Steiner Triple Systems with Automorphism Type [1,0,0,0,0,0,0,t,0,...,0,1,0,...,0]

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Abstract. Necessary and sufficient conditions are given for a Steiner triple system of order v admitting an automorphism consisting of one large cycle, t cycles of length 8, and a fixed point, with $t \leq 4$. Necessary conditions are given for all $t \geq 1$.

1. Introduction

A *Steiner triple system of order v*, denoted STS(v), is a pair (X, β), where X is a set of cardinality v, and β is a collection of 3-subsets of X, called *blocks*, such that any 2-subset of X is contained in a unique block. The notation (x, y, z) will be used for the block containing the subsets $\{x, y\}$, $\{y, z\}$, and $\{x, z\}$. It is well known that a STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6). If (X,β) is a Steiner triple system of size v, then a *subsystem* of (X,β) is an ordered pair (X', β') such that $X' \subseteq X$ and $\beta' \subseteq \beta$ and (X',β') is a STS(v'), where v' is the cardinality of X'.

An *automorphism* of a STS(v) is a permutation π of X that preserves the blocks in β. A Steiner triple system of order v admitting the automorphism π will be denoted $STS_\pi(v)$. A permutation of a v-element set is said to be of type [π] =

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$[\pi_1, \pi_2, \ldots, \pi_v]$ if the disjoint cyclic decomposition of has π_i cycles of length i. So $\sum i \cdot \pi_i = v$. The *orbit* of a block under an automorphism π is the image of the block under the powers of π. A set B of blocks is a set of *base blocks* or *starter blocks* for a STS(v) under the permutation π if the orbits of the blocks of B produce the STS(v) and exactly one block of B occurs in each orbit.

A question of concern has been that of given a particular automorphism type, does there exist an STS(v). Several types of automorphisms have been explored in connection with the question. Extensive results exist for Steiner triple systems. For a survey of these results, see [3]. Of particular interest to this paper is the *k-rotational* Steiner triple system, which admits an automorphism of type $[\pi] = [1, 0, \ldots, 0, k, 0, \ldots, 0]$. The permutation consists of a single fixed point and precisely k cycles of length $(v-1)/k$. The spectrum of a k-rotational Steiner triple system of order v has been completely determined [1,2,5].

The purpose of this paper is to address the problem of existence for Steiner triple systems admitting an automorphism with a disjoint decomposition consisting of one fixed point, t cycles of length 8, and one large cycle of length $d = v - 8t - 1$. Thus the automorphism is of type $[\pi] = [1, 0, 0, 0, 0, 0, 0, t, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_1 = 1$, $\pi_8 = t$, and $\pi_d = 1$. If $t = 0$, then the system is a 1-rotational Steiner triple system. Thus, we assume $t \geq 1$. Since d is the size of the larger cycle, we have $d \geq 9$.

2. STS(v) with $\pi = (\infty)(0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)(0, 1, 2, \ldots, d-1)$

Let π be an automorphism having one fixed point, one 8-cycle, and one cycle of length d. So $v = d + 9$, and π is of type $[\pi] = [1, 0, 0, 0, 0, 0, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_1 = 1$, $\pi_8 = 1$, and $\pi_d = 1$. The STS(v) can be defined on the v-set $X = Z_d \cup X$ where $X = \{\infty, 0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1\}$. The permutation of X can be defined as follows:

if $a = \infty$ then $\pi(a) = a$
if $a \in X \backslash \{\infty\}$ then $\pi(a_1) = [a + 1 \pmod 8]_1$
if $a \in \{0, 1, 2, \ldots, d-1\}$ then $\pi(a) = a + 1 \pmod d$.

Hence the disjoint cyclic decomposition of π on the set X can be represented by $\pi = (\infty)(0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)(0, 1, \ldots, d-1)$.

Lemma 2.1. *If (X,β) is a $STS_\pi(v)$, where $\pi = (\infty)(0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)(0, 1, 2, \ldots, d-1)$ and $d = v - 9$ is the length of the large cycle, then $8 \mid d$.*

Proof. By the definition of Steiner triple system, the set $\{x_1, 0\}$ must be contained in some block. It can be shown that there are no base blocks of the form (a, b_1, c_1); that is, with one element from the large cycle and two elements from the short cycle. Suppose that such a block exists. Then applying π to this block 8 times produces the block $(a + 8 \pmod d, b_1, c_1)$. We have stated that $d \geq 9$. Thus, $a + 8 \pmod d \neq a$, and the resulting block is different from (a, b_1, c_1), contradicting the fact that b_1 and c_1 must appear together in exactly one block. Therefore there can be no blocks of the form (a, b_1, c_1). Similarly, there can be no blocks of the form (∞, a, b_1). Thus, the block containing $\{x_1, 0\}$ must be of the form $(x_1, 0, y)$, with $y \in \{1, 2, 3, \ldots, d-1\}$. Applying the automorphism π to the block $(x_1, 0, y)$ d times yields the block $(\pi^d(x_1), \pi^d(0), \pi^d(y)) = ([x + d \pmod 8]_1, 0, y)$. Again using the definition of STS, note that the set $\{0, y\}$ can only appear as a subset of exactly one block. Therefore, $(x_1, 0, y) = ([x + d \pmod 8]_1, 0, y)$, and thus $x \equiv x + d \pmod 8$, since x_1 was chosen from the cycle of length 8. Then $8 \mid (x_1 + d) - x_1$; in other words, $8 \mid d$. ■

It is known that a Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$. Combining this with the above lemma gives $v \equiv 1$ or $9 \pmod{24}$. It can be shown that if $v = 1$ or 9 then the STS(v) is non-existent or trivial.

Lemma 2.2. *If a Steiner triple system of order v admitting the automorphism π exists, then $v \equiv 1$ or $9 \pmod{24}$.*

The following constructions on the set $X = \{\infty, 0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1\} \cup Z_d$ demonstrate that the necessary conditions of Lemma 2.2 are also sufficient.

Lemma 2.3. *A STS(v) on the set X admitting the automorphism π exists if $v \equiv 1 \pmod{24}$.*

Proof. Let $v = 24s + 1$. With slight modifications to the difference partitions described by Skolem, Peltesohn, and Rosa, we can ensure that every possible difference between elements is represented in exactly one base block [4,6,7,8]. It can be shown that for $s = 0$, the solution is non-existent. For all $s \geq 1$, the base blocks include $(\infty, 0, 12s - 4)$, $(\infty, 0_1, 4_1)$, $(0_1, 1_1, 3_1)$, and additional blocks as follows:

Case 1: $s \equiv 0 \pmod 4$ and $s \geq 4$.

$(0, 1, 10s - 4)$;
$(0, 12s - 6, 6_1)$; $(0, 12s - 5, 3_1)$; $(0, 2s + 1, 2_1)$; $(0, 8s - 6, 7_1)$;
$(0, 4 + 8m, 6s + 4m)$, $m = 0, 1, \ldots, (s/2) - 1$;
$(0, 8 + 8m, 10s + 4m)$, $m = 0, 1, \ldots, (s/2) - 2$;
$(0, 1 + 2m, 12s - 5 - 2m)$, $m = 1, 2, \ldots, 3s - 2$, $m \neq s$.

Case 2: $s \equiv 1 \pmod 4$.
$(0, 1, 10s - 5)$;
$(0, 12s - 6, 1_1)$; $(0, 12s - 5, 4_1)$; $(0, 2s + 1, 2_1)$; $(0, 8s - 6, 0_1)$;
$(0, 4 + 8m, 6s - 2 + 4m)$, $m = 0, 1, \ldots, [(s - 1)/2] - 1$;
$(0, 8 + 8m, 10s - 2 + 4m)$, $m = 0, 1, \ldots, [(s - 1)/2] - 1$;
$(0, 1 + 2m, 12s - 5 - 2m)$, $m = 1, 2, \ldots, 3s - 2$, $m \neq s$.

Case 3: $s \equiv 2 \pmod 4$.
$(0, 1, 10s - 4)$;
$(0, 12s - 6, 7_1)$; $(0, 12s - 5, 3_1)$; $(0, 2s + 1, 6_1)$; $(0, 8s - 6, 4_1)$;
$(0, 4 + 8m, 6s + 4m)$, $m = 0, 1, \ldots, (s/2) - 1$;
$(0, 8 + 8m, 10s + 4m)$, $m = 0, 1, \ldots, (s/2) - 2$;
$(0, 1 + 2m, 12s - 5 - 2m)$, $m = 1, 2, \ldots, 3s - 2$, $m \neq s$.

Case 4: $s \equiv 3 \pmod 4$.
$(0, 1, 10s - 5)$;
$(0, 12s - 6, 5_1)$; $(0, 12s - 5, 3_1)$; $(0, 2s - 1, 1_1)$; $(0, 8s - 6, 0_1)$;
$(0, 4 + 8m, 6s - 2 + 4m)$, $m = 0, 1, \ldots, [(s - 1)/2] - 1$;
$(0, 8 + 8m, 10s - 2 + 4m)$, $m = 0, 1, \ldots, [(s - 1)/2] - 1$;
$(0, 1 + 2m, 12s - 5 - 2m)$, $m = 1, 2, \ldots, 3s - 2$, $m \neq s$. ■

Lemma 2.4. *A STS(v) on the set X admitting the automorphism π exists if $v \equiv 9 \pmod{24}$ and $v \neq 9$.*

Proof. Let $v \equiv 9 \pmod{24}$, say $v = 24s + 9$. It can be shown that for $s = 0$, no STS(v) exists admitting the automorphism π. So let $s \geq 1$. Using similar techniques to those employed in Lemma 2.3, we find that the base blocks $(\infty, 0, 12s)$, $(0, 8s, 16s)$, $(\infty, 0_1, 4_1)$, $(0_1, 1_1, 3_1)$, and additional blocks as follows:

Case 1: $s \equiv 0 \pmod 8$ and $s \geq 8$.
$(0, 2s - 2, 12s - 2)$;
$(0, 1, 1_1)$; $(0, 5s + 1, 3_1)$; $(0, 7s - 1, 4_1)$; $(0, 12s - 1, 6_1)$;
$(0, 1 + 2m, 12s - 1 - 2m)$, $m = 1, 2, \ldots, 3s - 1$, $m \neq 5s/2$;
$(0, 4 + 8m, 6s + 4m)$, $m = 0, 1, \ldots, (s/2) - 1$;
$(0, 8 + 8m, 10s + 4 + 4m)$, $m = 0, 1, \ldots, (s/2) - 2$.

Case 2: $s \equiv 2 \pmod 8$.
$(0, 2s - 2, 12s - 2)$;
$(0, 1, 4_1)$; $(0, 5s + 1, 5_1)$; $(0, 7s - 1, 6_1)$; $(0, 12s - 1, 7_1)$;
$(0, 1 + 2m, 12s - 1 - 2m)$, $m = 1, 2, \ldots, 3s - 1$, $m \neq 5s/2$;
$(0, 4 + 8m, 6s + 4m)$, $m = 0, 1, \ldots, (s/2) - 1$;
$(0, 8 + 8m, 10s + 4 + 4m)$, $m = 0, 1, \ldots, (s/2) - 2$.

Case 3: $s \equiv 4 \pmod 8$.
$(0, 2s - 2, 12s - 2)$;
$(0, 1, 4_1)$; $(0, 5s + 1, 6_1)$; $(0, 7s - 1, 5_1)$; $(0, 12s - 1, 7_1)$;
$(0, 1 + 2m, 12s - 1 - 2m)$, $m = 1, 2, \ldots, 3s - 1$, $m \neq 5s/2$;
$(0, 4 + 8m, 6s + 4m)$, $m = 0, 1, \ldots, (s/2) - 1$;
$(0, 8 + 8m, 10s + 4 + 4m)$, $m = 0, 1, \ldots, (s/2) - 2$.

Case 4: $s \equiv 6 \pmod 8$.
$(0, 2s - 2, 12s - 2)$;
$(0, 1, 1_1)$; $(0, 5s + 1, 4_1)$; $(0, 7s - 1, 3_1)$; $(0, 12s - 1, 6_1)$;
$(0, 1 + 2m, 12s - 1 - 2m)$, $m = 1, 2, \ldots, 3s - 1$, $m \neq 5s/2$;
$(0, 4 + 8m, 6s + 4m)$, $m = 0, 1, \ldots, (s/2) - 1$;
$(0, 8 + 8m, 10s + 4 + 4m)$, $m = 0, 1, \ldots, (s/2) - 2$.

Case 5: $s \equiv 1$ or $5 \pmod 8$.
$(0, 1, 6s - 1)$;
$(0, 12s - 1, 5_1)$; $(0, 12s - 2, 3_1)$; $(0, 2, 6_1)$; $(0, 6s + 1, 7_1)$;
$(0, 1 + 2m, 12s - 1 - 2m)$, $m = 1, 2, \ldots, 3s - 2$;
$(0, 4 + 8m, 10s + 2 + 4m)$, $m = 0, 1, \ldots, (s - 1)/2 - 1$;
$(0, 8m, 6s - 2 + 4m)$, $m = 1, 2, \ldots, (s - 1)/2$.

Case 6: $s \equiv 3$ or $7 \pmod 8$.
$(0, 1, 6s - 1)$;
$(0, 12s - 1, 4_1)$; $(0, 12s - 2, 2_1)$; $(0, 2, 7_1)$; $(0, 6s + 1, 6_1)$;
$(0, 1 + 2m, 12s - 1 - 2m)$, $m = 1, 2, \ldots, 3s - 2$;
$(0, 4 + 8m, 10s + 2 + 4m)$, $m = 0, 1, \ldots, (s - 1)/2 - 1$;
$(0, 8m, 6s - 2 + 4m)$, $m = 1, 2, \ldots, (s - 1)/2$. ■

In each case, the collection of blocks form a set of base blocks for a STS(v). Combining Lemmas 2.2, 2.3 and 2.4, we get the following result:

Theorem 2.5. *A Steiner triple system of order v with $v = d + 9$ admitting an automorphism whose disjoint decomposition is a fixed point, a cycle of length 8, and a cycle of length d exists if and only if $v \equiv 1$ or $9 \pmod{24}$ and $v \neq 9$.*

3. STS(v) with $(\pi) = (\infty)\,(0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)\,(0_2, 1_2, 2_2, 3_2, 4_2, 5_2, 6_2, 7_2)\,(0, 1, 2, \ldots, d - 1)$

Theorem 3.1. *There can be no Steiner triple system STS(v) that admits an automorphism of the type $[\pi] = [1, 0, 0, 0, 0, 0, 0, t, 0, \ldots, 0, 1, 0, \ldots, 0]$ with $t \equiv 2 \pmod 3$.*

Proof. Let $v = d + 8t + 1$. Such a STS has a subsystem of size $8t + 1$. Since a subsystem is also a STS, $8t + 1 \equiv 1$ or $3 \pmod 6$. Thus, $t \equiv 0$ or $1 \pmod 3$. ■

4. STS(v) with $\pi = (\infty)(0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)(0_2, 1_2, 2_2, 3_2, 4_2, 5_2, 6_2, 7_2)(0_3, 1_3, 2_3, 3_3, 4_3, 5_3, 6_3, 7_3)(0, 1, \ldots, d - 1)$

Let π be the automorphism defined as follows: $\pi = (\infty)(0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)(0_2, 1_2, 2_2, 3_2, 4_2, 5_2, 6_2, 7_2)(0_3, 1_3, 2_3, 3_3, 4_3, 5_3, 6_3, 7_3)(0, 1, \ldots, d - 1)$ with $v = d + 25$.

As in the proof of Lemma 2.1, it can be shown that the large cycle must be divisible by 8. Thus we have:

Lemma 4.1. *If (X, β) is a STS(v), where $\pi = (\infty)(0_1, 1_1, \ldots, 7_1)(0_2, 1_2, \ldots, 7_2)(0_3, 1_3, \ldots, 7_3)(0, 1, \ldots, d - 1)$ and $d = v - 25$ is the length of the large cycle, then $8 | d$.*

Combining this lemma with the fact that a Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$ gives the following necessary condition (analogous to Lemma 2.2).

Lemma 4.2. *If a Steiner triple system of order v admitting the automorphism π exists, then $v \equiv 1$ or $9 \pmod{24}$.*

Suppose, however that $t \equiv 0 \pmod 3$, say $t = 3k$, and $v \equiv 1 \pmod{24}$, say $24s + 1$. Then $d = 24s - 24k$ and the set of all differences between pairs of elements in the larger cycle can be characterized as $D = \{1, 2, 3, \ldots, 12(s - k)\}$. Each of these differences must occur in starter blocks. Each of the 8-cycles will require 4 of these differences and the $12(s - k)$ difference must be paired with the fixed point. The remaining $12(s - k) - 4(3k) - 1$ differences must occur in triples in base blocks consisting of elements from the larger cycle. Thus we must have that 3 divides $12s - 24k - 1$ or else 3 divides $12s - 24k - 2$. Neither of these being the case, it is not possible that $v \equiv 1 \pmod{24}$. We have then in conclusion the following theorem (a refinement of Lemma 4.2):

Theorem 4.3. *If a Steiner triple system of order $v = d + 8t + 1$ admitting the automorphism of type $[\pi] = [1, 0, 0, 0, 0, 0, 0, t, 0, \ldots, 0, 1, 0, \ldots, 0]$ exists with $t \equiv 0 \pmod 3$, then $v \equiv 9 \pmod{24}$.*

The following constructions on the set $X = \{\infty\} \cup Z_d \cup \{0_i, 1_i, 2_i, 3_i, 4_i, 5_i, 6_i, 7_i\}$, $i = 1,2,3$, demonstrate that the necessary conditions of Lemma 4.3 are also sufficient.

Lemma 4.4. *A STS(v) on the set X admitting the automorphism exists if $v \equiv 9 \pmod{24}$ and $v \neq 9$ and $v \neq 33$.*

Proof. Let $v = 24s + 9$. Then $d = v - 25 = 24s - 16$. It can be shown that for $s = 0$ or $s = 1$, no STS(v) exists admitting the automorphism . So $s \geq 2$. Then the base blocks are as follows:

$(\infty, 0, 12s - 8)$; $(\infty, 0_1, 4_1)$; $(\infty, 0_2, 4_2)$; $(\infty, 0_3, 4_3)$; $(0_1, 0_2, 0_3)$; $(0_1, 2_2, 6_3)$;

$(0_1, 1_1, 6_2)$; $(0_1, 2_1, 3_2)$; $(0_1, 3_1, 7_2)$; $(0_3, 1_3, 6_1)$; $(0_3, 2_3, 3_1)$; $(0_3, 3_3, 7_1)$;
$(0_2, 1_2, 7_3)$; $(0_2, 2_2, 3_3)$; $(0_2, 3_2, 5_3)$;

$(0, 4s - 2m - 8, 8s - m - 9)$, $m = 1, 2, \ldots, 2s - 5$;

$(0, 4s - 2m - 9, 12s - m - 13)$, $m = 1, 2, \ldots, 2s - 5$;.

$(0, 4s - 4, 12s - 10)$; $(0, 4s - 7, 12s - 12)$; $(0, 4s - 6, 8s - 8)$;
$(0, 8s - 9, 7_2)$; $(0, 4s - 9, 3_3)$; $(0, 8s - 5, 5_3)$.

If $s \equiv 0 \pmod 4$ and $s \geq 4$, include: $(0, 6s - 5, 1_1)$; $(0, 8s - 7, 4_1)$; $(0, 4s - 1, 7_1)$; $(0, 12s - 13, 5_1)$.

If $s \equiv 1 \pmod 4$ and $s \geq 5$, include: $(0, 6s - 5, 3_1)$; $(0, 8s - 7, 6_1)$; $(0, 4s - 1, 4_1)$; $(0, 12s - 13, 7_1)$.

If $s \equiv 2 \pmod 4$, include: $(0, 6s - 5, 7_1)$; $(0, 8s - 7, 3_1)$; $(0, 4s - 1, 5_1)$; $(0, 12s - 13, 4_1)$.

If $s \equiv 3 \pmod 4$, include: $(0, 6s - 5, 6_1)$; $(0, 8s - 7, 4_1)$; $(0, 4s - 1, 5_1)$; $(0, 12s - 13, 7_1)$.

If $s \equiv 0 \pmod 2$ and $s \geq 2$, include: $(0, 4s - 5, 5_2)$; $(0, 12s - 11, 6_2)$; $(0, 4s - 7, 4_2)$;
$(0, 4s - 3, 6_3)$; $(0, 12s - 9, 7_3)$.

If $s \equiv 1 \pmod 2$ and $s \geq 3$, include: $(0, 4s - 5, 4_2)$; $(0, 12s - 11, 3_2)$; $(0, 4s - 7, 6_2)$; $(0, 4s - 3, 7_3)$; $(0, 12s - 9, 4_3)$.

Combining Theorem 4.3 and Lemma 4.4, we get the following result:

Theorem 4.5. *A Steiner triple system of order v with $v = d + 25$ admitting an automorphism whose disjoint decomposition is a fixed point, three cycles of length 8, and a cycle of length d exists if and only $v \equiv 9 \pmod{24}$ and $v \neq 9$ and $v \neq 33$.*

5. STS(v) with $\pi = (\infty)(0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)(0_2, 1_2, 2_2, 3_2, 4_2, 5_2, 6_2, 7_2)$

$(0_3, 1_3, 2_3, 3_3, 4_3, 5_3, 6_3, 7_3)$ $(0_4, 1_4, 2_4, 3_4, 4_4, 5_4, 6_4, 7_4)$ $(0, 1, 2, \ldots, d - 1)$

Let π be the automorphism defined as follows: $\pi = (\infty)\ (0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)\ (0_2, 1_2, 2_2, 3_2, 4_2, 5_2, 6_2, 7_2)\ (0_3, 1_3, 2_3, 3_3, 4_3, 5_3, 6_3, 7_3)\ (0_4, 1_4, 2_4, 3_4, 4_4, 5_4, 6_4, 7_4)\ (0, 1, 2, \ldots, d - 1)$ with $v = d + 33$.

The lemmas which precede the proof for four 8-cycles are similar in nature to those of Section 2. We can again determine that $8 \mid d$, and thus the following necessary condition.

Lemma 5.1. *If a Steiner triple system of order v admitting the automorphism exists, then* $v \equiv 1$ *or 9 (mod 24).*

Let A be the following set of base blocks for a STS: $A = \{(\infty, 0_1, 4_1); (\infty, 0_2, 4_2), (\infty, 0_3, 4_3); (\infty, 0_4, 4_4); (0_1, 0_2, 1_4); (0_2, 0_3, 2_1); (0_3, 0_4, 2_2); (0_4, 1_1, 2_3); (0_1, 0_3, 3_3); (0_2, 0_4, 3_4); (0_3, 1_1, 4_1); (0_4, 1_2, 4_2); (0_1, 0_4, 2_4); (0_2, 1_1, 3_1); (0_3, 1_2, 3_2); (0_4, 1_3, 3_3); (0_1, 1_1, 3_2); (0_2, 1_2, 3_3); (0_3, 1_3, 3_4); (0_4, 1_4, 4_1); (0_1, 1_2, 3_4); (0_2, 1_3, 4_1); (0_3, 1_4, 4_2); (0_4, 2_1, 4_3)\}$. In fact, these are exactly the base blocks for a 4-rotational STS on 33 elements, which indicates an automorphism consisting of one fixed point and 4 cycles of length eight. These blocks are a subsystem of the desired STS(v).

The following constructions on the set $X = \{\infty\} \cup Z_d \cup \{0_i, 1_i, 2_i, 3_i, 4_i, 5_i, 6_i, 7_i\}$, i = 1, 2, 3, 4, demonstrate that the necessary conditions of Lemma 5.1 are also sufficient.

Lemma 5.2. *A STS(v) on the set X admitting the automorphism exists if* $v \equiv 1$ *(mod 24) and* $v \geq 97$.

Proof. Let $v = 24s + 1$. It can be shown that for $s = 0, 1, 2$, or 3, no STS(v) exists admitting the automorphism π. So $s \geq 4$. Then the base blocks include $(\infty, 0, 12s - 16)$, all of the base blocks in set A, and additional blocks as follows:

Case 1: s is even and $s \geq 6$.

$(0, 2s - 1, 4_1)$ and $(0, 12s - 17, 2_1)$ only when $s \equiv 2 \pmod 4$;

$(0, 2s - 1, 1_1)$ and $(0, 12s - 17, 3_1)$ only when $s \equiv 0 \pmod 4$;

$(0, 8s - 14, 7_1)$; $(0, 12s - 18, 6_1)$; $(0, 1, 10s - 15)$; $(0, 3, 5_2)$; $(0, 5, 6_2)$; $(0, 7, 7_2)$; $(0, 9, 4_2)$; $(0, 12s - 22, 0_3)$; $(0, 12s - 26,$

1_3); $(0, 12s - 30, 4_3)$; $(0, 12s - 34, 5_3)$; $(0,12s - 19,6_4)$; $(0,12s - 21, 5_4)$; $(0, 12s - 23, 4_4)$; $(0, 12s - 25, 7_4)$;

$(0, 4 + 8m, 6s - 8 + 4m)$, $m = 0, 1, \ldots, (s/2) - 2$;

$(0, 8 + 8m, 10s - 12 + 4m)$, $m = 0, 1, \ldots, (s/2) - 2$;

$(0, 2m + 11, 12s - 27 - 2m)$, $m = 0, 1, \ldots, 3s - 10$, $m \neq s - 6$.

Case 2: s is odd and $s \geq 7$.

$(0, 2s - 1, 5_1)$ and $(0, 12s - 17, 2_1)$ only when $s \equiv 3 \pmod 4$;

$(0, 2s - 1, 0_1)$ and $(0, 12s - 17, 5_1)$ only when $s \equiv 1 \pmod 4$;

$(0, 8s - 14, 3_1)$; $(0, 12s - 18, 6_1)$; $(0, 1, 10s - 14)$;

$(0, 3, 5_2)$; $(0, 5, 6_2)$; $(0, 7, 7_2)$; $(0, 9, 4_2)$,

$(0, 12s - 22, 6_3)$; $(0, 12s - 26, 4_3)$; $(0, 12s - 30, 7_3)$;

$(0, 12s - 34, 5_3)$; $(0, 12s - 19, 4_4)$; $(0, 12s - 21, 7_4)$;

$(0, 12s - 23, 6_4)$; $(0, 12s - 25,5_4)$;

$(0, 4 + 8m, 6s - 6 + 4m)$, $m = 0,1, \ldots, (s - 3)/2$;

$(0, 8 + 8m, 10s - 10 + 4m)$, $m = 0\ 1, \ldots, [(s - 3)/2] - 1$;

$(0, 2m + 11, 12s - 27 - 2m)$, $m = 0,1, \ldots, 3s - 10$, $m \neq s - 6$.

Case 3: $s = 4$.

$(0, 30, 6_1)$; $(0, 31, 1_1)$; $(0, 7, 3_1)$; $(0, 18, 7_1)$;

$(0, 3, 3_2)$; $(0, 5, 1_2)$; $(0, 9, 7_2)$; $(0, 11, 5_2)$;

$(0, 26, 2_3)$; $(0, 22, 1_3)$; $(0, 14, 4_3)$; $(0, 10, 7_3)$;

$(0, 29, 1_4)$; $(0, 27, 5_4)$; $(0, 23, 7_4)$; $(0, 21, 3_4)$;

$(0, 4, 16)$; $(0, 8, 28)$; $(0, 1, 25)$; $(0, 13, 19)$; $(0, 15, 17)$.

Case 4: $s = 5$.

$(0,9,2_1)$; $(0,26,6_1)$; $(0,42,7_1)$; $(0,43,3_1)$;

$(0, 3, 0_2)$; $(0, 5, 6_2)$; $(0, 7,3_2)$; $(0,11, 2_2)$;

$(0, 38, 0_3)$; $(0, 34, 3_3)$; $(0, 30,4_3)$; $(0,22,5_3)$;

$(0, 41, 3_4)$; $(0, 39, 7_4)$; $(0, 37, 6_4)$; $(0, 33, 5_4)$;

$(0, 4, 24)$; $(0, 12, 28)$; $(0, 8, 40)$; $(0, 1, 36)$; $(0, 13, 31)$;

$(0, 15, 29)$; $(0, 17, 27)$; $(0, 19, 25)$; $(0, 21, 23)$. ■

Lemma 5.3. *A STS(v) on the set X admitting the automorphism* π *exists if* $v \equiv 9 \pmod{24}$ *and* $v \geq 81$.

Proof. Let $v = 24s + 9$. It can be shown that for $s = 0$, 1, or 2, no STS(v) exists admitting the automorphism. So $s \geq 3$. Then the base blocks include $(\infty, 0, 12s - 12)$, $(0, 8s - 8, 16s - 16)$, all of the base blocks in set A, and additional blocks as follows:

Case 1: $s \equiv 0 \pmod 4$ and $s \geq 4$.

$(0, 12s - 13, 4_1)$; $(0, 12s - 14, 2_1)$; $(0, 2, 7_1)$; $(0, 6s - 5, 6_1)$;
$(0, 6s - 11, 5_2)$; $(0,3, 4_2)$; $(0, 6s - 13, 6_2)$; $(0, 5, 7_2)$;
$(0, 10, 2_3)$; $(0, 12s - 18, 4_3)$; $(0, 14, 5_3)$; $(0, 12s - 22, 3_3)$
$(0, 6s - 1, 4_4)$; $(0, 12s - 15, 1_4)$; $(0, 6s + 1, 3_4)$; $(0, 12s - 17, 6_4)$; $(0, 6s - 9, 6s - 3)$; $(0, 1, 6s - 7)$;
$(0, 2m + 7, 12s - 19 - 2m), m = 0, 1, \ldots, 3s - 11$;
$(0, 4 + 8m, 10s - 8 + 4m), m = 0, 1, \ldots, s/2 - 2$;
$(0, 8m, 6s - 8 + 4m), m = 1, 2, \ldots, s/2 - 1$.

Case 2: $s \equiv 1$ or $3 \pmod 4$ and $s \geq 3$.
$(0, 5s - 4, 3_1)$ and $(0, 7s - 8, 6_1)$ only when $s \equiv 1 \pmod 8$;
$(0, 5s - 4, 6_1)$ and $(0, 7s - 8, 7_1)$ only when $s \equiv 3 \pmod 8$;
$(0, 5s - 4, 7_1)$ and $(0, 7s - 8, 6_1)$ only when $s \equiv 5 \pmod 8$;
$(0, 5s - 4, 6_1)$ and $(0, 7s - 8, 3_1)$ only when $s \equiv 7 \pmod 8$;
$(0, 1, 1_1)$; $(0, 12s - 13, 4_1)$; $(0, 2s - 4, 12s - 14)$; $(0, 3, 5_2)$;
$(0, 4s - 3, 4_2)$; $(0, 5, 6_2)$; $(0,4s - 5, 7_2)$; $(0, 12s - 18, 2_3)$;
$(0, 4s - 6, 4_3)$; $(0, 12s - 22, 5_3)$; $(0, 4s - 2, 3_3)$; $(0, 12s - 15, 6_4)$;
$(0, 8s - 9, 7_4)$; $(0, 12s - 17, 5_4)$; $(0, 8s - 7, 4_4)$;
$(0, 2m + 7, 12s - 9 - 2m), m = 0, 1, \ldots, 3s - 7, m \neq (5s - 5)/2s - 3, 2s - 5, 2s - 6$;
$(0, 4 + 8m, 6s - 6 + 4m), m = 0, 1, \ldots, [(s-1)/2] - 1$;
$(0, 8 + 8m, 10s - 6 + 4m), m = 0, 1, \ldots, [(s-1)/2] - 2$.

Case 3: $s \equiv 2 \pmod 4$ and $s \geq 6$.
$(0, 12s - 13, 3_1)$; $(0, 12s - 14, 6_1)$; $(0, 2, 7_1)$; $(0, 6s - 5, 1_1)$;
$(0, 6s - 11, 4_2)$; $(0, 3, 5_2)$; $(0, 6s - 13, 7_2)$; $(0, 5, 6_2)$;
$(0, 10, 2_3)$; $(0, 12s - 18, 4_3)$; $(0, 14, 5_3)$; $(0, 12s - 22, 3_3)$;
$(0, 6s - 1, 5_4)$; $(0, 12s - 15, 4_4)$; $(0, 6s + 1, 64)$; $(0, 12s - 17, 7_4)$; $(0, 6s - 9, 6s - 3)$; $(0, 1, 6s - 7)$;
$(0, 2m + 7, 12s - 18 - 2m), m = 0, 1, \ldots, 3s - 11$;
$(0,4 + 8m, 10s - 8 + 4m), m = 0, 1, \ldots, s/2 - 2$;
$(0, 8m, 6s - 8 + 4m), m =1,2, \ldots, s/2 - 1$. ■

Combining Lemmas 5.1,5.2, and 5.3, we get the following result:

Theorem 5.4. *A Steiner triple system of order v with $v = d+33$ admitting an automorphism whose disjoint decomposition is a fixed point, four cycles of length 8, and a cycle of length d exists if and only $v \equiv 1$ or $9 \pmod{24}$ and $v \geq 81$.*

6. Concluding Remarks

In closing, we state previous results, that being the case of t=0, and summarize the new results. For $t \leq 4$, a Steiner triple system of size v admitting π as an automorphism of type $[\pi] = [1, 0, 0, 0, 0, 0, 0, t, 0, \ldots, 0, 1, 0, \ldots, 0]$ exists if and only if $v \equiv 3$ or 9 (mod 24) and $t = 0$, $v \equiv 1$ or 9 (mod 24) and $t = 1$ or 4, or $v \equiv 9$ (mod 24) and $t = 3$. If $t = 2$ then no such Steiner triple system exists.

In generalizing, for all t let $t \geq 1$ and $v = d + 8t + 1$. As was shown in Theorem 3.1, $t \equiv 0$ or 1 (mod 3). By Theorem 4.3, if $t \equiv 0$ (mod 3) then $v \equiv 9$ (mod 24). We have in conclusion the following:

Theorem 6.1. *If a Steiner triple system of order $v = d+8+1$ admitting an automorphism whose disjoint cyclic decomposition consists of a fixed point, t cycles of length 8, and one cycle of length d exists, then $v \equiv 1$ (mod 24) and $t \equiv 1$ (mod 3) or $v \equiv 9$ (mod 24) and $t \equiv 0$ or 1(mod 3).*

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