Minimum Degree Growth of the Iterated Line Graph

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Abstract

Let δ_k denote the minimum degree of the k^{th} iterated line graph $L^k(G)$. For any connected graph G that is not a path, the inequality $\delta_{k+1} \geq 2\delta_k - 2$ holds. Niepel, Knor, and Šoltés [5] have conjectured that there exists an integer K such that, for all $k \geq K$, equality holds; that is, the minimum degree δ_k attains the least possible growth. We prove this conjecture by extending the methods we used in [2] for a similar conjecture about the maximum degree.

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1 Introduction

The line graph $L(G)$ of a graph G is defined as the graph whose vertices are the edges of G and where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident to a common vertex. Define the iterated line graph $L^k(G)$ recursively as $L^0(G) = G$, and $L^k(G) = L(L^{k-1}(G))$ for $k \geq 1$. Though line graphs themselves are well studied, iterated line graphs have received comparatively little attention. See [3] for a survey of results on line graphs, and [4], [5], and [2] for some recent results on iterated line graphs.

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If v is a vertex in $L(G)$ and u and w are the endpoints of the edge in G that corresponds to v, then $\deg_{L(G)}(v) = \deg_G(u) + \deg_G(w) - 2$. Thus, the maximum degree $\Delta(L(G))$ of $L(G)$ satisfies

$$\Delta(L(G)) \leq 2\Delta(G) - 2,$$

and the minimum degree $\delta(L(G))$ satisfies

$$\delta(L(G)) \geq 2\delta(G) - 2.$$

The maximum degree Δ_k and minimum degree δ_k of the iterated line graph $L^k(G)$ are bounded by the inequalities ([4], [5])

$$2^k(\delta_0 - 2) + 2 \leq \delta_k \leq \Delta_k \leq 2^k(\Delta_0 - 2) + 2.$$

In [5], Niepel, Knor, and Šoltés conjectured that, for any graph except a path, the minimum degree of the iterated line graph eventually attains the minimum growth rate of $\delta_{k+1} = 2\delta_k - 2$. Following the use of the Maximum Degree Growth Property (MDGP) in [2], we say that a graph G has the *Minimum Degree Growth Property (mDGP)* if $\delta(L(G)) = 2\delta(G) - 2$. In this paper, we prove that there exists an integer K such that, for all $k \geq K$, $L^k(G)$ possesses the mDGP, and thus establish the conjecture.

The corresponding conjecture for the growth of the maximum degree was proved by Hartke and Higgins in [2]. Section 2 of this paper lists definitions and results that mirror those in [2] when the obvious changes in notation and inequalities are made from "maximum degree" to "minimum degree." Section 2 also contains the proof of the minimum degree conjecture. Although the proof from [2] cannot be directly extended to the minimum degree case, the proof in this paper, while explicitly written for the minimum degree case, can be extended to prove the maximum degree conjecture. We conclude this paper with some questions of interest about iterated line graphs in Section 3.

For the remainder of the paper, we consider only finite simple connected graphs with no loops. Note that the iterated line graph of a path eventually becomes the empty graph and that the iterated line graphs of cycles and $K_{1,3}$ (whose line graph is a triangle) trivially satisfy the conjecture. Therefore, we consider only those graphs which are not contained in these classes.

2 The Minimum Degree of the Iterated Line Graph

For each statement in [2], there exists a dual statement about minimum or locally minimum degrees. To parallel the development of the maximum degree case, we shall number our statements correspondingly, but with primes. Except for Lemmas 20, 21, 22, and 23 of [2], the dual statements can be obtained by reversing all inequalities involving degrees. We shall present here Lemmas 22′ and 23′ using different methods of proof; however, the lemmas fulfill the same function in the overall proof. Lemma 20′ is an extension of Lemma 20 which encapsulates a technique used in Lemmas 22 and 24 of [2] for use in Lemmas 22′ and 23′, and Lemma 21′ is a rewording of Lemma 21.

Definition 1′. Let $\Delta(G)$ be the maximum degree among the vertices of G, and $\delta(G)$ be the minimum degree. Let Δ_k denote $\Delta(L^k(G))$ and δ_k denote $\delta(L^k(G))$.

Definition 2′. A graph G has the *Minimum Degree Growth Property (mDGP)* if $\delta(L(G)) = 2\delta(G) - 2$.

The minimum degree conjecture of Niepel, Knor, and Šoltés can now be stated as follows.

Conjecture 3′. [5] *Let G be a connected graph that is not a path. Then there exists an integer K such that, for all $k \geq K$, the mDGP holds; that is,*

$$\delta_{k+1} = 2\delta_k - 2.$$

We introduce the minimum degree induced subgraph to characterize those graphs which possess the mDGP.

Definition 4′. Let the *minimum degree induced subgraph* $m(G)$ be the subgraph of G induced by the vertices of G that have minimum degree $\delta(G)$. Let m_k denote $m(L^k(G))$.

The following lemmas have corresponding statements for maximum degree, and the proofs are analogous to those presented in [2].

Lemma 5′. *The mDGP holds for a graph G if and only if $m(G)$ contains an edge.*

Lemma 6′. *If H is a subgraph of G, then $L(H)$ is an induced subgraph of $L(G)$.*

Lemma 7′. *The mDGP holds for a graph G if and only if $L(m(G)) \cong m(L(G))$.*

Corollary 8′. *If $L^k(m(G))$ contains an edge for all $k \geq 0$, then the mDGP will hold for all $L^k(G)$, $k \geq 0$.*

Although Corollary 8′ proves the conjecture for many graphs, it does not help in cases where $m(G)$ is a path or a union of paths. The local minimum induced subgraph provides the key concept in finishing our proof.

Definition 9′. Let the *neighborhood* $N_G(v)$ of a vertex v in G be the set of vertices in G adjacent to v. Note that $v \notin N_G(v)$.

Let the *neighborhood* $N_G(S)$ of a subgraph S of G be the set of vertices adjacent to vertices in S but not contained in S. Thus, $N_G(S) = (\bigcup_{v \in S} N_G(v)) \setminus S$.

Definition 10′. A vertex v in G is a *local minimum* if $\deg_G(v) \leq \deg_G(w)$ for all $w \in N_G(v)$.

Definition 11′. A vertex $v \in L(G)$ is *generated by a vertex* $u \in G$ if the edge e in G that corresponds to v is incident to u. A subgraph J of $L(G)$ is *generated by a subgraph H of G* if, for each vertex $v \in J$, v is generated by a vertex in H.

We extend this concept to several iterations: Let $0 \leq t < s$ be given integers. A vertex $v \in L^s(G)$ is *generated by a vertex* $u \in L^t(G)$ if there exists a sequence of vertices $u = v_t, v_{t+1}, \ldots, v_{s-1}, v_s = v$, where $v_k \in L^k(G)$ and v_k is generated by v_{k-1} for $t < k \leq s$. A subgraph J of $L^s(G)$ is *generated by a subgraph H of $L^t(G)$* if, for each vertex $v \in J$, v is generated by a vertex in H.

Lemma 12′. *Every local minimum v in $L(G)$ is generated by a local minimum w in G. Moreover, v is generated by w and a vertex in G that is minimum in $N_G(w)$.*

Definition 13′. Let the *local minimum induced subgraph* $\mathfrak{Lm}(G)$ be the subgraph of G induced by local minimum vertices. Let $\mathfrak{Lm}_k$ denote $\mathfrak{Lm}(L^k(G))$.

Lemma 14′. *Let C be a component of $\mathfrak{Lm}(G)$. Then all vertices in C have the same degree in G.*

In [2], Lemmas 15, 16, and 17 were used solely to prove Corollary 18. The lemmas are technical and the corresponding changes for minimum degree are straightforward, and left to the interested reader. We state the result corresponding to Corollary 18.

Corollary 18′. *There exists an integer J_1 such that every component of $\mathfrak{Lm}_k$ generates a component of $\mathfrak{Lm}_{k+1}$ for $k \geq J_1$, and every component of $\mathfrak{Lm}_{k+1}$ is generated by exactly one component of $\mathfrak{Lm}_k$.*

Definition 19′. Let J_1 be as in Corollary 18′. For $k \geq J_1$, let $\{C_k\}$ be a sequence, where C_k is a component of $\mathfrak{Lm}_k$ and C_{k+1} is generated by C_k. Let $r_k = \deg_{L^k(G)}(v)$ for all $v \in C_k$.

Lemma 20′. *Let J_1 be as in Corollary 18′. Assume also that $R \geq J_1$ and that there does not exist an integer $Q \geq R$ such that C_k contains an edge for all $k \geq Q$. If $\{v_1, v_2, \ldots, v_n\}$ is a set of distinct vertices in $N_{L^R(G)}(C_R)$*

with the same degree in $L^R(G)$, then there exists an integer $T > R$ such that C_T contains a complete subgraph of n vertices generated by the induced subgraph of $\{v_1, v_2, \ldots, v_n\}$.

Proof. Let J_1 be as in Corollary 18′. Assume also that $R \geq J_1$, and assume that there does not exist an integer $Q \geq R$ such that C_k contains an edge for all $k \geq Q$. Let $\{v_R^1, v_R^2, \ldots, v_R^n\}$ be a set of distinct vertices in $N_{L^R(G)}(C_R)$ with $\deg_{L^R(G)}(v_R^1) = \ldots = \deg_{L^R(G)}(v_R^n) = a_R$. For $k \geq R$, inductively define a sequence of vertices as follows: let v_{k+1}^i be the vertex generated by $v_k^i \in N_{L^k(G)}(C_k)$ and by a vertex in C_k. Since $v_k^i \neq v_k^j$ for $i \neq j$, $v_{k+1}^i \neq v_{k+1}^j$. Also,

$$\begin{aligned}\deg_{L^{k+1}(G)}(v_{k+1}^i) &= \deg_{L^k(G)}(v_k^i) + r_k - 2 \\ &= \deg_{L^k(G)}(v_k^j) + r_k - 2 \quad \text{for all } j \\ &= \deg_{L^{k+1}(G)}(v_{k+1}^j).\end{aligned}$$

Thus, all of the vertices v_{k+1}^i have the same degree in $L^{k+1}(G)$. Let a_{k+1} denote $\deg_{L^{k+1}(G)}(v_{k+1}^i)$. Now, if a_k is not the minimum degree of a vertex in $N_{L^k(G)}(C_k)$, or if C_k contains an edge, then $v_{k+1}^i \in N_{L^{k+1}(G)}(C_{k+1})$ for every i. If a_k is minimum and C_k does not contain an edge, then $v_{k+1}^i \in C_{k+1}$.

Note that if C_k has an edge, then

$$\begin{aligned}a_{k+1} - r_{k+1} &= (a_k + r_k - 2) - (2r_k - 2) \\ &= a_k - r_k.\end{aligned}$$

If C_k does not have an edge, then a shrinking separation of degrees occurs:

$$\begin{aligned}a_{k+1} - r_{k+1} &< (a_k + r_k - 2) - (2r_k - 2) \\ &= a_k - r_k.\end{aligned}$$

Since we assume there are an infinite number of k's where C_k does not contain an edge, there exists an integer $T - 1$ such that C_{T-1} does not contain an edge and where a_{T-1} is the minimum degree of vertices in $N_{L^{T-1}(G)}(C_{T-1})$. Thus, C_T contains the n vertices $\{v_T^1, \ldots, v_T^n\}$. Since C_{T-1} does not contain an edge, it consists of a single vertex u. The vertex u generates v_T^i for all i, and so all the v_T^i are adjacent. Thus, C_T contains a complete subgraph of n vertices. □

Lemma 21′. *Let G be a connected graph that is not a path, cycle, or $K_{1,3}$. Then for all integers s there exists an integer Q such that $\delta_q \geq s$ for all $q \geq Q$. In particular, there exists an integer $J_2 \geq J_1$ such that $\delta_k \geq 4$ for all $k \geq J_2$.*

We now proceed with our characterization of the components of $\mathfrak{L}\mathrm{m}_k$.

Lemma 22′. *There exists an integer $J_3 \geq J_2$ such that C_{J_3} contains an edge.*

Proof. Assume that there does not exist an integer $J_3 \geq J_2$ such that C_{J_3} contains an edge. Let v be a vertex in $L^{J_2}(G)$ of maximum degree Δ_{J_2}. Let $\mathcal{P}_{J_2}$ be a path of minimum length from v to $w \in C_{J_2}$. The expression $\Delta_{J_2} - \delta_{J_2} + 1$ represents the maximum number of distinct degrees of vertices in $N_{L^{J_2}(G)}(v)$. Now, there are $\Delta_{J_2} - 1$ vertices in $N_{L^{J_2}(G)}(v) \setminus \mathcal{P}_{J_2}$. Since $\Delta_{J_2} - 1 > \Delta_{J_2} - \delta_{J_2} + 1$, then by the Pigeonhole Principle and the fact that $\delta_{J_2} \geq 4$, there exist two vertices z_1 and z_2 in $N_{L^{J_2}(G)}(v) \setminus \mathcal{P}_{J_2}$ such that $\deg_{L^{J_2}(G)}(z_1) = \deg_{L^{J_2}(G)}(z_2)$. Thus there exists a path $\mathcal{P}_{J_2}$ from $w \in C_{J_2}$ to a vertex v, where two vertices in $N_{L^{J_2}(G)}(v) \setminus \mathcal{P}_{J_2}$ have the same degree.

Suppose that $\mathcal{P}_k = (p_k^1, p_k^2, \ldots, p_k^{i_k}, w_k)$ is a path of length i_k in $L^k(G)$, $k \geq J_2$, where $w_k \in C_k$ and where there exist two vertices z_1 and z_2 in $N_{L^k(G)}(p_k^1) \setminus \mathcal{P}_k$ of the same degree. By Lemma 6′, $L(\mathcal{P}_k)$ is a path in $L^{k+1}(G)$. If $p_k^{i_k} w_k \in C_{k+1}$, then set $\mathcal{P}_{k+1} = L(\mathcal{P}_k) = (p_{k+1}^1 = p_k^1 p_k^2, \ldots, p_{k+1}^{i_{k+1}} = p_{k+1}^{i_k - 1} = p_k^{i_k} w_k)$. Otherwise, set $\mathcal{P}_{k+1}$ equal to $L(\mathcal{P}_k)$ extended to $w_{k+1} \in C_{k+1}$. Since w_k generates w_{k+1}, the extension involves adding only one edge. Note that $z_1 p_k^1$ and $z_2 p_k^1$ are vertices in $N_{L^{k+1}(G)}(p_{k+1}^1) \setminus \mathcal{P}_{k+1}$ with

$$\begin{aligned}
\deg_{L^{k+1}(G)}(z_1 p_k^1) &= \deg_{L^k(G)}(z_1) + \deg_{L^k(G)}(p_k^1) - 2 \\
&= \deg_{L^k(G)}(z_2) + \deg_{L^k(G)}(p_k^1) - 2 \\
&= \deg_{L^{k+1}(G)}(z_2 p_k^1).
\end{aligned}$$

Thus, $\mathcal{P}_{k+1}$ is a path $(p_{k+1}^1, \ldots, p_{k+1}^{i_{k+1}}, w_{k+1})$ in $L^{k+1}(G)$ where $w_{k+1} \in C_{k+1}$ and where there exist two vertices in $N_{L^{k+1}(G)}(p_{k+1}^1) \setminus \mathcal{P}_{k+1}$ of the same degree. If $p_k^{i_k} w_k \notin C_{k+1}$, then the length of $\mathcal{P}_{k+1}$ is $\ell(\mathcal{P}_{k+1}) = \ell(L(\mathcal{P}_k)) + 1 = \ell(\mathcal{P}_k) = i_k$. If $p_k^{i_k} w_k \in C_{k+1}$, then $\ell(\mathcal{P}_{k+1}) = \ell(L(\mathcal{P}_k)) = \ell(\mathcal{P}_k) - 1 = i_k - 1$. By Lemma 20′ applied to the single vertex $p_k^{i_k}$, there exists an integer $T > k$ such that $p_{T-1}^{i_{T-1}} w_{T-1} \in C_T$. Thus, $\ell(\mathcal{P}_T) = i_k - 1 < i_k = \ell(\mathcal{P}_{T-1})$.

By iteratively applying the above observations to $\mathcal{P}_{J_2}$, we notice that the length of $\mathcal{P}_k$ decreases as k increases, $k \geq J_2$. Specifically, for any given length $1 \leq t \leq \ell(\mathcal{P}_{J_2})$ there exists an integer T_t where $p_{T_t-1}^{i_{T_t}-1} w_{T_t-1} = p_{T_t-1}^t w_{T_t-1} \in C_{T_t}$, and so $\ell(\mathcal{P}_{T_t}) = t - 1 < t = \ell(\mathcal{P}_{T_t-1})$. Hence there exists an integer $T_1 \geq J_2$ where $\ell(\mathcal{P}_{T_1}) = 0$, and so there exist two vertices in $N_{L^{T_1}}(C_{T_1})$ generated by z_1 and z_2 that have the same degree. Applying Lemma 20′ again to these two vertices, there exists an integer J_3 where C_{J_3} contains two adjacent vertices, and thus has an edge. □

Lemma 23′. *If there does not exist an integer Q such that C_k contains an edge for all $k \geq Q$, then there exists an integer J_4 such that there are three*

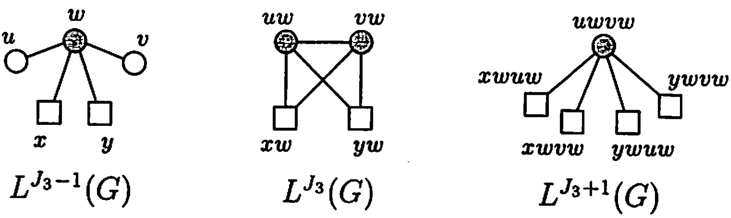

Figure 1: C_{J_3} contains an edge.

vertices in $N_{L^{J_4}(G)}(C_{J_4})$ where all three vertices have the same degree and are adjacent to the same vertex in C_{J_4}.

Proof. By Lemma 22′, there exists an integer J_3 such that C_{J_3} contains an edge. So that we may apply Lemma 21′ to $L^{J_3-1}(G)$, we assume that $J_3 > J_2$. This assumption is valid since we may prove Lemma 22′ with J_2 replaced by $J_2 + 1$. Let uw and vw be the adjacent vertices in C_{J_3}, where $w \in C_{J_3-1}$ and u and v have minimum degree in $N_{L^{J_3-1}(G)}(w)$. Note that u and v may be in C_{J_3-1}. By Lemma 21′, $\delta_{J_3-1} \geq 4$, and so there exist two vertices x and y in $N_{L^{J_3-1}(G)}(w) \setminus \{u, v\}$. Assume that $\deg_{L^{J_3-1}(G)}(x) \leq \deg_{L^{J_3-1}(G)}(y)$. The vertices xw and yw are in $N_{L^{J_3}(G)}(uw)$, and xw and yw are in $N_{L^{J_3}(G)}(vw)$. Then $(uw)(vw) \in C_{J_3+1}$, and $(xw)(uw), (xw)(vw), (yw)(uw), (yw)(vw) \in N_{L^{J_3+1}(G)}((uw)(vw))$. Figure 1 shows these vertices and the relevant edges for $k = J_3 - 1, J_3, J_3 + 1$.

If $\deg_{L^{J_3-1}(G)}(x) = \deg_{L^{J_3-1}(G)}(y)$, then $N_{L^{J_3+1}(G)}((uw)(vw))$ already contains four vertices of the same degree. Otherwise, since there does not exist an integer Q such that C_k contains an edge for all $k \geq Q$, we can apply Lemma 20′, and so there exists an integer $T > J_3 + 1$ where $(xw)(uw)$ and $(xw)(vw)$ generate two vertices in C_T. Following the proof of Lemma 20′, there are two vertices of the same degree in $N_{L^T(G)}((xw)(uw))$ generated by $(yw)(uw)$ and $(yw)(vw)$, and similarly in $N_{L^T(G)}((xw)(vw))$. Thus, in $N_{L^{T+1}(G)}(C_{T+1})$ there are actually four vertices with the same degree. □

A similar approach would be to construct a path from C_k to a maximum degree component that always contains an edge. Such a component exists by Theorem 25 of [2]. Three vertices of the same degree can then be found "dangling" off the end of the path. Applying the same shrinking path technique of Lemma 22′ would then prove Lemma 23′; however, this method relies on the proof of the maximum degree conjecture given in [2], and so cannot be extended to prove that conjecture. As given, the first part of Lemma 22′ is, in fact, Lemma 22. The proof of Lemma 23′ can be extended to a dual proof for the maximum degree case, thus providing an alternate proof of the original Lemma 23.

Lemma 24′. *There exists an integer J_4 such that C_k contains an edge for all $k \geq J_4$.*

Proof. If there does not exist such an integer J_4, then a contradiction results by applying Lemma 20′ to the three vertices in $N_{L^{J_3}(G)}(C_{J_3})$ obtained by Lemma 23′. □

Theorem 25′. *For all connected graphs G that are not paths, there exists an integer K such that the mDGP will hold for $L^k(G)$ for all $k \geq K$.*

Proof. By Lemma 24′ and since $\mathfrak{Lm}_k$ has a finite number of components, there exists a K such that every component of $\mathfrak{Lm}_k$ contains an edge for all $k \geq K$. By Lemma 14′, m_k is a subset of the components in $\mathfrak{Lm}_k$. Thus, m_k contains an edge for all $k \geq K$, and, by Lemma 5′, the mDGP will hold for all $k \geq K$. □

3 Open Questions

As noted in the introduction, line graphs are a well-studied area in graph theory, but few properties of iterated line graphs have been investigated. In [5], Niepel, Knor, and Šoltés presented three conjectures about iterated line graphs. Buckley and Ojeda used a construction in [1] to address the subject of the first two conjectures, and in [2], Hartke and Higgins proved the maximum degree case of the third conjecture. In this paper, we prove the minimum degree case. Our investigations into the iterated line graph when proving these conjectures have suggested the following open questions.

The primary concepts used in the proof of the growth rates of the maximum and minimum degrees are the local maximum induced subgraph and the local minimum induced subgraph. These subgraphs are useful because they are local structures that persist under the iterated line graph operator. Indeed, any subgraph induced by vertices of the same degree that is not a path is persistent. The question of how prevalent such persistent "regular" subgraphs are naturally arises: What proportion of vertices in $L^k(G)$ are in such subgraphs as $k \to \infty$? Does there exist an integer K such that every vertex in $L^K(G)$ is generated by a vertex in $L^{K-1}(G)$ in such a subgraph?

Theorem 25′ proves the existence of an integer K such that the mDGP holds for all $k \geq K$. However, the proof provides no useful method of determining the least integer K such that the mDGP holds for all iterations past K. The calculation of this tight bound for a given graph remains an open question. Similarly, the calculation of the least integer K such that the Maximum Degree Growth Property holds for all $k \geq K$ is likewise open.

From the results of this paper and [2], we know that there exists an integer K' where

$$2^{k-K'}(\delta_{K'} - 2) + 2 = \delta_k \leq \Delta_k = 2^{k-K'}(\Delta_{K'} - 2) + 2$$

for all $k \geq K'$. Thus, at each iteration, we know how big or small the degree of any given vertex can be. How are the vertices distributed among the degrees between the maximum and minimum degree? One formulation of this question can be obtained by defining the degree set $D_k(G)$ of $L^k(G)$ as $D_k(G) = \{\deg_{L^k(G)}(v) : v \in L^k(G)\}$, and letting $D_\infty(G) = \bigcup_{k=0}^{\infty} D_k$. For a given graph G, we can ask what the density in $\mathbb{N}$ of $D_\infty(G)$ is. Is the density of $D_\infty(G)$ always defined? Clearly, for a regular graph G, the density of $D_\infty(G)$ is zero. What are the conditions on G such that the density is 1? Other constructive questions about $D_\infty(G)$ are also interesting. For instance, let S be a set of positive integers. Does there exist a graph G with $\Delta(G) < \min S$ such that $S \cap D_\infty(G) = \emptyset$?

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