $\sum_{i=1}^{r-1} p_i(d_i+1) < n \leq \sum_{i=1}^{r} p_i(d_i+1)$.

This implies $\sum_{i=1}^{k} m_i(d_i+1) \geq n > \sum_{i=1}^{r-1} p_i(d_i+1)$.

Then $\sum_{i=1}^{r-1} (m_i - p_i)(d_i+1) + \sum_{i=r}^{k} m_i(d_i+1) \geq n - \sum_{i=1}^{r-1} p_i(d_i+1) > 0$.

But $\sum_{i=1}^{r-1} (m_i-p_i)(d_i+1) + (d_r+1) \sum_{i=r}^{k} m_i \geq \sum_{i=1}^{r-1} (m_i-p_i)(d_i+1) + \sum_{i=r}^{k} m_i(d_i+1)$.

Therefore $\sum_{i=1}^{r-1} (m_i - p_i)(d_i+1) + (d_r+1) \sum_{i=r}^{k} m_i \geq n - \sum_{i=1}^{r-1} p_i(d_i+1)$.

Then $\sum_{i=1}^{r-1} \frac{(m_i-p_i)(d_i+1)}{d_r+1} + \sum_{i=r}^{k} m_i \geq \frac{n - \sum_{i=1}^{r-1} p_i(d_i+1)}{d_r+1}$.

Since $m_i \leq p_i$ for $1 \leq i \leq r-1$(by lemma 3.1) and $d_i > d_r$ for $1 \leq i \leq r-1$,

we have $\sum_{i=1}^{r-1} (m_i - p_i) + \sum_{i=r}^{k} m_i \geq \sum_{i=1}^{r-1} \frac{(m_i-p_i)(d_i+1)}{d_r+1} + \sum_{i=r}^{k} m_i > \frac{n - \sum_{i=1}^{r-1} p_i(d_i+1)}{d_r+1}$.

That is $\sum_{i=1}^{k} m_i - \sum_{i=1}^{r-1} p_i \geq \left\lceil \frac{n - \sum_{i=1}^{r-1} p_i(d_i+1)}{d_r+1} \right\rceil = n_r' \geq p_r$.

Therefore $\gamma(G) = \sum_{i=1}^{k} m_i \geq \sum_{i=1}^{r} p_i = \sum_{i=1}^{k} p_i$. □

Remark Since $p_1 = \lceil \frac{n}{1+\Delta(G)} \rceil$, the result that $\gamma(G) \geq \lceil \frac{n}{1+\Delta(G)} \rceil$ in [7] can be written as $\gamma(G) \geq p_1$. The proposition 3.2 shows that this lower bound can be improved. For a regular graph, the proposition 3.2 gives the same result proved by Walikar et al [7]. Since $\gamma(G) \leq \gamma_s(G)$, we have

Corollary 3.3 $\gamma_s(G) \geq \sum_{i=1}^{k} p_i$, *where* p_i*'s are as in lemma 3.1.*

Acknowledgement

The authors would like to thank the referee for his careful reading of the paper and ... excellent suggestions.

References

[1] Johannes H. Hattingh and Renu C. Laskar, On weak domination in graphs, Ars Combinatoria **49** (1998) 205-216.

[2] F.Harary, Graph Theory, Addison-Wesley, Reading, Mass (1972).

[3] O.Ore, Theory of Graphs, Amer. Math. Soc. Colloq. Publ. **38** Providence (1962).

[4] E. Sampathkumar and L. Pushpalatha, Strong weak domination and domination balance in a graph, Discrete Math. **161** (1996) 235-242.

[5] Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker Inc. (1998).

[6] Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater, Domination in Graphs, Advanced Topics, Marcel Dekker Inc. (1998).

[7] H.B. Walikar, B.D. Acharya and E. Sampathkumar, Recent developments in the theory of domination in graphs, MRI Lecture Notes in Math. **1** (1976).