ON SOME PARAMETERS RELATED TO UNIQUELY VERTEX–COLOURABLE GRAPHS AND DEFINING SETS

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Abstract

We consider two possible methods of embedding a (simple undirected) graph into a uniquely vertex colourable graph. The first method considered is to build a k–chromatic uniquely vertex colourable graph from a k–chromatic graph G on $G \cup K_k$ by adding a set of new edges between the two components. This gives rise to a new parameter called *fixing number* (Daneshgar (1997)). Our main result in this direction is to prove that a graph is uniquely vertex colourable if and only if its fixing number is equal to zero (which is a counterpart to the same kind of result for defining numbers proved by Hajiabolhassan *et.al.* (1996)).
In our second approach, we try a more subtle method of embedding which gives rise to the parameters t_r*-index* and τ_r*-index* ($r = 0, 1$) for graphs. In this approach we show the existence of certain classes of u–cores, for which, the existence of an extremal graph provides a counter example for Xu's conjecture.

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1 Basic Goals and Preliminaries

In this section, first, we try to introduce the main background, objectives and results of this paper. Also, after that we go through some basic definitions and known results which will be used in the rest of the paper. The organization of the paper is explained in what follows.

1.1 Basic Goals

Uniquely colourable graphs (or UCG's for short) are interesting for their many different connections to important graph colouring problems (e.g. see [8] and its list of references). On the other hand, since it is known that the characterization of chromatic structures is essentially not a well–defined problem, one may try to introduce a hierarchy of these structures and then try to analysis the most simple ones. One approach to this is introduced in [11] (also see [8]), where the *core* of a UCG is defined and minimal cores are studied.

Study of cores is important in the study of colouring structures, and specially, the construction of infinite family of them with prescribed parameters is an interesting and important problem in the study of extremal UCG's. The main theme of this paper is to try to obtain an extension of this scenario to general graphs, and filling the relationship in between, by introducing appropriate extensions of graphs to uniquely colourable ones.

In Section 2 we consider one of the most simple forms of this embedding process by means of complete graphs. This will introduce a new parameter for graphs which is called the fixing number[1] [5]. We investigate some properties of this parameter and our main result in this section is that a graph is a UCG if and only if its fixing number is equal to zero.

Also, in Section 3 we consider a more subtle method of embedding which introduces the new parameters t_r–index and τ_r–index ($r = 0, 1$) for graphs. Our main result in this regard is to prove that under certain conditions on these parameters, some specific classes of cores exist.

1.2 Some Basic Definitions and Concepts

In the sequel, $\mathrm{N} = \{1, 2, \ldots\}$ is the set of *natural numbers*. For any finite set X, $|X|$ is the *size* of X, i.e. the number of elements of X, and $P(X)$ is the *power set* of X, i.e. the set of all subsets of X. We consider finite

[1] At the time of writing this paper the first author was notified that T. Morrill and D. Pritikin had also come to the definition of the fixing number through list–colouring concepts (what they call the *list–defining number* is a generalization of fixing number for an arbitrary number of colours). Also, part (b) of Proposition 1 is stated by T. Morrill and D. Pritikin as a theorem with a proof based on alternating paths [16].

simple undirected graphs such as $G = (V(G), E(G))$ with the *vertex set* $V(G)$ and the *edge set* $E(G)$. $|V(G)|$ and $|E(G)|$ represent the *order* and the *size* of the graph G, respectively. In what follows we go through some definitions and results which will be used in the sequel, while we refer to [2, 18] for the basic concepts and backgrounds in graph theory.

K_k is the complete graph on k vertices and a subgraph of a graph G which is isomorphic to K_k is called a *k-clique* of G. The *clique number* of G, $cl(G)$, is the maximum number k such that G contains a k-clique.

The *cartesian product* of two graphs G and H is a graph, $K = G\Box H$, with $V(K) = V(G) \times V(H)$ in which two vertices (u_1, v_1) and (u_2, v_2) are adjacent in K if and only if u_1 is adjacent to u_2 in G and $v_1 = v_2$ or v_1 is adjacent to v_2 in H and $u_1 = u_2$.

Consider a graph G and a collection of nonvoid subsets of $P(V(G))$ such as $\mathcal{F} = (W_1, \ldots, W_l)$ with $W_i \in P(V(G))$ for all $1 \leq i \leq l$. Note that in this setting, it is possible to have a subset of $P(V(G))$ appears more than once in the collection. Also, there are situations throughout this paper that such collections appear naturally as domains of some maps. Therefore, we make this a rule to consider a collection $\mathcal{F} = (W_1, \ldots, W_l)$ as a set of ordered pairs as (i, W_i) in which the first component is used as a counter. In this paper, this is called a *list* $\mathcal{F} \subseteq \mathrm{N} \times P(V(G))$ of subsets of $P(V(G))$ [10].

A *proper k-colouring* (or simply a k-colouring for short) of a graph G is an assignment of *colours* from a set of colours, namely $\{1, 2, \ldots, k\}$, to the vertices of G such that adjacent vertices take different colours; and a graph which admits a k-colouring is said to be a *k-colourable* graph. Note that any k-colouring of G induces a k-partition on the vertex set of G, $V(G)$, such that there is no adjacent vertices in any class of this partition. These classes are called *colour-classes* of G (with respect to this k-colouring) and $[i]$ denotes the colour-class of colour i. $\chi(G)$, the *chromatic number* of G, is the minimum integer k such that G admits a k-colouring, and a *k-chromatic* graph G is a graph with $\chi(G) = k$. Also, $ccl(G) = \chi(G) - cl(G)$ will be called the *coclique number* of G.

List colouring is one of the generalizations of the above concept in which each vertex has its own list of legal colours. In this regard, a list colouring problem for a graph G with lists $\mathcal{L} = \{L_v\}_{v \in V(G)}$ is to assign a colour i_v to each vertex v of G such that $i_v \in L_v$ and adjacent vertices take different colours.

While finding a necessary and sufficient condition for a list colouring problem is quite hard in general, for the complete graph K_k the problem reduces to the well known Marriage Theorem of P. Hall. Also, as a consequence of this theorem we have the following theorem of M. Hall which has already been used in the study of unique colourability and defining sets of graphs [14, 15].

Theorem A. [13] *If n sets $A_1, \ldots, A_n$ have a system of distinct representative* (SDR) *and the smallest of these sets contains t objects, then if $t \geq n$, there are at least $t(t-1)\ldots(t-n+1)$ different* SDR*'s, and if $t < n$, there are at least $t!$ different* SDR*'s.*

A graph G is said to be *k–uniquely–vertex–colourable* (or a k–UCG for short) if $\chi(G) = k$ and any k–colouring of G induces a unique k–partition on $V(G)$. The following theorem of M. Truszczyński and S.J. Xu shows that there is a lower bound for $|E(G)|$ when G is a k–UCG.

Theorem B. [19, 17] *The minimum number of edges for a k–*UCG*, G, is*

$$|V(G)|(k-1) - \binom{k}{2};$$

and equality holds if and only if the subgraph induced on any two colour–classes of G is a tree.

It is easy to see that this lower bound is tight by the existence of q–trees [3]. In this regard, we define

Definition 1. For any graph, G, $\Lambda(G) = |E(G)| - |V(G)|(k-1) + \binom{k}{2}$. $\diamond$

Note that Theorem B essentially means that for any k–UCG, G, we have $\Lambda(G) \geq 0$. Also, as examples, it is easy to check that $\Lambda(K_n) = 0$ for any $n > 0$, $\Lambda(T) = 0$ for any tree T, $\Lambda(C_{2n}) = 1$ and $\Lambda(C_{2n+1}) = -2(n-1)$. Moreover, we have the following conjecture of S.J. Xu for minimal UCG's.

Conjecture 1.[19] *Xu's Conjecture*
*If G is a k–*UCG *with $\Lambda(G) = 0$ then $ccl(G) = 0$.*

Although, recently, using a computer search, it has been shown that there exists a counter example for Conjecture 1 which is actually a 3-chromatic core on 24 vertices [1], we believe that constructing other k–chromatic cores which are counter examples of Conjecture 1 for $k > 3$ is an important task since such graphs should contain important chromatic structures (also see [4]).

Another concept which is closely related to the vertex colouring of graphs is the concept of a *defining set* for a graph G. A defining set for a graph G is a set $S \subseteq V(G)$ of vertices along with their colours such that it uniquely extends to a $\chi(G)$–colouring of G. The *defining number*, $d(G)$, is the minimum of $|S|$ such that S is a defining set for G. We refer the interested reader to [15] for more details and the background.

To emphasis on the relationship between these concepts note that when one fixes the colours of some vertices in a set $S \subseteq V(G)$, then the extension problem to a k–colouring is actually a list colouring problem for which each vertex $v \notin S$ has a list of colours as $L_v = \{1, \dots, k\} - A$, where A is the colours of all the vertices in S which are adjacent to v (note that A may be an empty set). Clearly, S is a defining set if and only if this list colouring problem has a unique solution. Also, it is evident that a graph is a k–UCG if and only if the list colouring problem with $L_v = \{1, \dots, k\}$ for any vertex v has $k!$ solutions (unique up to permutation of colours or, equivalently, unique up to the induced partition).

A k–UCG with no vertex of degree $k - 1$ whose colour–classes all have more than one vertex is called a *u–core*[2] (or a *core* for short). K_1 is, pathologically, defined to be a core and also it is known that any UCG, G, has a unique core $cor(G)$ (as its induced subgraph) [8, 11]. The following simple lemma is also used in the sequel.

Lemma 1. *If we construct a new graph G^* from a graph G by adding a new vertex and connecting it to all vertices in $V(G)$, then $ccl(G) = ccl(G^*)$ and $\Lambda(G) = \Lambda(G^*)$.*

2 The Fixing Number

In this section we consider a simple method of embedding a k-chromatic graph G into a k–UCG $\tilde{H}$ by trying to add some new edges appropriately to $G \cup K_k$. This scenario gives rise to the concept of fixing number which is first introduced in [5, 8] (also see [16]).

To be more precise, let G be a k–chromatic graph and consider the graph $H = G \cup K_k$ such that $V(K_k) = \{v_1, \dots, v_k\}$ and assume that we have fixed the colours of the component K_k such that $v_i \in [i]$ for all $i \in \{1, \dots, k\}$. Then a class of (new) edges as Φ, such that for each edge $e \in \Phi$ one end of e is in $\{v_i\}_{i=1}^{k}$ and the other end is in $V(G)$, is called a *fixing set* (of edges) for G if the graph $\tilde{H}$ with the vertex set $V(H)$ and the edge set $E(H) \cup \Phi$ is a k–UCG. Also, $\phi_0(G)$ is defined as

$$\phi_0(G) = \min\{|\Phi| \mid \Phi \text{ is a fixing set for } G\},$$

and $\phi(G) = \phi_0(G) - \binom{k}{2}$ is called the *fixing number* of G. Before we proceed we focus on some simple examples.

Example 1. As a simple example consider the graph G_1 in Figure 1; and note that by the fixing depicted in this figure we have $\phi(\mathsf{G}_1) \leq 1$. Also, it is fairly easy to check that $\phi(\mathsf{G}_1) = 1$ (for a simpler proof see Proposition 1 and Theorem 2). ◇

[2] This is different from the categorical definition of *cores*.

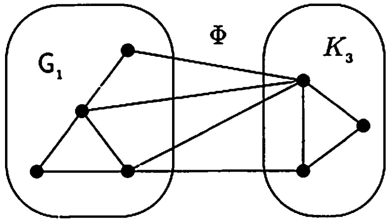

Figure 1: $\phi(G_1) = 1$ (see Example 1).

Example 2. For the next example we focus on the cycles C_n $(n > 2)$. It is clear that for even cycles we have $\phi(C_{2n}) = 0$ (we will see in the sequel that the fixing number of any UCG is zero). Also, for the odd cycles we claim that $\phi(C_{2n+1}) = 2(n-1)$. To see this, we first note that $\phi(C_{2n+1}) \leq 2(n-1)$, since in Figure 2 we have introduced a fixing set of size $2n + 1$.
On the other hand, although, it is not very hard to prove that $\phi(C_{2n+1}) \geq 2(n-1)$ by direct reasoning, but this inequality is a direct consequence of our next results since $\phi(C_{2n+1}) + \Lambda(C_{2n+1}) \geq 0$ (see Proposition 1(b)). $\diamond$

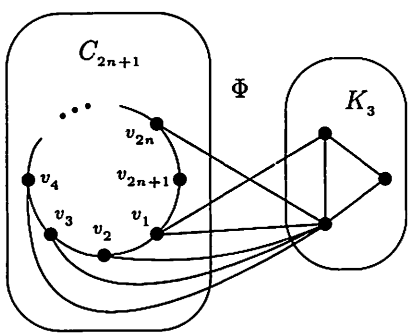

Figure 2: $\phi(C_{2n+1}) = 2(n-1)$ (see Example 2).

Note that the concept of fixing number can also be considered as a generalization (weakened form) of the concept defining number. ¿From this point of view, consider a k-chromatic graph G and a list colouring problem on G in which all vertices of G receive the list $\{1, 2, \ldots, k\}$. Then $\phi_0(G)$ is the minimum total number of colours that one can exclude from the lists in

such a way that the new list colouring problem has a unique solution (also see [16]).
The following proposition contains the most basic properties of the fixing number, and an upper bound in terms of defining number, which are actually direct consequences of the definition.

Proposition 1. *For any k-chromatic graph G,*

a) $0 \leq \phi(G) \leq (k-1)d(G) - \binom{k}{2}$.

b) $\phi(G) + \Lambda(G) \geq 0$ *and if equality holds then the subgraph induced on any two colour–classes of the minimal colouring is a forest.*

Proof. First, for the left hand inequality of part (a) note that the subgraph induced on any two colour–classes of the k–UCG $\tilde{H}$ which contains G and the k–clique is connected. Consequently, there exists at least one edge between any two vertices of the k–clique and the corresponding colour–classes in G. This shows that at least $\binom{k}{2}$ edges are needed for the fixing. Now, for the right hand side, assume that G has a defining set of size d. Then it is sufficient to connect each vertex, v, of the defining set to all vertices of the k–clique except the one which has the same colour as v; in order to fix the colours of the vertices of the defining set which specifies a fixed colouring of G by itself.
On the other hand, for (b) just note that

$$\Lambda(G) + \phi(G) = \Lambda(\tilde{H})) \geq 0;$$

since $\tilde{H}$ is a UCG (see Theorem B). □

It is also interesting to note that the following bound for the defining number, which has first appeared in [15], is a direct corollary of the above proposition.

Corollary 1.[15] *For any graph G,*

$$d(G) \geq |V(G)| - \frac{|E(G)|}{\chi(G) - 1}.$$

On the other hand, considering the sharpness of inequalities in Proposition 1, let us recall that $d(K_2 \square C_{2n+1}) = n + 1$ [15], and, therefore, we have the following example.

Example 3. Let $G = K_2 \square C_{2n+1}$, then from Proposition 1 we have

$$2n - 2 \leq \phi(G) \leq 2n - 1.$$

◇

Also, the following theorem can be considered as the counterpart of Theorem 2 in [15] and has essentially the same method of proof.

Theorem 1. *Let G be a simple graph on n vertices with degree sequence $(d_i)_{i=1}^n$ and let m be an integer such that $\chi(G) \leq m$. Then*

$$\phi_0(K_m \Box G) \geq \sum_{i=1}^{n} \binom{m - d_i}{2}.$$

Corollary 2. *If $m > n$ then $\phi_0(K_m \Box K_n) \geq n\binom{m-n+1}{2}$.*

One also can prove the following theorem which may be considered as the counterpart of the corresponding theorem for defining sets proved by Hajiabolhassan *et.al.* [12].

Theorem 2. *$\phi(G) = 0$ if and only if G is a* UCG.

Proof. First, assume that G is a k–UCG. Then choose vertices v_i $(i = 1, \ldots, k)$ each from one of the colour–classes and note that $\{v_i \mid i = 1, \ldots, k\}$ is a defining set. Now, it is easy to see that $\binom{k}{2}$ edges is enough to fix the colours of vertices in G and Proposition 1(a) says that this is also necessary. Hence $\phi(G) = 0$.
Conversely, let $\phi(G) = 0$ and assume that $\tilde{H}$ is the k–UCG which contains G and the k–clique. Let σ be the colouring of G which is induced by the unique colouring of $\tilde{H}$. Then, by direct forcing, this colouring induces some lists of (admissible) colours on the vertices of the k–clique. Since this list–colouring problem on the k–clique has only one solution, by M. Hall's Theorem (Theorem A), one of the vertices, say v_1, has a list which only contains one colour; and this means that v_1 is connected to at least $k - 1$ vertices in G. Now if we delete the colour of v_1 from the other lists we face a new list–colouring problem with $k - 1$ colours and by the same kind of reasoning there is a vertex v_2 in the k–clique which is connected to at least $k - 2$ vertices of G and so forth. This along with $\phi(G) = 0$ shows that there is an ordering of vertices of the k–clique such as $\{v_i \mid i = 1, \ldots, k\}$, such that each v_i is connected to exactly $k - i$ vertices in G $(i = 1, \ldots, k)$.
Hence, we know the structure of $\tilde{H}$. Assume that G has another k–colouring different from σ. Then it is clear that we can extend this colouring to a k–colouring of $\tilde{H}$ using the above structure; and this is a contradiction since $\tilde{H}$ is a k–UCG. □

At the end of this section it is instructive to add some remarks about the extremal condition $\phi + \Lambda = 0$. First note that from Proposition 1 we have some information about the structure of these graphs, however, it should be noted that classification of graphs for which $\phi(G) + \Lambda(G) = 0$, is an extremely hard problem. To see this, note that if G is a UCG, then $\phi(G) = 0$ and equality holds if and only if $\Lambda(G) = 0$. In this case we know a lot about such graphs with $ccl(G) = 0$, however, we do not know a lot about $ccl(G) > 0$ [1, 4, 5, 6, 7, 8, 11][3]. In the next section we try to turn around this problem by considering a more subtle embedding which can also generate UCG's with $ccl > 0$.

3 t–indices and τ–indices

In this section we consider the embedding of a $(k-1)$–chromatic graph G into a k–UCG $\tilde{H}$ and, this time, we wish to have $ccl(\tilde{H}) > 0$; which means that we should avoid any k–clique in our construction. Our new method of construction can be described as follows. We first assume that

1) G is $(k-1)$–chromatic.
2) G has a special structure which makes it possible to add some new set of vertices such as V^{*} to G and make a new graph H such that in any k–colouring of H, V^{*} is a fixed colour–class and $ccl(H) > 0$.
3) This fixed colour–class can be used to fix the colour of the rest of vertices in such a way that for the new graph $\tilde{H}$ we have $ccl(\tilde{H}) > 0$.

To begin let us recall the following definition from [10].

Definition 2.[10] Consider a $(k-1)$–chromatic graph G. Then a list

$$\mathcal{F} = \{(i, W_i) \mid 1 \leq i \leq l\}$$

of subsets of $P(V(G))$ is called a *transverse system* for G if both of the following conditions are satisfied.

- For every $(k-1)$–colouring σ of G, if $(i, W_i) \in \mathcal{F}$ then W_i has nonempty intersection with all colour classes of σ.
- For every k–colouring $\sigma : V(G) \stackrel{onto}{\longrightarrow} \{1, \ldots, k\}$ of G, there exists $(i, W_i) \in \mathcal{F}$ such that W_i has nonempty intersection with all colour classes of σ.

◇

Example 4. Consider the prism $\mathsf{P} = \mathsf{D4} - \{b_1, b_2, b_3\}$ (see Figure 3) and note that

$$\mathcal{F} = \{(1, \{a_1, a_2, c_1, c_2\}), (2, \{a_2, a_3, c_2, c_3\}), (3, \{a_3, a_1, c_3, c_1\})\}$$

is a transverse system for P. ◇

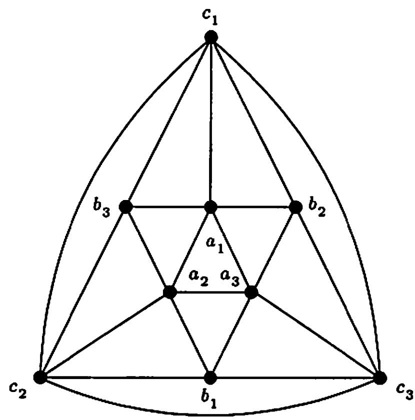

Figure 3: The graph D4.

The following theorem is actually a generalization of a theorem of M. Truszczyński [17] which has first appeared in [8] and has been already used in construction of k–chromatic graphs with some extremality conditions (e.g. see [10]). This theorem is our first step toward our second assumption.

Theorem 3.[10] *Let H be a k–chromatic graph such that in every k–colouring of H there is a fixed colour-class V^* consisting of m specified vertices $v_1, \ldots, v_m$, $(m \geq 1)$; and consider $G = H - V^*$. Also define $\mathcal{F} = \{(i, N_G(v_i)) \mid v_i \in V^*\}$. Then,*

a) $\chi(G) = k - 1$ and $\mathcal{F}$ is a transverse system for G.

Moreover if $cl(H) \leq k - 1$ then,

b) $cl(G[W_i]) \leq k - 2$ for every $(i, W_i) \in \mathcal{F}$ (and also clearly $cl(G) \leq k - 1$).

Conversely, let G be a $(k-1)$–chromatic graph and let $\mathcal{F} \subseteq \mathrm{N} \times P(V(G))$ be a transverse system for G. Then the graph H obtained by adding to G new vertices v_i, for each $((i, W_i) \in \mathcal{F})$, and joining each v_i to all vertices in W_i is a k–chromatic graph such that in any one of its k–colourings the class $V^* = \{v_i \mid (i, W_i) \in \mathcal{F}\}$ is fixed. If in addition (b) is also fulfilled then $cl(H) \leq k - 1$.

[3]T. Morrill and D. Pritikin in [16] consider this problem where they call these graphs *k–optimal*.

Note that by condition (a) for any $(i, W_i) \in \mathcal{F}$ we have $|W_i| \geq k-1$. Also, as an application of this theorem we can consider the transverse system of Example 4 and deduce that $\{b_1, b_2, b_3\}$ is a fixed colour-class in any 4-colouring of D4 (see Figure 3). For more on this theorem and related subjects see [8].
Now, we try to formulate the necessary definitions which will be used to compute $\Lambda(\tilde{H})$ in the final step.

Definition 3. Let G be a $(k-1)$-chromatic graph. Then, a list $\mathcal{F} \subseteq \mathbf{N} \times P(V(G))$ is said to be *admissible of type* 0 (resp. *admissible of type* 1) if $\mathcal{F}$ satisfies condition (a) (resp. conditions (a) and (b)) of Theorem 3, $|\mathcal{F}| > 1$ and

$$|\{(i, W_i) \in \mathcal{F} \ | \ |W_i| = k-1\}| < k.$$

The t_0*-index* (resp. t_1*-index*) of G,$t_0(G)$ (resp. $t_1(G)$), is defined to be the minimum of $||\mathcal{F}|| - (k-1)|\mathcal{F}|$ where

$$||\mathcal{F}|| = \sum_{(i,W_i)\in\mathcal{F}} |W_i|,$$

and $\mathcal{F}$ is an admissible list of type 0 (resp. of type 1). Moreover, in the case of $t_1(G)$, if there is no such admissible list we define $t_1(G) = -\infty$. Also, for any graph G, $\tau_r(G) = \phi(G) + t_r(G)$, is called the τ_r*-index* of G $(r = 0, 1)$. ◇

Example 5. As a trivial case, consider a $(k-1)$-chromatic graph G with $k > 2$ and consider the following list

$$\mathcal{F} = \{(1, V(G)), (2, V(G))\}.$$

Now, note that this list is a transverse system for G which trivially satisfies the conditions of an admissible list of type 0. This observation shows that $0 \leq t_0(G) \leq 2(|V(G)| - k + 1)$. ◇

Example 6. As it is more or less clear from the definition, t-indices and τ-indices are usually quite hard to compute. To set forward some examples, let us consider odd cycles C_{2n+1} $(n > 1)$. Then it is fairly easy to see that having two new vertices which are connected to all vertices of the cycle provide a minimal admissible list of type 1, and consequently we have

$$t_0(C_{2n+1}) = t_1(C_{2n+1}) = 4(n-1) \quad (n > 1).$$

Also considering Example 2 we can deduce that

$$\tau_0(C_{2n+1}) = \tau_1(C_{2n+1}) = 6(n-1) \quad (n > 1).$$

As one more example, consider the prism $\mathsf{P} = \mathsf{D4} - \{b_1, b_2, b_3\}$ (see Figure 3) and note that

$$\mathcal{F} = \{(1, \{a_1, a_2, c_1, c_2\}), (2, \{a_2, a_3, c_2, c_3\}), (3, \{a_3, a_1, c_3, c_1\})\}$$

provides a minimal admissible list of type 1. Hence, D4 is actually the 4–chromatic graph which is obtained by adding the three new vertices $\{b_1, b_2, b_3\}$ to P through $\mathcal{F}$, and consequently

$$t_0(\mathsf{P}) = t_1(\mathsf{P}) = 12 - 3 \times 3 = 3.$$

Also,

$$\phi(\mathsf{P}) = 1 \quad \text{and} \quad \tau_0(\mathsf{P}) = \tau_1(\mathsf{P}) = 4.$$

The graph D4 was already obtained as a special case of a general construction using forcing [4, 6, 7, 9] (for more on this subject see [5, 8]). ◇

Now, we are ready to apply our embedding as follows.

Theorem 4. *Let G be a $(k-1)$–chromatic graph with $\tau_r > -\infty$ $(r = 0, 1)$. Then there exists a core, U, such that*

$$|V(U)| \leq |V(G)| + |\mathcal{F}| + 2(k-1), \quad ccl(U) = r, \quad \chi(U) = k,$$

$$\Lambda(U) = \Lambda(G) + \tau_r(G) - |V(G)| + k - 1;$$

in which $\mathcal{F}$ is a minimal admissible list of type r.

Proof. Fix an admissible list of type r for G $(r = 0, 1)$ such as $\mathcal{F}$ and introduce new vertices $v_{(i,W_i)}$'s for each $(i, W_i) \in \mathcal{F}$. Then connect each $v_{(i,W_i)}$ to all vertices of $W_i \subseteq V(G)$ and note that in any k–colouring of the new graph, $v_{(i,W_i)}$'s form a fixed colour–class (by the hypothesis and Theorem 3).

Now consider the general pattern of the graph U, depicted in Figure 4, where a_j's are the vertices of the $(k-1)$–clique K_{k-1} for which we assume that $a_j \in [j]$ $(j = 1, \ldots, k-1)$. By the definition of an admissible list we know that $|\mathcal{F}| > 1$ and we also know that the number of (i, W_i)'s with $|W_i| = k - 1$ is less than or equal to $k - 1$. Hence, we can join a_j's to $v_{(i,W_i)}$'s such that

- Each a_j is connected to exactly one vertex in $\{v_{(i,W_i)} \mid (i, W_i) \in \mathcal{F}\}$.
- There is no vertex $v_{(i,W_i)}$ which is connected to all vertices in $\{a_j \mid j = 1, \ldots, k-1\}$.
- Each vertex $v_{(i,W_i)}$ with $|W_i| = k - 1$ is connected to at least one vertex in $\{a_j \mid j = 1, \ldots, k-1\}$.

Moreover, add $k-1$ vertices b_j's ($j = 1, \ldots, k-1$) and join each b_j to one of the vertices in $\{v_{(i,W_i)} \mid (i, W_i) \in \mathcal{F}\}$ and all vertices in $\{a_i \mid i \neq j\}$. Now note that in any k–colouring of this graph $\{a_j, b_j\} \subseteq [j]$ $(j = 1, \ldots, k-1)$ and $\{v_{(i,W_i)} \mid (i, W_i) \in \mathcal{F}\} \subseteq [k]$.
In order to complete the structure it is sufficient to fix the colour–classes of G and this can be done by adding a fixing set of edges between $\{b_j \mid j = 1, \ldots, k-1\}$ and $V(G)$. Using this structure in a minimal case provides a k–UCG which clearly does not have a colour–class of size 1. Therefore, the core of this k–UCG is a graph with the desired parameters. □

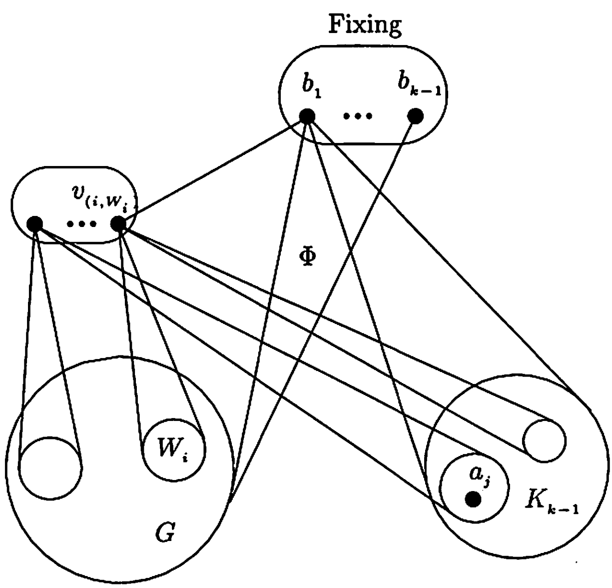

Figure 4: General pattern of U (Theorem 4).

Note that the theorem holds for both cases $r = 0$ and $r = 1$, however, the importance of the theorem is for the case $r = 1$.

Corollary 3. *Let G be a k–chromatic graph with $\tau_r > -\infty$ $(r = 0$ or $1)$. Then $\tau_r(G) + \Lambda(G) \geq |V(G)| - k$.*

Proof. Note that $\Lambda(U) \geq 0$ for any UCG, U. □

It is clear from Theorem 3 that, $\{v\}$ is a fixed colour–class of size one in a k–chromatic graph G if and only if the vertex v is connected to all vertices in $V(G) - \{v\}$; and from this point of view, Lemma 1 states that *ccl* and Λ are invariant under deletion or addition of such colour–classes. The next proposition shows that ϕ, t_r and τ_r $(r = 0, 1)$ have the same property.

Proposition 2. *If we construct a new graph G^* from a graph G by adding a new vertex and connecting it to all vertices in $V(G)$, then $\phi(G) = \phi(G^*)$, $\tau_r(G) = \tau_r(G^*)$ and $t_r(G) = t_r(G^*)$ $(r = 0, 1)$.*

Proof. First, consider the fixing number and assume that v is the vertex which is added to the k–chromatic graph G. In order to compute the fixing number of G^*, we should add the minimum number of edges to $G^* \cup K_{k+1}$ in such a way that we obtain a $(k + 1)$–UCG.
Now if we connect v to k vertices in K_{k+1}, v is forced to take the remaining colour; and then it suffices to use a fixing set of G to fix the rest of colours. This shows that $\phi(G^*) \leq \phi(G)$.
On the other hand, consider a fixing of G^* as a $(k + 1)$–UCG $\tilde{H}$, and assume that $[k + 1] = \{v, u\}$ for some vertex $u \in V(K_{k+1})$. Now, since in any UCG the subgraph induced on any two colour–classes is connected [17], we may deduce that there are at least k edges in the fixing set of G^* which are adjacent to u or v. Also, we know that $\tilde{H} - [k+1]$ is a UCG and consequently $\phi(G^*) \geq \phi(G)$. This proves that $\phi(G^*) = \phi(G)$.
For t_r $(r = 0, 1)$ note that if $\mathcal{F}$ is an admissible list of type r for G, then one can add the new vertex v to any W_i for $(i, W_i) \in \mathcal{F}$ to obtain

$$\mathcal{F}^* = \{(i, W \cup \{v\}) \mid (i, W_i) \in \mathcal{F}\},$$

which is clearly an admissible list of type r for G^*. This shows that $t_r(G^*) \leq t_r(G)$.
Conversely, let $\mathcal{F}^*$ be an admissible list of type r for G^*. Then by condition (a) of Theorem 3 we know that v is in any W for $(i, W) \in \mathcal{F}^*$. Therefore, if we exclude v from classes of $\mathcal{F}^*$ to obtain $\mathcal{F}$, then it is clear that $\mathcal{F}$ is an admissible list of type r for G and consequently $t_r(G) = t_r(G^*)$ $(r = 0, 1)$.
The equalities $\tau_r(G) = \tau_r(G^*)$ $(r = 0, 1)$ follows from the definition. □

Corollary 4. *Let G be a $(k_0 - 1)$–chromatic graph that satisfies the conditions of Theorem 4 ($r = 0$ or 1). Then there exists a class of cores $\{U_k\}_{(k \geq k_0)}$ such that for each $k \geq k_0$ we have*

$$|V(U_k)| \leq |V(G)| + |\mathcal{F}| + 3k - k_0 - 2, \quad ccl(U_k) = r, \quad \chi(U_k) = k,$$

$$\Lambda(U_k) = \Lambda(G) + \tau_r(G) - |V(G)| + k_0 - 1.$$

Proof. Use Lemma 1 on G to obtain a $(k - 1)$–chromatic graph, then apply Proposition 2 and Theorem 4 to this graph. □

To begin, consider odd cycles $G = C_{2n+1}$ for $(n > 1)$ and apply Theorem 4. Then since $\Lambda(C_{2n+1}) = -2(n - 1)$ and $\tau_1(C_{2n+1}) = 6(n - 1)$ we obtain a new 4–UCG $\tilde{H}_n$ with

$$|V(\tilde{H}_n)| = 2n + 9, \quad ccl(\tilde{H}_n) = 1, \quad \chi(\tilde{H}_n) = 4 \text{ and } \Lambda(\tilde{H}_n) = 2(n - 1).$$

Now note that $ccl(\tilde{H}_2) = 1$, $\Lambda(\tilde{H}_2) = 2$ and applying Corollary 4 we obtain the following proposition.

Proposition 3. *There exists a class of cores* $\{\mathsf{U}_k^2\}_{(k\geq 4)}$ *such that*

$$|V(\mathsf{U}_k^2)| = 3k+1, \quad ccl(\mathsf{U}_k^2) = 1, \quad \chi(\mathsf{U}_k^2) = k, \quad \text{and} \quad \Lambda(\mathsf{U}_k^2) = 2,$$

for each $k \geq 4$.

On the other hand, consider the prism $\mathsf{P} = \mathsf{D4} - \{b_1, b_2, b_3\}$ (see Figure 3) with $\tau_1(\mathsf{P}) = 4$ and $\Lambda(\mathsf{P}) = 0$. Then if we apply Theorem 4 we obtain a graph $\tilde{H}_{\mathsf{P}}$ with

$$|V(\tilde{H}_{\mathsf{P}})| = 15, \quad ccl(\tilde{H}_{\mathsf{P}}) = 1, \quad \chi(\tilde{H}_{\mathsf{P}}) = 4, \quad \text{and} \quad \Lambda(\tilde{H}_{\mathsf{P}}) = 1.$$

Applying Corollary 4 yields,

Proposition 4. *There exists a class of cores* $\{\mathsf{U}_k^1\}_{(k\geq 4)}$ *such that*

$$|V(\mathsf{U}_k^1)| = 3k+3, \quad ccl(\mathsf{U}_k^1) = 1, \quad \chi(\mathsf{U}_k^1) = k, \quad \text{and} \quad \Lambda(\mathsf{U}_k^1) = 1,$$

for each $k \geq 4$.

On the other hand, it is easy to see that, in general, the above construction is very far from being vertex–minimal. For this, note that we could add a new vertex d to the graph D4 of Figure 3 and join it to vertices a_2, a_3, b_2 and c_3 in order to obtain a 4–UCG on 10 vertices (where we have used a one vertex fixing structure which uses a 3–clique in the graph itself [4, 6]). Also, for the case of odd cycles too, one can easily check that only one extra vertex is sufficient for the fixing process. This gives rise to an enhancement of the previous result as follows.

Proposition 5. *There exists a class of cores* $\{\tilde{\mathsf{U}}_k^2\}_{(k\geq 4)}$ *such that*

$$|V(\tilde{\mathsf{U}}_k^2)| = 2k+3, \quad ccl(\tilde{\mathsf{U}}_k^2) = 1, \quad \chi(\tilde{\mathsf{U}}_k^2) = k, \quad \text{and} \quad \Lambda(\tilde{\mathsf{U}}_k^2) = 2,$$

for each $k \geq 4$.

The following conjecture has been set forward in [8].

Conjecture 2.[8] *If* G *is a* k–UCG *such that* $5 \leq k \leq |V(G)| \leq 2(k+1)$, $ccl(G) > 0$ *and* $\Lambda(G) \leq 2$, *then* $\chi(cor(G)) < k$.

In our new approach in this section, the extremal graphs are those with $\chi = k$ and $\tau_1 + \Lambda = |V| - k$; while we know that if such a graph exists then Xu's conjecture is wrong. Needless to say, finding methods of construction which generate such extremal graphs is of great importance since this kind of graphs (or even graphs which are nearly extremal in this sense) have very interesting colouring properties (e.g. see [9]). To sum up, we introduce the following problem,

Problem 1. *Find a k-chromatic graph, G, such that* $\tau_1(G) + \Lambda(G) = |V(G)| - k$.

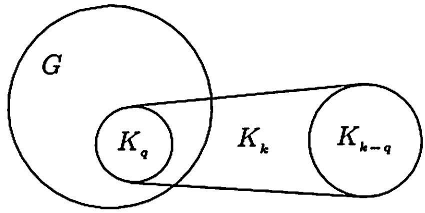

Figure 5: Using a maximum clique in fixing (see Section 4).

4 A More General Overview

In this paper we considered the problem of embedding a graph G, as an induced subgraph, into a UCG $\tilde{H}$; and we obtained some parameters for graph G by applying Truszczyński–Xu Theorem for $\tilde{H}$. However, this setup can be generalized even for the cases when G is just a subgraph of $\tilde{H}$ (which is not necessarily induced); since we have already encountered cases in which the construction of Theorem 4 is not vertex minimal.
As an example of this approach we can consider cases in which one maximum clique of the graph is used instead of the extra cliques used in our previous embeddings. To see this, assume that G is a k-chromatic graph with $cl(G) = q$, and consider the graph $G \cup K_{k-q}$ with one maximum q-clique of G (as in Figure 5). Now, if we assume that $\{v_1, \ldots, v_q\}$ are the vertices of the q-clique and $\{v_{q+1}, \ldots, v_k\}$ are the vertices of K_{k-q} and if we form a K_k on these vertices, then one can talk about the minimum number of edges between the set of vertices $\{v_1, \ldots, v_k\}$ and $V(G)$ which turns $G \cup K_{k-q}$ into a k-UCG. Also, we can use this new parameter in an embedding similar to that of Theorem 4 and obtain UCG's with a smaller number of vertices.
Note that we could even consider G itself, without any additional vertices, and try to turn it into a k-UCG by adding new edges. Then one can talk about the minimum number of such edges as a new parameter for G.
We do not follow these ideas here, however, we note that these new parameters are more difficult to handle.

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