Triangle-Free Regular Graphs as an Extremal Family

K. J. Roblee
Department of Mathematics and Statistics
Youngstown State University
Youngstown, Ohio 44555

Abstract

It has been shown that if $G = (V, E)$ is a simple graph with n vertices, m edges, an average (per edge) of t triangles occurring on the edges, and $J = \max_{uv \in E}|N(u) \cup N(v)|$, then $4m \leq n(J+t)$. The extremal graphs for this inequality for $J = n$ and $J = n - 1$ have been determined. For $J = n$, the extremal graphs are the Turán graphs with parts of equal size; notice that these are the complements of the strongly regular graphs with $\mu = 0$. For $J = n - 1$, the extremal graphs are the complements of the strongly regular graphs with $\mu = 1$. (The only such graphs known to exist are the Moore graphs of diameter 2).

For $J = n - 2$ and $t = 0$, it has recently been shown that the only extremal graph (except when $n = 8, 10$) is $K_{\frac{n}{2},\frac{n}{2}}$ – (1-factor). Here, we use a well-known theorem of Andrásfai, Erdős, and Sós to characterize the extremal graphs for $t = 0$, any given value of $n - J$, and n sufficiently large (they are the regular bipartite graphs). Then we give some examples of extremal non-bipartite graphs for smaller values of n.

1 Introduction

Suppose that $G = (V, E)$ is a simple graph with $|V| = n$ and $|E| = m$. For each $u \in V$, let $N(u) = \{v \in V \mid uv \in E\}$; for each $e = uv \in E$, let $t(e) = |N(u) \cap N(v)|$, and let $J(e) = |N(u) \cup N(v)|$. Finally, let $t = t(G) = \frac{1}{m}\sum_{e \in E} t(e)$, and let $J = J(G) = \max_{e \in E} J(e)$.

It is known that $4m \leq n(J+t)$ with equality if and only if G is regular and $e \mapsto t(e)$ is a constant function [3]. This also holds if J and t are redefined so that J is an average and t is a maximum.

If $e = uv \in E$, $t(e) + J(e) = d_G(u) + d_G(v)$, with d_G denoting degree in G. It follows that if G is regular, then $e \mapsto t(e)$ is constant if and only if $e \mapsto J(e)$ is constant. It also follows that the degree of G in this case (that is, G is regular and $e \mapsto t(e)$ is constant) is $\frac{t+J}{2}$.

We say that $G \in ET(n, J, t)$ if and only if $J = J(G)$, $t = t(G)$, $n = |V(G)|$, $m = |E(G)|$, and $4m = n(J + t)$; that is, $ET(n, J, t)$ is the set of extremal graphs for the above inequality, with parameters n, J, and t. Henceforth, we refer to these graphs as extremal graphs.

For $J = n$ and $J = n - 1$, the extremal graphs for the above inequality have been characterized. For $J = n$, the extremal graphs are the Turán graphs with parts of equal sizes [2]. It is worth noting that these are the complements of the strongly regular graphs with $\mu = 0$. (A strongly regular graph is a regular graph such that any pair of adjacent vertices have λ common neighbors and any pair of non-adjacent vertices have μ common neighbors, for some non-negative integers λ and μ. It is easy to see that the complement of a strongly regular graph is strongly regular, and also that the complement of a strongly regular graph on n vertices, with parameters λ and μ, is in $ET(n, n - \mu, t)$, for some t.) For $J = n - 1$, the extremal graphs are the complements of the strongly regular graphs with $\mu = 1$ [3]. Only three of these strongly regular graphs with $\mu = 1$ are known for sure to exist: C_5, the Petersen graph, and the Hoffman-Singleton graph ($n = 50, t = 35$). See [3] for further discussion.

Recently, the extremal graphs with parameters $J = n - 2$ and $t = 0$ have been characterized. Except for the cases $n = 8, 10$, these graphs are precisely $K_{\frac{n}{2},\frac{n}{2}}$ – (1-factor) [4]. These graphs are not strongly regular when $n \geq 6$.

2 A Characterization of Graphs in ET(n, J, 0) For Fixed n – J and n Sufficiently Large

The following is a well-known corollary of a theorem of Andrásfai, Erdős, and Sós [1].

Theorem AES *If G is a triangle-free graph of order n with $\delta(G) > 2n/5$, then G is bipartite.*

Theorem *For $J = n - (2k + 1)$, where $k \geq 1$, $t = 0$, and $n > 10k + 5$, we have $ET(n, J, t) = \emptyset$. For $J = n - 2k$, where $k > 1$, n even, $t = 0$, and $n > 10k$, we have that if $G \in ET(n, J, t)$, then G is a graph of the form $K_{\frac{n}{2},\frac{n}{2}}$ – (k-factor).*

Proof. Observe that if $G \in ET(n, J, 0)$, then G is $J/2$-regular, and $n > 10k$,

$J = n - 2k$, and $n > 10k + 5$, $J = n - (2k + 1)$, both imply $J > 4n/5$. Thus the degree d of every vertex is greater than $2n/5$, in either case, and $t = 0$ implies that G is triangle-free, so G is bipartite. Since G is regular of positive degree, n must be even. If $J = n - (2k + 1)$ then J is odd, which is impossible, since G is $J/2$-regular. If $J = n - 2k$ and $n > 10k$ then G is bipartite and regular of degree $\frac{n}{2} - k$, which implies that G is of the form described.

□

The inequality in the theorem is sharp. The following graph is the standard example for this: Replace every vertex in the 5-cycle with a stable set of size $2k + 1$ (respectively $2k$ for the second case) and make vertices adjacent if their template vertices were adjacent. It is easy to see this graph is non-bipartite, regular, and triangle-free. In fact, it is an easy consequence of various proofs of Theorem AES that this is the only non-bipartite graph in $ET(10k + 5, 8k + 4, 0)$, respectively, $ET(10k, 8k, 0)$.

The Theorem and the remarks in the preceeding paragraph, applied in the special case $J = n - 2$, $k = 1$, imply most of the main result in [4]. The one claim of that main result which is not implied by results here is that there is a unique non-bipartite graph in $\bigcup_{n=2}^{9} ET(n, n - 2, 0)$, in $ET(8, 6, 0)$, which is described there. It is worth noting that the non-bipartite triangle-free graph with eight vertices, of degree 3, is not among the graphs described in the next section.

3 Non-Bipartite Extremal Graphs of Smaller Order

In this section we present three constructions of non-bipartite graphs in $ET(n, J, 0)$ with either $J = n - 2k$, $n < 10k$ or $J = n - (2k+1)$, $n < 10k+5$. Whether or not these graphs are unique is an open problem.

In each construction we start with two adjacent vertices u, v; A will be the set of vertices of the graph being constructed which are to be adjacent to u, other than v, and similarly B will be the neighbor set of v, excluding u. Letting $d = J/2$ denote the degree of the graph, we will have, in each case, $|A| = |B| = d - 1$, and there will be no edges among the vertices of A, nor among those of B. Let $X = V \backslash (A \cup B \cup \{u, v\})$; note that $|V \backslash X| = J$.

Construction for $\mathbf{n = 4k + 4}$, k even, $\mathbf{J = n - 2k = 2k + 4}$. Let $A = \{a_1, a_2, \ldots, a_{k+1}\}$ and $B = \{b_1, b_2, \ldots, b_{k+1}\}$ be as above. Let one vertex $x \in X$ be adjacent to the first $\frac{k+2}{2}$ vertices in A and in B; let another vertex $y \in X$ be adjacent to the last $\frac{k+2}{2}$ vertices in A and in B. Next join a_i with vertex b_m where $m = i + \frac{k+2}{2}$, for all $i \leq \frac{k}{2}$ (similarly for the b_i's).

Then make a perfect matching with the remaining $2k-2$ vertices in X; for each edge in this matching, join one endpoint to each vertex in A and the other to each vertex in B. It is left to the reader to see that this graph is triangle-free and regular. It is also seen to be non-bipartite by noticing that u, a, x, b, v, u, where $a \in A$ and $b \in B$ are both adjacent to x, form a 5-cycle. We exhibit such a graph in the figure below.

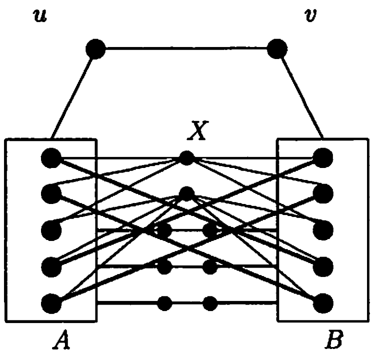

Figure 1: A graph in $ET(n = 20, n - 2k = 12, 0)$

Construction for $\mathbf{n = 4k + 6,\ k > 1}$ odd, $\mathbf{J = n - 2k = 2k + 6}$. Let $A = \{a_1, \ldots, a_{k+2}\}$ and $B = \{b_1, \ldots, b_{k+2}\}$. Form a perfect matching with $2k-4$ of the vertices in X, and for each edge in this matching, let one endpoint be adjacent to everything in A and the other be adjacent to everything in B. Let the remaining vertices in X be $x_1, \ldots, x_4$. Let x_1 and x_2 be adjacent to the first $\frac{k+3}{2}$ vertices in A and in B; let x_3 and x_4 be adjacent to the last $\frac{k+3}{2}$ vertices in A and in B. Finally, we form 2 disjoint perfect matchings between the first $\frac{k+1}{2}$ vertices of A and the last $\frac{k+1}{2}$ vertices of B (similarly for the first $\frac{k+1}{2}$ vertices of B and the last $\frac{k+1}{2}$ vertices of A). Again, the reader can verify this graph is regular, triangle-free, and non-bipartite.

Construction for $\mathbf{n = 6k + 5,\ J = n - (2k + 1) = 4k + 4}$. Again, form a perfect matching with $2k$ vertices in X. For each edge in this matching, let one endpoint be adjacent to everything in A, and let the other be adjacent to everything in B. Now let the leftover vertex $z \in X$ have half its neighbors in A (call the set of these vertices A_z) and the other half in B (call the set of these vertices B_z). Let everything in A_z be adjacent to everything in $B - B_z$ (similarly for B_z). Once more, it is left to the pleasure of the reader to verify the graph is triangle-free, regular, and non-bipartite.

Problem: Do there exist regular non-bipartite, triangle-free graphs G_i, $i = 1, 2, \ldots$ of order n_i and degree d_i, such that $2/5 > d_i/n_i \to 2/5$ as

$i \to \infty$?

References

[1] B. Andrásfai, P. Erdős, and V. T. Sós, On the connection between chromatic number, maximal clique and minimal degree of a graph, *Discrete Math.* 8 (1974), 205-218.

[2] C. E. Edwards, P. D. Johnson Jr., and Zs. Tuza, Another extremal property of some Turán graphs, *European J. Comb.* 8 (1987), 27-28.

[3] D. G. Hoffman and P. D. Johnson Jr., An extremal family of strongly regular graphs, *Congressus Numerantium* 78 (1990), 119-122.

[4] P. D. Johnson Jr. and K. J. Roblee, More extremal graphs for a maximum-joint-neighborhood, average-triangles-per-edge inequality. *Congressus Numerantium* 140 (1999), 87-95.