On the $n \times n$ Knight Cover Problem

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Abstract

A set of Knights covers a board if a Knight attacks every unoccupied square. *What is the minimum number of Knights in a cover of an $n \times n$ board?* For $n \leq 10$, we give a non-computational proof that the widely accepted answers are correct. For $n \leq 14$, fractional Knight packings are used in an exhaustive branch-and-bound program. This gives the first enumeration of minimum Knight covers for $11 \leq n \leq 14$. For $n \geq 15$, integer programs are used to find small (though not necessarily minimum) symmetric covers. This yields smaller covers for $16 \leq n \leq 19$, and new covers when $21 \leq n \leq 25$. Simulated annealing discovered yet smaller covers for $n = 19$ and $n = 21$. Guess work improved the results for $n = 20$ and $n = 25$.

1 Introduction

In Chess, a Knight moves (or attacks) as shown in figure 1. A Knight *covers* the squares it attacks plus the square it is on. A set of Knights *covers* a board if all of its squares are covered by the Knights (see figure 1).

In 1896, [15] asked: *What is the minimum number of Knights needed to cover an $n \times n$ board?* Ahren [2] in 1918, Gardner [6] in 1967, Gardner [9] in 1977, and Jackson and Pargas [12] in 1991 gave the best answers known at the time. This paper has yet better answers to this century-old puzzle.

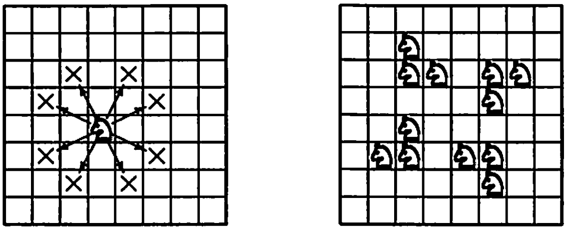

Figure 1. A Knight (♘) moves two squares horizontally or vertically and then one square orthogonal (shown with $\times$). The right is the only way (up to symmetry) for 12 Knights to cover an 8×8 board.

2 Up to 10×10 Boards

For up to 10×10 boards, the minimum number of Knights in a cover appears to have been settled by 1898 as [14] focuses on 11×11 boards (see table 1). However, below is what I believe to be the first published non-computational proof of these results. Hare and Hedetniemi [11] did give a computational proof, but the approach here is simpler.

n	1	2	3	4	5	6	7	8	9	10
Minimum Number in a Cover	1	4	4	4	5	8	10	12	14	16
Number of Minimum Covers	1	1	8	9	47	127	10	2	2	4
No. of Distinct Min. Covers	1	1	2	3	8	23	3	1	1	2

Table 1 – Knight Covers of an $n \times n$ Board. This show the minimum number of Knights needed to cover an $n \times n$ board, the number of minimum Knight covers, and the number of distinct (not counting reflections and rotations) Knight covers (see figure 8).

A *fractional Knight packing* puts nonnegative weights on the squares of an $n \times n$ board so the weights covered by a Knight add to at most one. Then the sum of the weights in a fractional Knight packing is a lower bound for the number of Knights in a cover. A fractional Knight packing is *maximum* if this sum is as large as possible[1]. Figure 16 shows how to find these.

Theorem. *The minimum number of Knights in a cover of an $n \times n$ board is 1, 4, 4, 4, 5, 8, 10, 12, 14, and 16 for n =1, 2, ..., 10, respectively.*

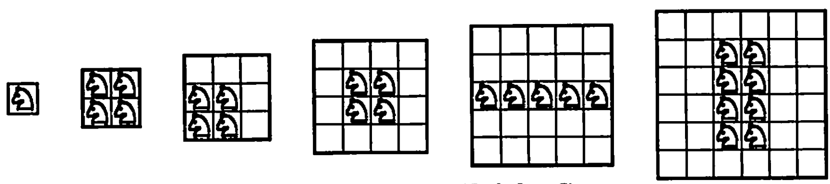

Figure 2 – Minimum Knight Covers.

Proof. Figures 1, 2, 8, and 9 show covers with the desired numbers of Knights. So we need to show that a cover cannot have fewer Knights.

In 7 of 10 cases, the proof is easy. Figure 3 gives maximum fractional Knight packings. Since these weights add to 1, 4, $3\frac{2}{3}$, $3\frac{7}{11}$, 10, 12, and $13\frac{5}{8}$ for 1×1, 2×2, 3×3, 4×4, 7×7, 8×8, and 9×9 boards, respectively, a cover has at least 1, 4, 4, 4, 10, 12, and 14 Knights. Proofs for the other cases are more difficult because the minimum number in a Knight cover is greater than one plus the maximum sum in a fractional Knight packing.

[1]The $n \times n$ *Knight graph* has nodes (i, j) where $1 \leq i \leq n$ and $1 \leq j \leq n$ with edges between nodes (i, j) and (k, ℓ) if and only if $|i - k||j - \ell| = 2$. Then a fractional Knight packing is a "fractional packing" of the Knight graph, and a Knight cover is a "domination" of the Knight graph. See Domke, Hedetniemi, and Laskar [3] for more on packing (also called 2-packing), domination, and their fractional analogues.

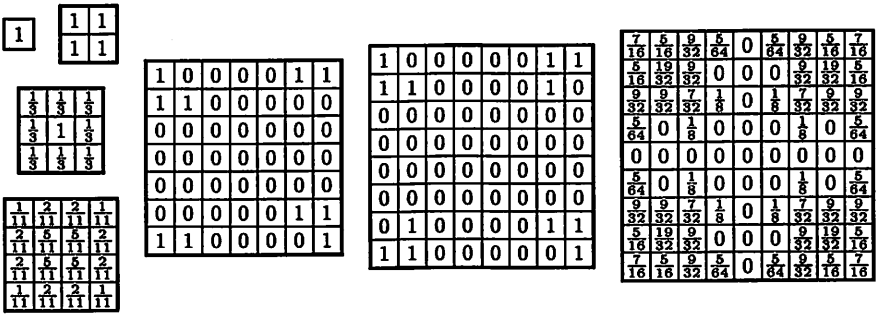

Figure 3 – Maximum Fractional Knight Packings. The sum of the weights which can be covered by a Knights is at most one.

For a 5 × 5 board, cover the board black and white in the usual way so the corners are white (see figure 4). Suppose a cover has fewer than 5 Knights. Since a Knight covers one square of its own color and at most 8 of the other, two Knights are on each color. Assume a Knight is in a corner, say on square (1, 1). Since this Knight covers 2 black squares and the black Knights cover 2 black squares, the other white Knight must cover 8 black squares and so is on square (3, 3) (see the left board of figure 4). Then the black Knights must be on squares (3, 4) and (4, 3) leaving square (4, 4) uncovered, a contradiction. Otherwise no Knights are in corners. To cover the corners, black Knights must be on squares (2, 3) and (4, 3), or squares (3, 2) and (3, 4). Either leaves 2 white Knights to cover 3 white squares (see the right board of figure 4), a contradiction. So at least 5 Knights are needed to cover a 5 × 5 board.

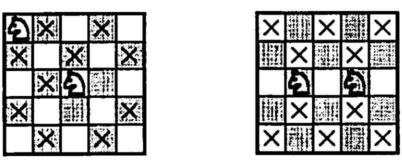

Figure 4 – Trying to Cover a 5 × 5 Board with 4 Knights. The left assumes there is a corner Knight. The right assumes there are no corner Knights.

For a 6 × 6 board, divide the board into quarters (see figure 5). Suppose a cover has fewer than 8 Knights. Then one quarter, say the top left, has at most one Knight. To cover square (1, 1), the Knight in the top left quarter must be on squares (1, 1), (2, 3), or (3, 2). Since squares (2, 1) and (1, 2) has to be covered by Knights not in the top left quarter, there must be Knights on squares (4, 2) and (2, 4). Since these two Knights cover squares (2, 3) and (3, 2), we can do no better than to place the Knight in the top left quarter on square (2, 3). Other Knights (which cannot be in the top left quarter) can only cover one of squares (1, 3), (4, 1), (5, 1), (5, 6), or (6, 6) (marked 1 in figure 5). Hence at least 5 more Knights are needed. Thus a 6×6 board cannot be covered with fewer than 8 Knights.

X	X	1	0	X	X
X	0	♘	♘	0	0
X	X	0	X	X	X
1	♘	X	X	X	0
1	0	0	X	0	1
X	0	X	0	0	1

Figure 5 – Trying to Cover a 6×6 Board with 7 Knights. Assume the top left quarter contains only one Knight. Then, without loss of generality, three Knights can be placed as shown. Since a Knight not in the top left quarter can only cover one square marked "1", at least five more Knights are needed.

For a 10×10 board, figure 6 shows a fractional Knight packing where the weights in each quarter add to $3\frac{1}{3}$. Squares with positive weight can only be covered by Knights in their quarter. So each quarter has 4 or more Knights giving at least 16 Knights in any cover of a 10×10 board. ■

$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Figure 6 – A Fractional Knight Packing of a 10×10 Board. Weights in each quarter sum to $3\frac{1}{3}$. So any cover has at least 4 Knights per quarter (the sum of the weights in a maximum fractional Knight packing is $14\frac{6}{11}$; the above with sum $13\frac{1}{3}$ is not maximum).

We also enumerate minimum Knight covers (see table 1 and figure 8). Gardner [9] gave the number of distinct minimum Knight covers for up to 8×8 boards. Jackson and Pargas [12] listed these covers. I wrote a branch-and-bound program in Pascal (shown in figure 7) to exhaustively search for Knight covers. The program recursively adds Knights to the right and then below previously placed Knights. At each step, we check if the remaining Knights can cover the rest of the board. This is determined with a fractional Knight packing: if the number of Knights left is less than the sum of the weights on the uncovered squares, the branch is abandoned.

```
1 program knight(input,output);
2 var sum,denom,g,i,j,k,l,n,uncovered:integer;
3     a:array[-1..20,-1..20] of boolean; r,c:array[1..9] of integer;
4     w:array[-1..20,-1..20] of integer; row,col:array[1..50] of integer;
5 procedure dom(left,frac,k,l:integer);
6 var newfrac,i,j:integer; ok:boolean; save:array[1..9] of boolean;
7 begin
8   if frac<=denom*left then repeat
9     l:=l+1; if l>n then begin l:=1; k:=k+1; end;
10    row[left]:=k; col[left]:=l; newfrac:=frac;
11    for i:=1 to 9 do begin
12      save[i]:=a[k+r[i],l+c[i]]; a[k+r[i],l+c[i]]:=true;
13      if not save[i] then newfrac:=newfrac-w[k+r[i],l+c[i]];
14    end;
15    if left=1 then begin
16      ok:=true; for i:=1 to n do for j:=1 to n do ok:=ok and a[i,j];
17      if ok then begin
18        for i:=1 to g do write(row[i]:0,',',col[i]:0,' '); writeln;
19      end;
20    end
21    else dom(left-1,newfrac,k,l);
22    for i:=1 to 9 do a[k+r[i],l+c[i]]:=save[i];
23  until ((k=n) and (l=n)) or not a[k-2,l-1]
24         or ((l=n-1) and not a[k-2,n]);
25 end;
26 begin
27  write('Board size: '); readln(n); write('No. of Knights: '); readln(g);
28  write('Denominator of Weights: '); readln(denom);
29  for i:=-1 to n+2 do for j:=-1 to n+2 do w[i,j]:=0;
30  writeln('Numerator of Weights: ');
31  for i:=1 to n do for j:=1 to n do read(w[i,j]);
32  sum:=0; for i:=1 to n do for j:=1 to n do sum:=sum+w[i,j];
33  r[1]:= 0; c[1]:= 0; r[2]:= 1; c[2]:= 2; r[3]:= 2; c[3]:= 1;
34  r[4]:= 2; c[4]:=-1; r[5]:= 1; c[5]:=-2; r[6]:=-1; c[6]:=-2;
35  r[7]:=-2; c[7]:=-1; r[8]:=-2; c[8]:= 1; r[9]:=-1; c[9]:= 2;
36  for i:=-1 to n+2 do for j:=-1 to n+2 do a[i,j]:=min(i-1,j-1,n-i,n-j)<0;
37  dom(g,sum,1,0);
38 end.
```

Figure 7 – Enumerating Covers of an $n \times n$ Board with g Knights. Inputs (27-31) are n, g, and a fractional Knight packing given as a matrix of numerators with a common denominator (*e.g.*, the fractional Knight packing of a 3×3 board in figure 4 has matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and denominator 3). After initialization (31-35), a recursive procedure is called whose arguments are the number of Knights left, the sum of weights left to be covered, and the last square where a Knight was added. The procedure checks if enough Knights are left to cover the remaining board (8). If so, a Knight is placed on squares beyond (k, l): for each square, altering the board (9-14), recursively calling the procedure (21), and changing the board back (22). A cover is displayed if the board is covered (15-20). This continues until we encounter a square which is the last opportunity to cover an uncovered square (23-24). Note all arithmetic is integer and hence exact.

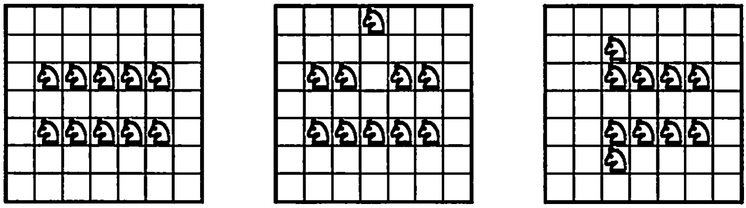

Figure 8 – Minimum Covers of a 7 × 7 Board. There are three "distinct" covers (different under symmetry) with 10 Knights. Since the left has 4-fold symmetry, the others have 2-fold symmetry, and the board has 8-fold symmetry, these represent $\frac{8}{4}+\frac{8}{2}+\frac{8}{2}=10$ covers.

The program confirms the results of [9] and [12] with one exception: both said there are 22 distinct minimum covers of a 6 × 6 board. Hare and Hedetniemi [11] gave a correct list of 23 covers (the 2nd board of the 3rd row in [11] is missing from [12]). Gardner [9] said it was believed that the covers in figure 9 are the only distinct minimum covers of 9 × 9 and 10 × 10 boards. The program verifies this (also verified in [11] and [4]).

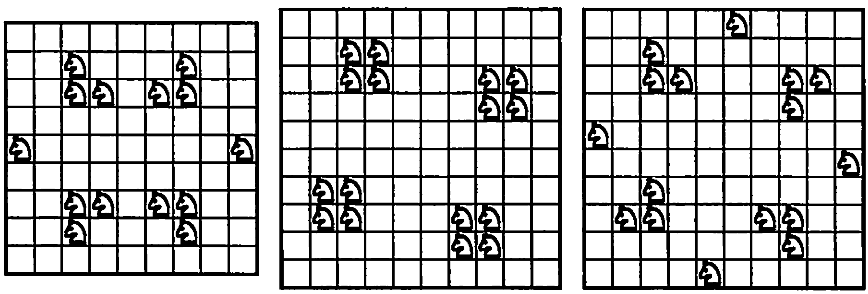

Figure 9 – Minimum Covers. The program shows these are the only distinct covers of a 9 × 9 board with 14 Knights and a 10 × 10 board with 16 Knights. The middle board is what was thought to be unique (up to symmetry) until 11 "readers" of Gardner [8] found a second minimum cover which is shown on the right.

3 11 × 11 through 14 × 14 boards

The program could find all minimum Knight covers up to 14 × 14 boards (see table 2). For 12 × 12 through 14 × 14 boards, this is also the first proof that what are believed to be the minimum values are indeed minimum.

Perhaps the most studied Knight cover problem is on 11 × 11 boards. In 1898, a cover with 23 Knights was given in [13]. Three years later, Ahrens [1] beat this with a cover of 22 Knights (assuming the top 12 Knights move down a row). This was thought to be minimum until 1973 when Lemaire

[15] found a cover with 21 Knights (see figure 10). Fricke, *et al.* [5] report that McRae has a lengthy unpublished proof that 21 is best. This settled the minimization problem for 11 × 11 boards.

Size of Board	11 × 11	12 × 12	13 × 13	14 × 14
Minimum in a Knight Cover	21	24	28	32
Number of Minimum Covers	800	2	152	4
No. of Distinct Min. Covers	100	1	20	1

Table 2 – Number of Covers. This gives the minimum number of Knights which cover an $n \times n$ board for $11 \le n \le 14$. In this range, this paper give the first census of minimum Knight covers.

Jackson and Pargas [12] stated that Steve Hedetniemi "lists over 100 variations" of Lemaire's cover (this list apparently had duplicates). In 8 hours, the program in figure 7 used the fractional Knight packing in figure 10 to show there are exactly 800 covers with 21 Knights (in 12 minutes, the program showed that no cover has 20 Knights, confirming McRae's result). Since all 800 lack symmetry, there are exactly 100 distinct minimum covers which are shown in figure 10. As Jackson and Pargas suspected, all minimum covers of an 11 × 11 board are "variations" of Lemaire's cover.

1	½	0	0	0	½	0	0	0	½	1
½	1	0	0	0	0	0	0	0	1	½
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	½	0	0	0	0	0
½	0	0	0	½	1	½	0	0	0	½
0	0	0	0	0	½	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
½	1	0	0	0	0	0	0	0	1	½
1	½	0	0	0	½	0	0	0	½	1

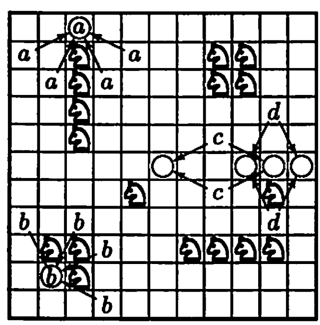

Figure 10 – Covering an 11 × 11 Board with 21 Knights. The left is a maximum fractional Knight packing whose sum is 17. The right shows all distinct minimum covers. They share 17 Knights leaving 6 uncovered squares (marked with circles). Four Knights can cover these: one each at a square marked *a*, *b*, *c*, and *d*. So there are $5^2 2^2 = 100$ distinct minimum covers. Lemaire's cover had Knights on the lower right *a*, the circled *b*, the top *c*, and the bottom *d*.

Since a 2 × 2 block of Knights covers a 4 × 6 block of squares, 24 Knights can cover a 12 × 12 board as in figure 11 (first found by Ahrens [2]). Gardner [9] said this cover was thought to be the unique (up to symmetry) minimum Knight cover. It took 10 minutes for the program in figure 7 to verify this using the fractional Knight packing in figure 11.

$\frac{11}{12}$	$\frac{5}{16}$	$\frac{1}{12}$	0	0	$\frac{11}{16}$	$\frac{11}{16}$	0	0	$\frac{1}{12}$	$\frac{5}{16}$	$\frac{11}{12}$
$\frac{5}{16}$	$\frac{37}{48}$	0	0	$\frac{7}{48}$	$\frac{7}{48}$	$\frac{7}{48}$	$\frac{7}{48}$	0	0	$\frac{37}{48}$	$\frac{5}{16}$
$\frac{1}{12}$	0	$\frac{1}{12}$	0	0	0	0	0	0	$\frac{1}{12}$	0	$\frac{1}{12}$
0	0	0	0	0	0	0	0	0	0	0	0
0	$\frac{7}{48}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	$\frac{7}{48}$	0
$\frac{11}{16}$	$\frac{7}{48}$	0	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	0	0	$\frac{7}{48}$	$\frac{11}{16}$
$\frac{11}{16}$	$\frac{7}{48}$	0	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	0	0	$\frac{7}{48}$	$\frac{11}{16}$
0	$\frac{7}{48}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	$\frac{7}{48}$	0
0	0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{12}$	0	$\frac{1}{12}$	0	0	0	0	0	0	$\frac{1}{12}$	0	$\frac{1}{12}$
$\frac{5}{16}$	$\frac{37}{48}$	0	0	$\frac{7}{48}$	$\frac{7}{48}$	$\frac{7}{48}$	$\frac{7}{48}$	0	0	$\frac{37}{48}$	$\frac{5}{16}$
$\frac{11}{12}$	$\frac{5}{16}$	$\frac{1}{12}$	0	0	$\frac{11}{16}$	$\frac{11}{16}$	0	0	$\frac{1}{12}$	$\frac{5}{16}$	$\frac{11}{12}$

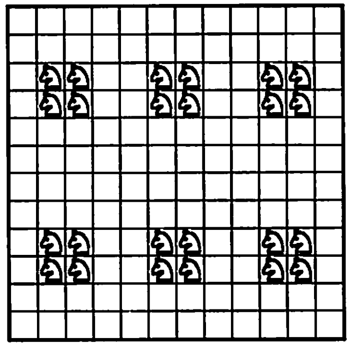

Figure 11 – Covering a 12 × 12 Board with 24 Knights. The left is a maximum fractional Knight packing whose weights sum to $21\frac{5}{12}$. The right is the only cover (up to symmetry) with 24 Knights.

Ahrens [2] covered a 13×13 board with 28 Knights. Using the fractional Knight packing in figure 14, the program in figure 7 took 15 minutes to show this is minimum, and to show there are 20 distinct (not counting reflections and rotations) minimum covers with 28 Knights. These fall into three "classes" which are shown in figures 12 and 13.

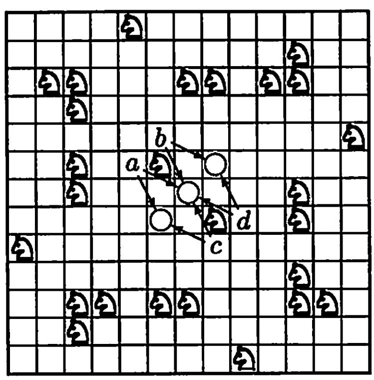

Figure 12 – Seventeen Covers of a 13 × 13 Board with 28 Knights. The 26 Knights shown cover all but 3 squares (marked with circles). At least one of the other two Knights is on square a, b, c or d as these are the only spots where a Knight can cover more than one circled square. If only one Knight is on a lettered square, there are 7 ways to cover the other circled square. So there are $\frac{4 \cdot 7}{2} = 14$ distinct minimum covers (halving accounts for symmetry). There are also 4 ways to cover the circled squares with two Knights on lettered squares: (a, b), (a, d), (b, c) and (c, d). However the first and last are reflections. So 17 distinct minimum covers share these 26 Knights.

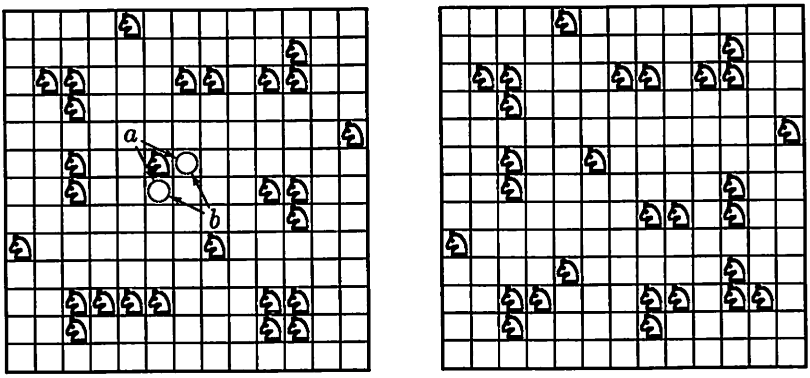

Figure 13 – The Other Three Covers of a 13 × 13 Board with 28 Knights. The left shows 27 Knights which cover all but 2 squares (marked with circles). These can be covered by a 28$^{\underline{th}}$ Knight at either square *a* or *b*. The right is one more distinct minimum cover. With the 17 in Figure 12, these are all 20 distinct minimum covers.

1	$\frac{1}{2}$	0	0	0	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{3}{8}$	0	0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{7}{8}$	0	0	0	$\frac{1}{8}$	$\frac{33}{56}$	$\frac{1}{8}$	0	0	0	$\frac{7}{8}$	$\frac{1}{2}$
0	0	0	0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0	0	0	0	0
0	0	0	0	0	0	$\frac{1}{28}$	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	0	0	$\frac{5}{14}$	$\frac{2}{7}$	$\frac{5}{14}$	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$\frac{3}{4}$	$\frac{33}{56}$	$\frac{1}{4}$	$\frac{1}{28}$	0	$\frac{2}{7}$	$\frac{11}{28}$	$\frac{2}{7}$	0	$\frac{1}{28}$	$\frac{1}{4}$	$\frac{33}{56}$	$\frac{3}{4}$
$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	0	0	$\frac{5}{14}$	$\frac{2}{7}$	$\frac{5}{14}$	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	$\frac{1}{28}$	0	0	0	0	0	0
0	0	0	0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0	0	0	0	0
$\frac{1}{2}$	$\frac{7}{8}$	0	0	0	$\frac{1}{8}$	$\frac{33}{56}$	$\frac{1}{8}$	0	0	0	$\frac{7}{8}$	$\frac{1}{2}$
1	$\frac{1}{2}$	0	0	0	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{3}{8}$	0	0	0	$\frac{1}{2}$	1

Figure 14 – A Fractional Knight Packing. This is a maximum fractional Knight packing of a 13×13 board. Its weights add to $25\frac{27}{28}$.

In 1918, Ahrens [2] covered a 14 × 14 board with 34 Knights. Davis (from [9]) beat this with a cover the 32 Knights shown in figure 15. With the fractional Knight packing in figure 15, the program took 18 hours to prove that Davis's cover is the unique (up to symmetry) minimum cover.

4 15 × 15 to 20 × 20 Boards

Beyond 14 × 14 boards, the program in figure 7 takes too long to run. So our goal shifts to just finding covers with fewer Knights, and in particular, to beat the results in Jackson and Pargas [12]. New records are set in 5 of these 6 cases (see table 3).

88	49	15	0	0	35	47	47	35	0	0	15	49	88
49	95	15	0	0	0	47	47	0	0	0	15	95	49
15	15	20	8	0	11	36	36	11	0	8	20	15	15
0	0	8	0	0	0	13	13	0	0	0	8	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	11	0	0	20	27	27	20	0	0	11	0	35
47	47	36	13	0	27	24	24	27	0	13	36	47	47
47	47	36	13	0	27	24	24	27	0	13	36	47	47
35	0	11	0	0	20	27	27	20	0	0	11	0	35
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	8	0	0	0	13	13	0	0	0	8	0	0
15	15	20	8	0	11	36	36	11	0	8	20	15	15
49	95	15	0	0	0	47	47	0	0	0	15	95	49
88	49	15	0	0	35	47	47	35	0	0	15	49	88

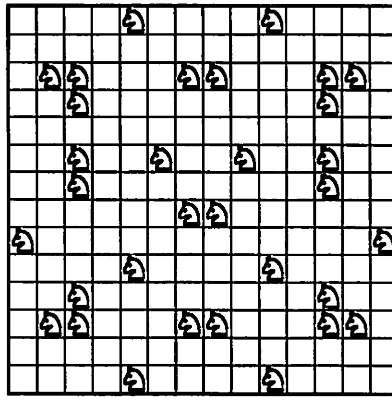

Figure 15 – Covering a 14 × 14 Board with 32 Knights. The left is the numerators of a maximum fractional Knight packing; the denominator is 118. It sums to $27\frac{49}{59}$. The right is the unique (up to symmetry) cover with 32 Knights.

	15 × 15	16 × 16	17 × 17	18 × 18	19 × 19	20 × 20
Jackson and Pargas	36	42	48	54	60	64
Integer Programming	36	40	46	52	59	64
Simulated Annealing	–	–	–	–	58	–
Guess work	–	–	–	–	–	63

Table 3 – Small Knight Covers. This compares covers in Jackson and Pargas [12] to those found with integer programs searching for symmetric covers and other means.

Since it is impractical to find all minimum Knight covers and since all square boards so far except for 11 × 11 boards have symmetric minimum covers, searches were limited to symmetric covers. A square board has three types of 2-fold symmetry: bilateral (invariant under horizontal or vertical reflections), diagonal (invariant under diagonal reflections), and rotational (invariant under 180° rotation). We search for covers of all three types.

The restricted search precludes proving a cover is minimum. So instead of using the program in figure 7, we looked for covers with integer programs (IP) solved with an optimization package (see figure 16). This is essentially a branch-and-bound algorithm where a maximum fractional Knight packing is found at each step (this can be much faster). However, rounding in the floating point arithmetic could cause a branch to be incorrectly abandoned (built-in tolerances reduce, but do not eliminate this problem). So we may not have a minimum symmetric cover even when the solver finishes.

In 1918, Ahrens [2] covered a 15×15 board with 37 Knights. This record stood until 1991 when Jackson and Pargas [12] used simulated annealing to find the delightful cover with 36 Knights shown in figure 17.

```
min AA+AB+AC+AD+AE+AF+AG+AH+AI+AJ+AK+AL+AM+AN+AO
   +BA+BB+BC+BD+BE+BF+BG+BH+BI+BJ+BK+BL+BM+BN+BO
   +CA+CB+CC+CD+CE+CF+CG+CH+CI+CJ+CK+CL+CM+CN+CO
   +DA+DB+DC+DD+DE+DF+DG+DH+DI+DJ+DK+DL+DM+DN+DO
   +EA+EB+EC+ED+EE+EF+EG+EH+EI+EJ+EK+EL+EM+EN+EO
   +FA+FB+FC+FD+FE+FF+FG+FH+FI+FJ+FK+FL+FM+FN+FO
   +GA+GB+GC+GD+GE+GF+GG+GH+GI+GJ+GK+GL+GM+GN+GO
   +HA+HB+HC+HD+HE+HF+HG+HH+HI+HJ+HK+HL+HM+HN+HO
   +IA+IB+IC+ID+IE+IF+IG+IH+II+IJ+IK+IL+IM+IN+IO
   +JA+JB+JC+JD+JE+JF+JG+JH+JI+JJ+JK+JL+JM+JN+JO
   +KA+KB+KC+KD+KE+KF+KG+KH+KI+KJ+KK+KL+KM+KN+KO
   +LA+LB+LC+LD+LE+LF+LG+LH+LI+LJ+LK+LL+LM+LN+LO
   +MA+MB+MC+MD+ME+MF+MG+MH+MI+MJ+MK+ML+MM+MN+MO
   +NA+NB+NC+ND+NE+NF+NG+NH+NI+NJ+NK+HL+NM+NN+NO
   +OA+OB+OC+OD+OE+OF+OG+OH+OI+OJ+OK+OL+OM+ON+OO
st
AA) AA+BC+CB>=1
AB) AB+BD+CA+CC>=1
AC) AC+BA+BE+CB+CD>=1
AD) AD+BB+BF+CC+CE>=1
AE) AE+BC+BG+CD+CF>=1
AF) AF+BD+BH+CE+CG>=1
AG) AG+BE+BI+CF+CH>=1
AH) AH+BF+BJ+CG+CI>=1
AI) AI+BG+BK+CH+CJ>=1
AJ) AJ+BH+BL+CI+CK>=1
AK) AK+BI+BM+CJ+CL>=1
AL) AL+BJ+BN+CK+CM>=1
AM) AM+BK+BO+CL+CN>=1
AN) AN+BL+CM+CO>=1
AO) AO+BM+CN>=1
BA) BA+AC+CC+DB>=1
BB) BB+AD+CD+DA+DC>=1
BC) BC+AA+AE+CA+CE+DB+DD>=1
BD) BD+AB+AF+CB+CF+DC+DE>=1
BE) BE+AC+AG+CC+CG+DD+DF>=1
BF) BF+AD+AH+CD+CH+DE+DG>=1
BG) BG+AE+AI+CE+CI+DF+DH>=1
BH) BH+AF+AJ+CF+CJ+DG+DI>=1
BI) BI+AG+AK+CG+CK+DH+DJ>=1
BJ) BJ+AH+AL+CH+CL+DI+DK>=1
BK) BK+AI+AM+CI+CM+DJ+DL>=1
BL) BL+AJ+AN+CJ+CN+DK+DM>=1
BM) BM+AK+AO+CK+CO+DL+DN>=1
BN) BN+AL+CL+DM+DO>=1
BO) BO+AM+CM+DN>=1
CA) CA+AB+BC+DC+EB>=1
CB) CB+AA+AC+BD+DD+EA+EC>=1
CC) CC+AB+AD+BA+BE+DA+DE+EB+ED>=1
CD) CD+AC+AE+BB+BF+DB+DF+EC+EE>=1
 :     :    :
 :     :    :
OO) OO+MN+NM>=1

int 225
go
```

Figure 16 – An Integer Program. This is designed to find a minimum Knight cover of a 15 × 15 board in Lindo. Variable AA being 1 means a Knight is on square (1, 1) (otherwise it is 0), etc. The goal is then to minimizes the sum of the variables. Constraint AA) ensures that a Knight covers square (1, 1), etc. (only some of the 225 constraints are shown). The "int 225" command makes the first 225 variables (*i.e.*, all of them) into Boolean variables (so they can only be 0 or 1). This took too long. Variants are useful, however. We find a maximum fractional Knight packing by replacing "min" with "max", reversing the inequalities, and removing "int 225". We can search for symmetric covers by making appropriate replacements (*e.g.*, for rotational symmetry: replace OO with AA, ON with AB, ..., HI with HG) and then remove redundant constraints (this only found covers with 37 or more Knights, but worked better for other sizes). What worked best for 15 × 15 boards was to force CC, MC, CM, MM, and BC to be in the cover (as, up to symmetry, they have been in all minimum covers of 6 × 6 and larger boards) by deleting these variables and all constraints that contain them. This found a new cover with 36 Knights (see figure 17).

I could not beat 36 Knights. The program in figure 7 would take months to find all (if any) covers with 35 Knights. An IP failed due to numerical problems. Searches confined to symmetric covers only found covers with at least 37 Knights. At least I found a new cover with 36 Knights. I divided the board into nine 5 × 5 subboards and used an IP to find a cover with

4 Knights per subboard. This resulted in the right cover in figure 17. It is quite different than Jackson and Pargas's cover suggesting that if 36 is minimum, then minimum covers of 15 × 15 boards are in several classes.

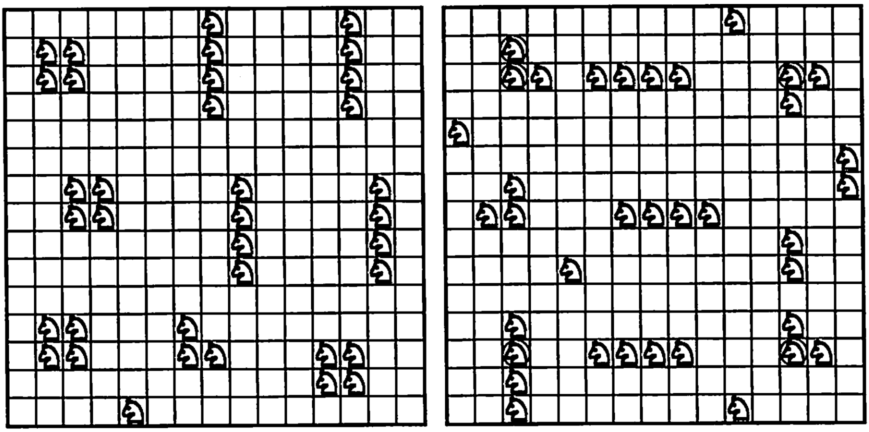

Figure 17 – Covering a 15 × 15 Board with 36 Knights. The left is from Jackson and Pargas. The right was found by an IP where the circled knights were forced to be in the cover.

For 16 × 16 boards, Jackson and Pargas [12] gave a cover using 42 Knights. Figure 18 is an elegant cover with 40 Knights having 90° rotational symmetry found with an IP searching for covers with only 180° symmetry.

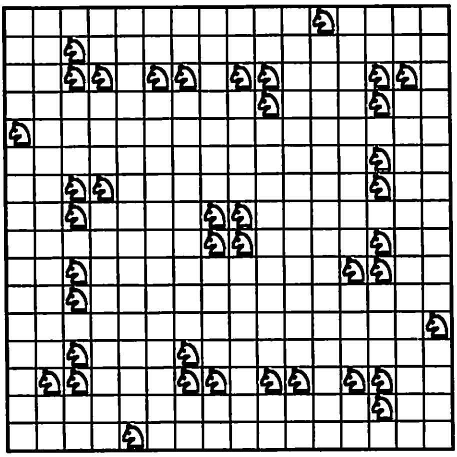

Figure 18 – Covering a 16 × 16 Board with 40 Knights.

For 17 × 17 boards, Jackson and Pargas [12] found a cover with 48 Knights. Figure 19 presents one with 46 Knights found using an IP searching for covers with 180° rotational symmetry. For 18 × 18 boards, [12] has a cover with 54 Knights. Figure 20 gives ones with 52 Knights found with an IP looking for bilaterally symmetric covers.

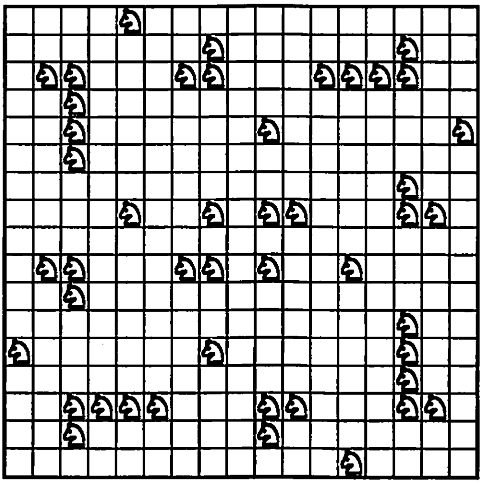

Figure 19 – Covering a 17 × 17 Board with 46 Knights.

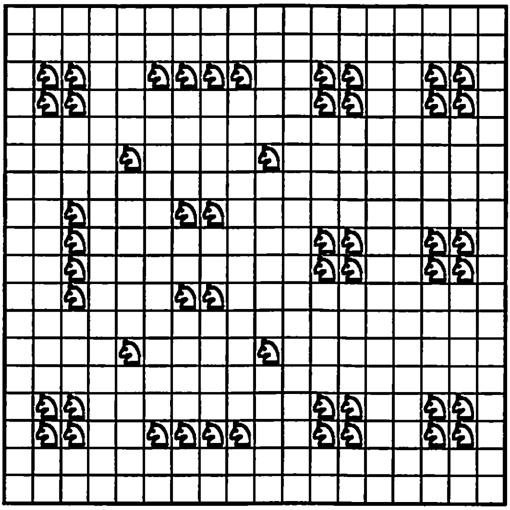

Figure 20 – Covering an 18 × 18 Board with 52 Knights.

Jackson and Pargas [**12**] covered a 19 × 19 board with 60 Knights. An IP gave a symmetric cover with 59 Knights. I proudly showed it to my 1999 *Computational Graph Theory* class. In perhaps my most thrilling moment as a teacher, Art Busch's presented a cover with 58 Knights (shown in figure 21) found using his simulated annealing program.

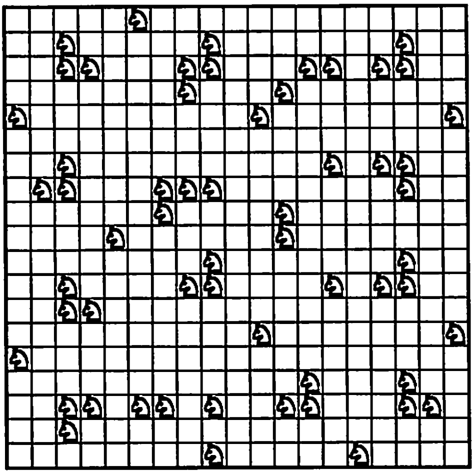

Figure 21 – Covering a 19 × 19 Board with 58 Knights.

In Jackson and Pargas [**12**], simulated annealing only found covers of a 20 × 20 board with 66 or more Knights. However, they noted that covers with 64 Knights can be formed from four minimum covers of a 10 × 10 board. IP searching for symmetric covers failed to beat this. Using various heuristics, Alice McRae (personal communication) could also only find covers with 64 or more Knights. Figure 22 shows a cover with 63 Knights that I found "by hand". It is interesting that a manual approach could beat such concerted computer efforts.

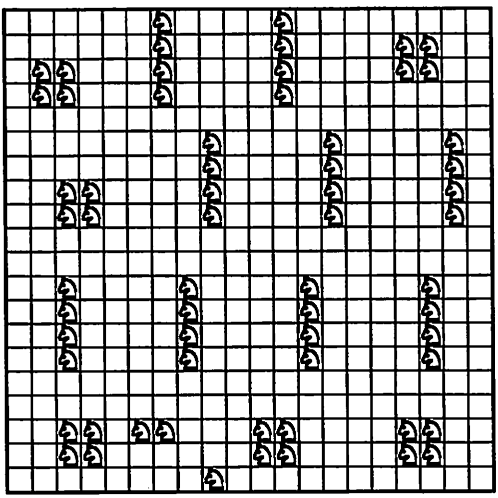

Figure 22 – Covering a 20 × 20 Board with 63 Knights. This was formed around a minimum 3 × 20 cover (from [4]) at the bottom.

5 21 × 21 to 25 × 25 Boards

There is no published work on Knight covers beyond 20 × 20 boards. My goal here is to set benchmarks for later work (see table 4). We continue using integer programs. But instead of searching for covers with 2-fold symmetry (which takes too long), we look for ones with 4-fold symmetry. A square board has three types of 4-fold symmetry: bilateral (invariant under horizontal and vertical reflections), diagonal (invariant under reflections about either diagonal), and rotational (invariant under 90° rotation).

Size of Board	21 × 21	22 × 22	23 × 23	24 × 24	25 × 25
Best Cover Found	71	76	84	88	97

Table 4 – New Knight Covers. The first was found by simulated annealing. The next three were found by an IP searching for minimum covers with 4-fold symmetry. The smallest cover known for a 25 × 25 board was found by "guess work".

For 21 × 21 boards, an IP found a cover of 72 Knights having 90° rotational symmetry. However, Art Busch beat this again with his simulated annealing program. It found the cover with 71 Knights shown in figure 23.

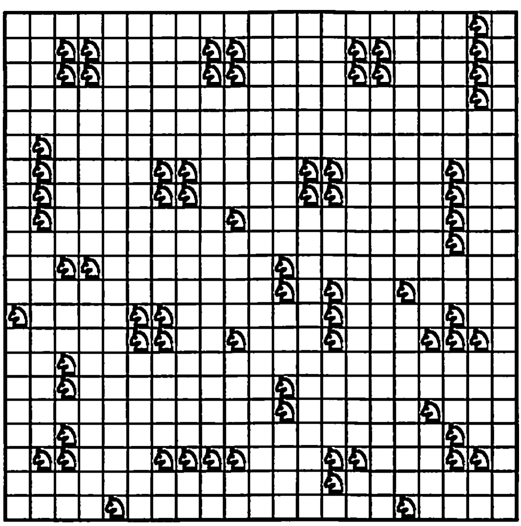

Figure 23 – Covering a 21 × 21 Board with 71 Knights.

For 22×22 boards, an IP found a 4-fold bilaterally symmetric cover with 76 Knights (figure 24). For a 23 × 23 board, an IP found a cover with 90° rotational symmetry having 84 Knights (figure 25). An IP looking for covers of a 24×24 board with 4-fold bilateral symmetry found one with 88 Knights (figure 26). It consists of 2 × 2 blocks of Knights (like the minimum cover of a 12 × 12 board, but unlike those since). For 25 × 25 boards, the best cover found with an IP (among those with 4-fold symmetry) had 105 Knights. Figure 27 (found mostly "by hand") shows a cover with 97 Knights.

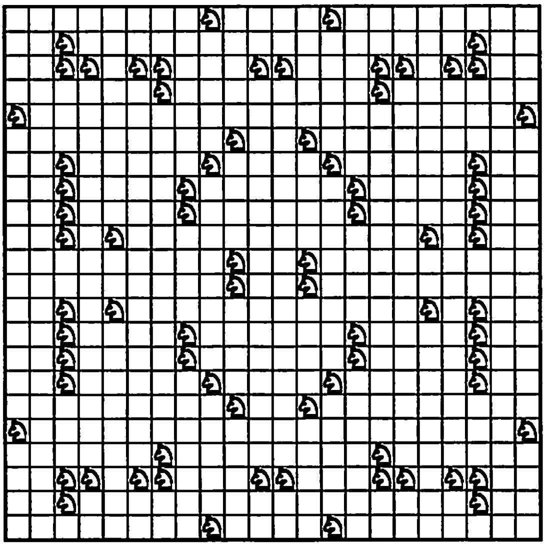

Figure 24 – Covering a 22 × 22 Board with 76 Knights.

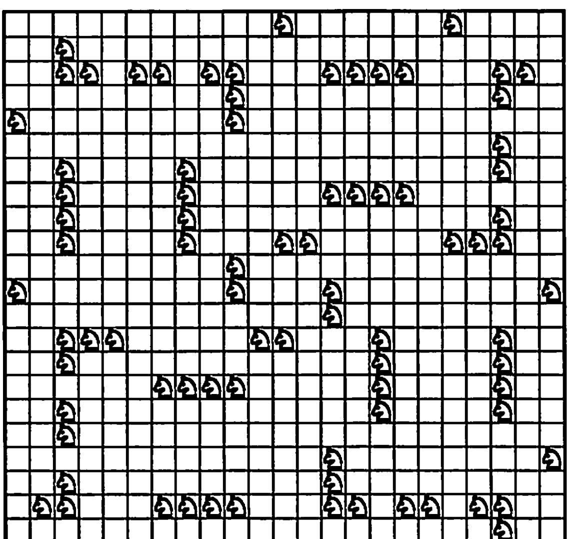

Figure 25 – Covering a 23 × 23 Board with 84 Knights.

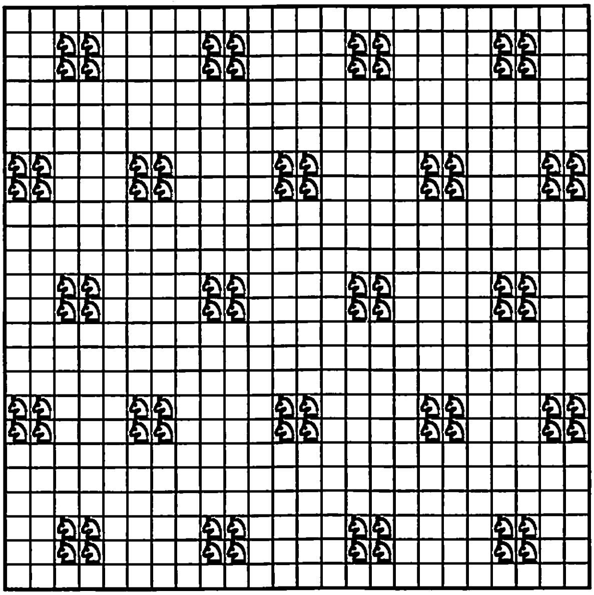

Figure 26 – Covering a 24 × 24 Board with 88 Knights.

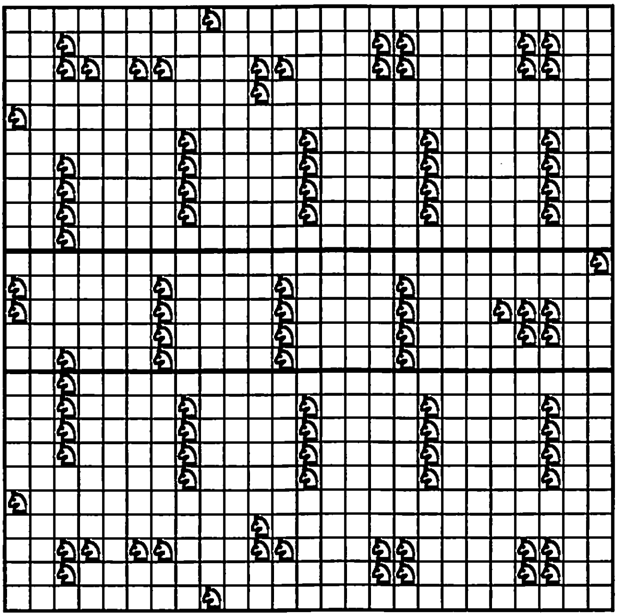

Figure 27 – Covering a 25 × 25 Board with 97 Knights. The bottom 10 rows roughly mimic the bottom 10 rows of figure 22; the top 10 are its reflection. An IP for a minimum cover of the uncovered squares placed the other Knights.

6 Comments and Reflections

Which records are easiest to break? Table 5 and figure 28 summarize the best known results. Up to 14 × 14 boards, these are proven to be the minimum value. I suspect the records for 15 × 15 to 18 × 18 boards are also unbeatable. Since several approaches gave the same cover, I believe the cover in figure 18 is the unique (up to symmetry) minimum cover of a 16 × 16 board. Numerologicly, the values for 21 × 21 and 23 × 23 boards appear easiest to beat.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
No. of Knights	1	4	4	4	5	8	10	12	14	16	21	24	28

n	14	15	16	17	18	19	20	21	22	23	24	25
No. of Knights	32	36	40	46	52	58	63	71	76	84	88	97

Table 5 – Knight Cover Records. This shows the smallest number of Knights known to be able to cover an $n \times n$ board. These are provably minimum for $n \leq 14$. They are plotted in figure 28.

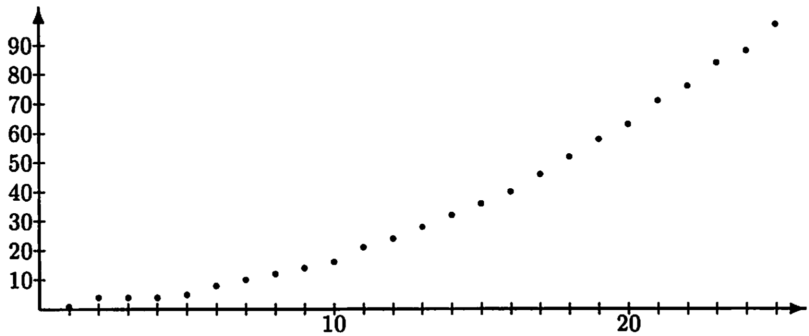

Figure 28 – A Plot. This plots n = 1, 2, ..., 25 against the best-known number of Knights in a cover of an $n \times n$ board.

What happens for rectangular boards? Hare and Hedetniemi [**11**] developed a dynamic programming (DP) algorithm for finding a minimum Knight cover of a $k \times n$ board. While exponential in k, the algorithm is linear in n. With it, they found minimum covers for all $k \leq 6$ and $n \leq 500$. From these results, conjectures were made for $k = 3$, 4, and 6 ($k = 1$ and 2 are trivial; $k = 5$ was not "fully analyzed"). By searching for periodicities in this algorithm, Fisher and Spalding [**4**] found minimum Knight covers when $k \leq 10$ and for all n (increases in k are due to algorithmic refinements and faster computers) verifying the conjectures in [**11**]. Since the program in figure 7 worked (up to 14 × 14 boards) where DP did not, perhaps combining these could succeed where neither did alone. For example, fractional Knight packings could be used to eliminate states in a DP algorithm.

What about larger boards? One might guess that staggered rows of 2×2 blocks as in figure 26 are ultimately best. Fisher and Spalding [4] showed they (with boundary alterations) cover a $k \times n$ board with at most $4 \lceil \frac{k+3}{6} \rceil \lceil \frac{n+1}{5} \rceil = \frac{2kn}{15} + O(k+n)$ Knights. Asymptoticly better are covers like in figure 29. Garnick and Nieuwejaar [10] proved (again with boundary alterations) they have at most $\left\lfloor \frac{(k+5)(n+6)}{8} \right\rfloor - 4 = \frac{kn}{8} + O(k+n)$ Knights. So if a_n is the minimum number of Knights in a cover of an $n \times n$ board, the limit of a_n/n^2 is between $\frac{1}{9}$ (a Knight covers at most 9 squares) and $\frac{1}{8}$. However for a 99×99 board, these bounds are $4 \lceil \frac{99+3}{6} \rceil \lceil \frac{99+1}{5} \rceil = 1360$ and $\left\lfloor \frac{(99+5)(99+6)}{8} \right\rfloor - 4 = 1361$ Knights. So if the covers from [10] are ultimately best, they may only be uniformly best for very large boards.

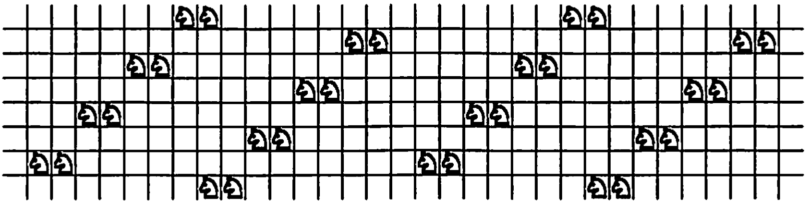

Figure 29 – An Asymptoticly Sparser Knight Cover

What inspired this paper? In a sense, I have worked on Knight covers for over 30 years. In 1968 (when I was 9), a friend of my parents, Darold Nelson, helped me read some *Mathematical Games* columns in *Scientific American* including Gardner [6–8]. I was fascinated, particularly with the sense of discovery conveyed with the follow-up publication of "repartee" from readers. Hoping to contribute, I unsuccessfully tried to cover an 11×11 board with fewer than 22 Knights, the record at the time (this seemed destined to be beaten as it was in 1973 by Lemaire [16]). In March 1999, Natasa Mateljević, a chemistry graduate student now at University of Denver, saw an early version of [4]. Seeing references to Martin Gardner, she said her father (a noted mathematician at the University of Belgrade) had also been inspired by Gardner's work. This prompted me to reexamine Knight covers on square boards. Upon finding better covers for 16×16 through 18×18 boards (the ones in figures 18-20), I put them in a draft of [4]. My coauthor, Anne Spalding (appalled by the paper's length), suggested I take them out and write a separate paper. That suggestion blossomed into this paper. Thanks to everyone mentioned.

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