A Note on multicolor bipartite Ramsey numbers for $K_{2,n}$

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Abstract

In this note we prove that the bipartite Ramsey number for $K_{2,n}$ with q colors does not exceed $(n-1)q^2+q+1-\lceil\sqrt{q}\rceil$, improving the previous upper bound by $\lceil\sqrt{q}\rceil-2$.
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1 Introduction

The *q-colors bipartite Ramsey number* for $K_{m,n}$, denoted by $b_q(m,n)$, is the smallest integer b such that in any q-coloring of the edges of $K_{b,b}$ there is a monochromatic subgraph isomorphic to $K_{m,n}$. In other words, $b_q(m,n)$ is the minimum b such that every $b\times b$ matrix with entries in $\{0,\ldots,q-1\}$ always contains a submatrix $m\times n$ or $n\times m$ all of whose entries are i, where $i\in\{0,\ldots,q-1\}$.

The bipartite Ramsey numbers for two colors were introduced by Beineke and Schwenk [1]. Afterwards, several authors [8, 10, 11, 12] have considered distinct approaches to generate these numbers, either studying related problems, generalizations, or investigating connections to other combinatorial structures (Hadamard matrices, Steiner systems, etc). See [8] for an overview and [11] for recent results. In particular, some classes of optimal values were established for the case where $q=2$ and $1\leq m\leq 3$, according to [1, 4, 12].

However, our knowledge on exact values of $b_q(m,n)$ is rather poor when $q=2$ and $m\geq 4$ or when $q\geq 3$ and $m\geq 2$. Even the case where $q=3$ seems to be a difficult problem. Indeed, the topic is so short of construction that until now, the only exact value known for this range is $b_3(2,2)=11$, due to Exoo [6].

For $m=2$, the following upper bounds are known: $b_2(2,n)\leq 4n-3$ [1]; $b_q(2,2)\leq q^2+q-1$ [11], which slightly sharpens $b_q(2,2)\leq q^2+q+1$ [5]; and $b_q(2,n)\leq(n-1)q^2+q-1$ [3].

Here, we improve all these results by proving that:

Theorem 1 *For every q*

$$b_q(2,n)\leq(n-1)q^2+q+1-\lceil\sqrt{q}\,\rceil \tag{1}$$

As far as we know, equality holds in (1) for all known optimal values. In fact, the bound in (1) is tight in the following cases: (i) $q=2$ and if there exists

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a Hadamard matrix of order $2n-2$, odd n (see [1]); (ii) $q = 2$ and if there exists a strongly regular graph with parameters $(4n-3, 2n-2, n-2, n-1)$ (see [4]); (iii) $b_2(2,2) = 5$ (see [1]) and $b_3(2,2) = 11$ (see [6]). Until now, an example is not known that yields a better bound than that given in (1).

The paper is organized as follows. In section 2, we consider the connection between Zarankiewicz numbers [8] and the q-color bipartite Ramsey problem. This connection is essential in the proof of Theorem 1. In section 3, we prove the Theorem 1. The section 4 is regarded to our final comments

2 Preliminaries

Let i, j, a and b be positive integers such that $i \leq a$ and $j \leq b$. The *Zarankiewicz number* $Z_{i,j}(a,b)$ denotes the smallest integer z such that every $0-1$ matrix of order $a \times b$ containing z 0's must have a $i \times j$ submatrix whose all entries are 0.

Since 1951 these numbers have been investigated by many authors, as an example, [7, 8, 9, 12]. Surveys can be found in [2, 8].

We first recall an useful sufficient condition to obtain an upper bound on $Z_{i,j}(a,b)$ (see proof in [2, VI.2, Lemma 2.1] or in [8, section 12].

Lemma 2 *If*

$$x\binom{v+1}{j} + (a-x)\binom{v}{j} - (i-1)\binom{b}{j} > 0 \tag{2}$$

then $Z_{i,j}(a,b) \leq av + x$

The next result establishes a connection between the q-color bipartite Ramsey numbers and the Zarankiewicz numbers (see [8] or [12]).

Proposition 3 *If* $Z_{i,j}(a,a) \leq \lceil a^2/q \rceil$, *then* $b_q(i,j) \leq a$.

By applying Lemma 2 and Proposition 3, it is proved in [3] that $b_q(2,n) \leq (n-1)q^2 + q - 1$ [4]. In order to show that $b_q(2,n) \leq (n-1)q^2 + q + 1 - \lceil \sqrt{q} \rceil$, we need the following refinement of Proposition 3:

Proposition 4 *Let* $\lceil a^2/q \rceil = au + b$, *where* $b \leq a$. *If* $Z_{i,j}(a,b) \leq b(u+1)$, *then* $b_q(i,j) \leq a$.

Proof. Given a q-coloring of an order a square matrix M, there exists a color, say color 0, that appears in at least $\lceil a^2/q \rceil = au + b$ entries of M.

Now, we prove that there is an $i \times j$ submatrix of M, all of whose entries are 0. Since the number of 0's in M is at least $au+b$, it follows from the pigeonhole principle that there is an $a \times b$ submatrix M' of M with at least $b(u+1)$ 0's. Since $Z_{i,j}(a,b) \leq b(u+1)$, then there exists an $i \times j$ submatrix of M' whose entries are all 0, which completes the proof. ■

3 A Better Upper Bound on $b_q(2, n)$

In this section, we consider $i = n$ and $j = m = 2$. Moreover, let k be a positive integer such that $k^2 < q$. We set $a = (n-1)q^2 + q - k$ and $b = (n-1)q^2 + (1-(n-1)k)q + (1-2k)$. Our aim is to prove that

$$b_q(2, n) \leq a.$$

Before proving it, we need an additional proposition.

Proposition 5

$$Z_{n,2}(a,\ b) \leq b((n-1)q + 1)$$

Proof: We use Lemma 2. Observe that $b((n-1)q+1) = av + x$, when $v = (n-1)(q-k)$ and $x = (n-1)q^2 + (n-(n-1)k)q + (1-2k-(n-1)k^2)$. Observe also that x is positive, since $k^2 < q$. Now, let d be left hand side of (2) for v and x fixed above. By performing some simple, but rather tedious algebraic manipulation, we obtain that

$$2d = (n-1)(q-k^2)\{(n-1)(q-k)+1\} \tag{3}$$

In order to establish the result, it remains to prove that $2d > 0$. Nevertheless, this is true since $q > k^2$. ■

Now, we deduce Theorem 1.

Proof of Theorem 1. Since $1 \leq k^2 < q$, then

$$\lceil a^2/q \rceil = (n-1)^2q^3 + 2(n-1)q^2 + (1-2(n-1)k)q - 2k + 1 = a(n-1)q + b.$$

By setting $u = (n-1)q$ in Proposition 4, we can conclude that $b_q(2, n) \leq a$ under the condition that $Z_{n,2}(a, b) \leq b((n-1)q+1)$. Nevertheless, this condition is assured by Proposition 5. In particular, for $k = \lceil \sqrt{q} \rceil - 1$, we obtain the desired bound.■

An improvement, if possible, on (1) is related to design theory. Let us illustrate this claim focusing on the boundary case $k^2 = q$.

Corollary 6 *$b_{k^2}(2, n) \leq (n-1)k^4 + k^2 - k$ provided there is no system $S_{n-1}(b, \{v, v+1\}, a)$, i.e, a collection of a blocks of b-set, each block with size v or $v+1$, such that every 2-tuples of b-set is contained in exactly $n-1$ blocks.*

Proof: By examining the proofs of Theorem 1 and Proposition 5 one sees that $d = 0$ for the boundary $k^2 = q$. A simple analysis of Lemma 2 shows that $Z_{n,2}(a, b) > av + x$ if and only if there is a system $S_{n-1}(b, \{v, v+1\}, a)$. ■

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