THE HEIGHT DISTRIBUTION OF NODES IN NON-CROSSING TREES

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ABSTRACT. We consider non-crossing trees and show, that the height of node ρn with $0 < \rho < 1$ in a non-crossing tree of size n is asymptotically Maxwell-distributed. We also give an asymptotic formula for the expected height of node ρn.

1. INTRODUCTION

A non-crossing tree is defined as a connected acyclic graph with the vertex set of n points in the plane forming the vertices of a convex polygon and whose edges are straight line segments that do not cross. We consider further the vertices labelled counter-clockwise from 1 to n with vertex 1 as the root of the tree.

The enumeration problem was considered first in [9] and also by Dulucq and Penaud [3]. It turns out, that the number of non-crossing trees of size $n+1$ is equal to the number of ternary trees of size n and therefore given by $\frac{1}{2n-1}\binom{3n-3}{n-1}$. A lot of parameters of non-crossing trees are studied quite recently by Noy [8], Flajolet and Noy [4] and Deutsch and Noy [1]. Among other parameters, in [1] the expected height of a non-crossing tree was studied under the assumption, that all trees of the same size are considered to be equally likely. The height of a node in a rooted tree is here always defined as the number of vertices on the direct path from the root to this node and the height of the tree is then the maximum of all heights of the nodes in this tree.

The behaviour of the height of several families of trees has already been analysed. Especially the so called family of simply generated trees were studied by Flajolet and Odlyzko [5]. In [1] a height-preserving bijection between the non-crossing trees and a member of the family of simply generated trees (the even trees) was established, and therefore results of [5] are applicable. It follows, that the expected height of non-crossing trees of size n is asymptotically given by $\frac{2}{3}\sqrt{3\pi n}$.

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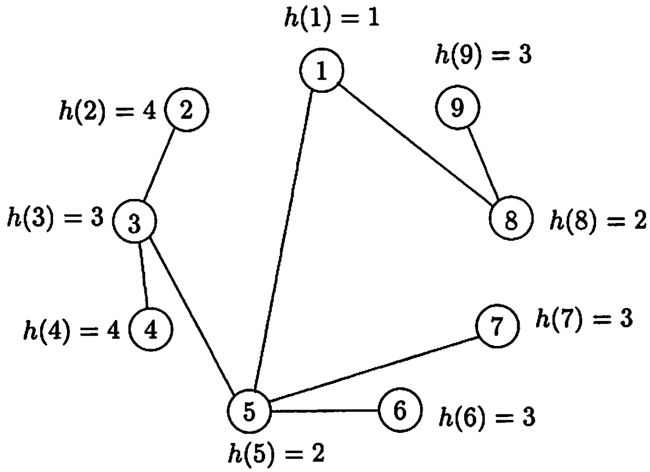

FIGURE 1. A non-crossing tree with heights $h(j)$ of the nodes j.

We will study here a somehow different parameter. Instead of considering the height of the whole tree, we are interested in the height $h(j)$ of a given node j. (see figure 1.)

For simply generated families of trees, the height of the leaves (the endnodes) was studied by Drmota [2] and by Gittenberger [7]. Although their results are not applicable for non-crossing trees (above mentioned height-preserving bijection does not preserve the labelling of the nodes), we can use their approach of using generating functions and extracting coefficients by means of a double Hankel-contour integral.

In our analysis, we will use the following combinatorial decomposition of a non-crossing tree as described in [4]. A non-crossing tree consists of a root, which is attached to a (possibly empty) sequence of butterflies, where a butterfly is a (ordered) pair of non-crossing trees, that share a common root. (see figure 2.)

This combinatorial decomposition can be translated immediately into an algebraic equation for the generating functions $T(z) = \sum_{n\geq 0} T_n z^n$ of the numbers T_n of non-crossing trees of size n and $B(z) = \sum_{n\geq 0} B_n z^n$ of the numbers B_n of butterflies with n nodes. We get the system

$$T(z) = \frac{z}{1 - B(z)}, \quad B(z) = \frac{T^2(z)}{z}$$

and therefore the equation

$$T^3(z) - zT(z) + z^2 = 0. \tag{1}$$

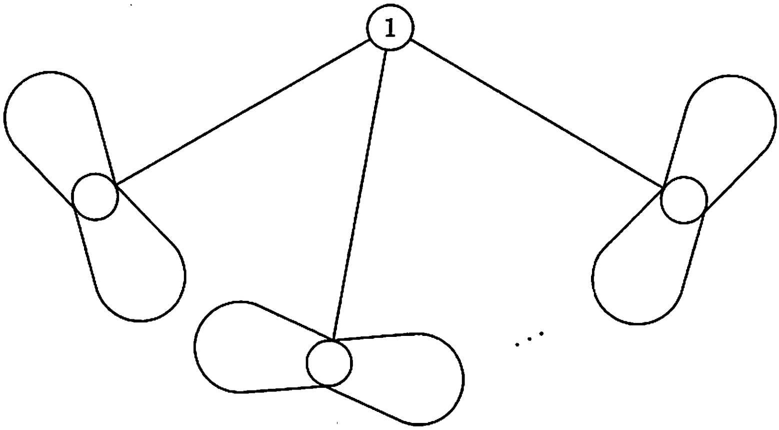

FIGURE 2. The combinatorial decomposition of a non-crossing tree.

Expanding these generating functions around their dominant algebraic singularity $z = \frac{4}{27}$, either by solving the algebraic equation (1) or by using the Weierstrass preparation theorem, we get

$$T(z) = T_0(z) + T_1(z)\sqrt{1 - \frac{27}{4}z} \sim \frac{2}{9} - \frac{2\sqrt{3}}{27}\sqrt{1 - \frac{27}{4}z}, \tag{2a}$$

$$B(z) = B_0(z) + B_1(z)\sqrt{1 - \frac{27}{4}z} \sim \frac{1}{3} - \frac{2\sqrt{3}}{9}\sqrt{1 - \frac{27}{4}z}, \tag{2b}$$

with functions $T_0(z)$, $T_1(z)$, $B_0(z)$ and $B_1(z)$ analytic in a neighborhood of the singularity.

2. THE GENERATING FUNCTION FOR THE HEIGHT DISTRIBUTION

We want to study here the random variable $X_{n,j}$, which counts the height $h(j)$ of node j in a non-crossing tree of size n, under the model, that all non-crossing trees of this size occur equally likely. To do this, we use trivariate generating functions $F(z, u, v) = \sum_{n,j,m} F_{n,j,m} z^n u^j v^m$ for the number of non-crossing trees of size n, where node j has height m. We also use the auxiliary functions $B(z, u, v) = \sum_{n,j,m} B_{n,j,m} z^n u^j v^m$ of the number of butterflies of size n, where node j has height m and $\tilde{F}(z, u, v) = \sum_{n,j,m} \tilde{F}_{n,j,m} z^n u^j v^m$ of the number of non-crossing trees with node n as root and where node j has height m (with respect to this root n).

The above described combinatorial decomposition of a non-crossing tree leads immediately to the following equations:

$$F(z,u,v) = \frac{zuv}{1-B(z)} + \frac{zuvB(z,u,v)}{(1-B(z))(1-B(zu))}, \tag{3a}$$

$$B(z,u,v) = \frac{\tilde{F}(z,u,v)T(z)}{z} + \frac{T(zu)F(z,u,v)}{zu} - \frac{vT(zu)T(z)}{z}. \tag{3b}$$

By symmetry arguments we have $\tilde{F}_{n,j,m} = F_{n,n+1-j,m}$ and $B_{n,j,m} = B_{n,n+1-j,m}$ or equivalently $\tilde{F}(z,u,v) = uF(zu,u^{-1},v)$ and $B(z,u,v) = uB(zu,u^{-1},v)$. With this considerations, we obtain from equation (3a)

$$\tilde{F}(z,u,v) = \frac{zuv}{1-B(zu)} + \frac{zvB(z,u,v)}{(1-B(z))(1-B(zu))}. \tag{3c}$$

The algebraic system (3) gives as solution for the generating function $F(z,u,v)$:

$$\begin{aligned} F(z,u,v) = {} & \frac{vuT(z)}{1-v\left(\frac{T(z)+T(zu)}{(1-B(z))(1-B(zu))}\right)} - \\ & - \frac{v^2uT^2(z)}{(1-B(z))(1-B(zu))} \frac{1}{1-v\left(\frac{T(z)+T(zu)}{(1-B(z))(1-B(zu))}\right)}. \end{aligned} \tag{4}$$

3. Extracting coefficients by a double contour integral

We are now interested in the coefficients

$$F_{n,j,m} = [z^n u^j v^m]F(z,u,v),$$

especially in the most interesting case $\frac{j}{n} = \rho$ for a fixed ratio $0 < \rho < 1$. We will also restrict ourselves to values $m \le C\sqrt{n}$ for a fixed constant $C > 0$. This is quite natural, because we know from [1], that the expected height of the non-crossing tree is of order $\Theta(\sqrt{n})$.

For the number of non-crossing trees of size n, where node ρn has exactly height m, we get from equation (4) with the substitution $w = zu$:

$$\begin{aligned} F_{n,\rho n,m} = {} & [z^{(1-\rho)n+1}w^{\rho n-1}]T(z)\left(\frac{T(z)+T(w)}{(1-B(z))(1-B(w))}\right)^{m-1} - \\ & - [z^{(1-\rho)n+1}w^{\rho n-1}]\frac{T^2(z)}{(1-B(z))(1-B(w))}\left(\frac{T(z)+T(w)}{(1-B(z))(1-B(w))}\right)^{m-2}. \end{aligned}$$

To extract coefficients, we will basically use Cauchy's formula

$$\begin{aligned} F_{n,\rho n,m} = {} & \frac{1}{(2\pi i)^2}\int_{\Gamma_1}\int_{\Gamma_2} \frac{1}{z^{(1-\rho)n+2}w^{\rho n}} T(z)\left(\frac{T(z)+T(w)}{(1-B(z))(1-B(w))}\right)^{m-1} dzdw - \\ & - \frac{1}{(2\pi i)^2}\int_{\Gamma_1}\int_{\Gamma_2} \frac{1}{z^{(1-\rho)n+2}w^{\rho n}} \frac{T^2(z)}{(1-B(z))(1-B(w))} \times \\ & \times \left(\frac{T(z)+T(w)}{(1-B(z))(1-B(w))}\right)^{m-2} dzdw \end{aligned}$$

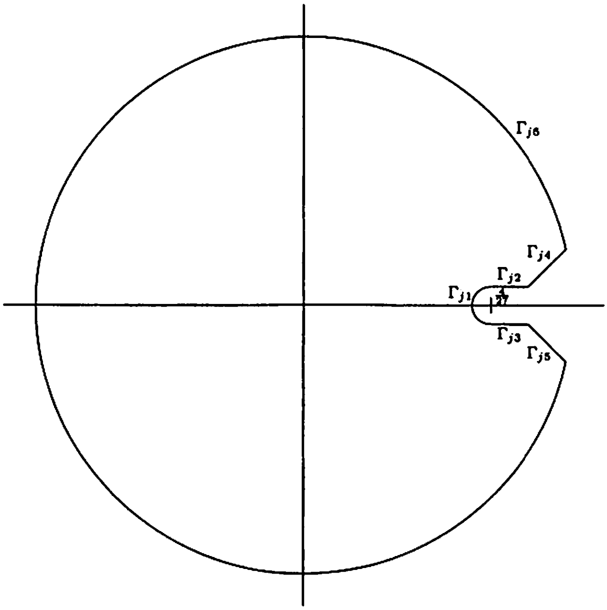

FIGURE 3. The integration paths Γ_j for $j = 1, 2$.

$$= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{z^{(1-\rho)n+2} w^{\rho n}} \frac{T(z)T(w)}{T(z)+T(w)} \times \\ \times \left(\frac{T(z)+T(w)}{(1-B(z))(1-B(w))} \right)^{m-1} dz dw \,, \tag{5}$$

where the paths of integration $\Gamma_1 = \Gamma_{11} + \Gamma_{12} + \Gamma_{13} + \Gamma_{14} + \Gamma_{15} + \Gamma_{16}$ and $\Gamma_2 = \Gamma_{21} + \Gamma_{22} + \Gamma_{23} + \Gamma_{24} + \Gamma_{25} + \Gamma_{26}$ are given by*(compare with figure 3):

$$\Gamma_{11} = \left\{ z = \tfrac{4}{27}(1 + \tfrac{t}{(1-\rho)n}) \,\middle|\, \Re t < 0 \,, |t| = 1 \right\} ,$$

$$\Gamma_{12} = \left\{ z = \tfrac{4}{27}(1 + \tfrac{t}{(1-\rho)n}) \,\middle|\, 0 \le \Re t \le \log^2 n, \Im t = 1 \right\} ,$$

$$\Gamma_{13} = \overline{\Gamma_{12}} \,,$$

$$\Gamma_{14} = \left\{ z = \tfrac{4}{27}(1 + \tfrac{t}{(1-\rho)n}) \,\middle|\, \Re t = \log^2 n + \tfrac{\sqrt{2}}{2} r, \Im t = 1 + \tfrac{\sqrt{2}}{2} r, 0 \le r \le n^{\frac{1}{2}+\epsilon} \right\} ,$$

$$\Gamma_{15} = \overline{\Gamma_{14}} \,,$$

*For some technical reasons, it seems here, that such an "artifical" contour is more convenient than a simpler one, used by other authors. Otherwise we would run into troubles when estimating the remainder terms.

$$\Gamma_{16} = \left\{ z \,\middle|\, |z| = \tfrac{4}{27} \left| 1 + \tfrac{\log^2 n + \frac{\sqrt{2}}{2} n^{\frac{1}{2}+\epsilon} + i(1 + \frac{\sqrt{2}}{2} n^{\frac{1}{2}+\epsilon})}{(1-\rho)n} \right| , \right.$$
$$\left. \arg\left(1 + \tfrac{\log^2 n + \frac{\sqrt{2}}{2} n^{\frac{1}{2}+\epsilon} + i(1 + \frac{\sqrt{2}}{2} n^{\frac{1}{2}+\epsilon})}{(1-\rho)n}\right) \le |\arg z| \le \pi \right\} .$$

and

$$\Gamma_{21} = \left\{ w = \tfrac{4}{27}(1 + \tfrac{s}{\rho n}) \,\middle|\, \Re s < 0 \,, |s| = 1 \right\} ,$$
$$\Gamma_{22} = \left\{ w = \tfrac{4}{27}(1 + \tfrac{s}{\rho n}) \,\middle|\, 0 \le \Re s \le \log^2 n, \Im s = 1 \right\} ,$$
$$\Gamma_{23} = \overline{\Gamma_{22}} ,$$
$$\Gamma_{24} = \left\{ z = \tfrac{4}{27}(1 + \tfrac{s}{\rho n}) \,\middle|\, \Re s = \log^2 n + \tfrac{\sqrt{2}}{2} r, \Im s = 1 + \tfrac{\sqrt{2}}{2} r, 0 \le r \le n^{\frac{1}{2}+\epsilon} \right\} ,$$
$$\Gamma_{25} = \overline{\Gamma_{24}} ,$$
$$\Gamma_{26} = \left\{ z \,\middle|\, |z| = \tfrac{4}{27} \left| 1 + \tfrac{\log^2 n + \frac{\sqrt{2}}{2} n^{\frac{1}{2}+\epsilon} + i(1 + \frac{\sqrt{2}}{2} n^{\frac{1}{2}+\epsilon})}{\rho n} \right| , \right.$$
$$\left. \arg\left(1 + \tfrac{\log^2 n + \frac{\sqrt{2}}{2} n^{\frac{1}{2}+\epsilon} + i(1 + \frac{\sqrt{2}}{2} n^{\frac{1}{2}+\epsilon})}{\rho n}\right) \le |\arg z| \le \pi \right\} .$$

In the paths Γ_{16} and Γ_{26} the ϵ can be an arbitrary positive value smaller than $\frac{1}{2}$.

4. Evaluation of the integral

We will show in section 5, that the contribution of the paths $z \in \Gamma_{14} \cup \Gamma_{15} \cup \Gamma_{16}$ or $w \in \Gamma_{24} \cup \Gamma_{25} \cup \Gamma_{26}$ in the integral (5) is asymptotically neglectable. We will denote the contribution of this part of the integral (the remainder integral) with $R_{n,\rho,m}$. That means, we will get the main contribution of the integral for the paths $z \in \Gamma_{11} \cup \Gamma_{12} \cup \Gamma_{13} \wedge w \in \Gamma_{21} \cup \Gamma_{22} \cup \Gamma_{23}$, which we will denote with $I_{n,\rho,m}$. Thus we will show, that

$$F_{n,\rho n,m} = I_{n,\rho,m} + R_{n,\rho,m} \sim I_{n,\rho,m} \,, \tag{6}$$

where we are going to evaluate the integral $I_{n,\rho,m}$ this section.

First we will study the integrand of (5) in the neighborhoods of the singularities $z = \frac{4}{27}$ and $w = \frac{4}{27}$. With the substitutions

$$z = \frac{4}{27}(1 + \frac{t}{(1-\rho)n}) \quad \text{and} \quad w = \frac{4}{27}(1 + \frac{s}{\rho n}) \tag{7}$$

we get from the asymptotic expansions (2):

$$T(z) = \left(\tfrac{2}{9} - \tfrac{2\sqrt{3}}{27\sqrt{1-\rho}}\sqrt{-\tfrac{t}{n}}\right)\cdot\left(1 + \mathcal{O}(\tfrac{|t|}{n})\right), \; T(w) = \left(\tfrac{2}{9} - \tfrac{2\sqrt{3}}{27\sqrt{\rho}}\sqrt{-\tfrac{s}{n}}\right)\cdot\left(1 + \mathcal{O}(\tfrac{|s|}{n})\right),$$
$$B(z) = \left(\tfrac{1}{3} - \tfrac{2\sqrt{3}}{9\sqrt{1-\rho}}\sqrt{-\tfrac{t}{n}}\right)\cdot\left(1 + \mathcal{O}(\tfrac{|t|}{n})\right), \; B(w) = \left(\tfrac{1}{3} - \tfrac{2\sqrt{3}}{9\sqrt{\rho}}\sqrt{-\tfrac{s}{n}}\right)\cdot\left(1 + \mathcal{O}(\tfrac{|s|}{n})\right) . \tag{8a}$$

When we define the abbreviations

$$\phi_1(z,w) = \frac{T(z)+T(w)}{(1-B(z))(1-B(w))}, \quad \phi_2(z,w) = \frac{T(z)T(w)}{T(z)+T(w)},$$

we get

$$\begin{aligned}\phi_1(z,w) &= \frac{\left(\frac{2}{9}-\frac{2\sqrt{3}}{27\sqrt{1-\rho}}\sqrt{-\frac{t}{n}}\right)\cdot\left(1+\mathcal{O}\left(\frac{|t|}{n}\right)\right)+\left(\frac{2}{9}-\frac{2\sqrt{3}}{27\sqrt{\rho}}\sqrt{-\frac{s}{n}}\right)\cdot\left(1+\mathcal{O}\left(\frac{|s|}{n}\right)\right)}{\left(\frac{2}{3}+\frac{2\sqrt{3}}{9\sqrt{1-\rho}}\sqrt{-\frac{t}{n}}\right)\cdot\left(1+\mathcal{O}\left(\frac{|t|}{n}\right)\right)\cdot\left(\frac{2}{3}+\frac{2\sqrt{3}}{9\sqrt{\rho}}\sqrt{-\frac{s}{n}}\right)\cdot\left(1+\mathcal{O}\left(\frac{|s|}{n}\right)\right)} \\ &= 1-\frac{\sqrt{3}}{2\sqrt{1-\rho}}\sqrt{-\frac{t}{n}}-\frac{\sqrt{3}}{2\sqrt{\rho}}\sqrt{-\frac{s}{n}}+\mathcal{O}\left(\frac{|s|}{n}+\frac{|t|}{n}\right). \qquad (8b)\end{aligned}$$

and

$$\begin{aligned}\phi_2(z,w) &= \frac{\left(\frac{2}{9}-\frac{2\sqrt{3}}{27\sqrt{1-\rho}}\sqrt{-\frac{t}{n}}\right)\cdot\left(1+\mathcal{O}\left(\frac{|t|}{n}\right)\right)\cdot\left(\frac{2}{9}-\frac{2\sqrt{3}}{27\sqrt{\rho}}\sqrt{-\frac{s}{n}}\right)\cdot\left(1+\mathcal{O}\left(\frac{|s|}{n}\right)\right)}{\left(\frac{2}{9}-\frac{2\sqrt{3}}{27\sqrt{1-\rho}}\sqrt{-\frac{t}{n}}\right)\cdot\left(1+\mathcal{O}\left(\frac{|t|}{n}\right)\right)+\left(\frac{2}{9}-\frac{2\sqrt{3}}{27\sqrt{\rho}}\sqrt{-\frac{s}{n}}\right)\cdot\left(1+\mathcal{O}\left(\frac{|s|}{n}\right)\right)} \\ &= \frac{1}{9}+\mathcal{O}\left(\sqrt{\frac{|s|}{n}}+\sqrt{\frac{|t|}{n}}\right). \qquad (8c)\end{aligned}$$

This leads further to the expansion

$$\begin{aligned}\phi_1(z,w)^{m-1} &= e^{(m-1)\cdot\log\left(1-\frac{\sqrt{3}}{2\sqrt{1-\rho}}\sqrt{-\frac{t}{n}}-\frac{\sqrt{3}}{2\sqrt{\rho}}\sqrt{-\frac{s}{n}}+\mathcal{O}\left(\frac{|s|}{n}+\frac{|t|}{n}\right)\right)} \\ &= e^{-m\frac{\sqrt{3}}{2\sqrt{1-\rho}}\sqrt{-\frac{t}{n}}}\cdot e^{-m\frac{\sqrt{3}}{2\sqrt{\rho}}\sqrt{-\frac{s}{n}}}\cdot\left(1+\mathcal{O}\left(m\frac{|s|}{n}+m\frac{|t|}{n}+\sqrt{\frac{|s|}{n}}+\sqrt{\frac{|t|}{n}}\right)\right). \qquad (8d)\end{aligned}$$

We also use the expansions

$$z^{-(1-\rho)n-2} = \left(\frac{27}{4}\right)^{(1-\rho)n+2} e^{-t}\cdot\left(1+\mathcal{O}\left(\frac{|t|^2}{n}\right)\right), \qquad (8e)$$

$$w^{-\rho n} = \left(\frac{27}{4}\right)^{\rho n} e^{-s}\cdot\left(1+\mathcal{O}\left(\frac{|s|^2}{n}\right)\right). \qquad (8f)$$

With the expansions (8) in the neighborhood of the singularities $z=\frac{4}{27}$ and $w=\frac{4}{27}$ we are now able to evaluate the integral (5) for the paths of integration $z\in\Gamma_{11}\cup\Gamma_{12}\cup\Gamma_{13}$ and $w\in\Gamma_{21}\cup\Gamma_{22}\cup\Gamma_{23}$. With the abbreviations

$$C_1 = \{t \,|\, |t|=1, \Re t<0\}\cup\{t \,|\, 0\le\Re t\le\log^2 n, \Im t=\pm 1\},$$
$$C_2 = \{s \,|\, |s|=1, \Re s<0\}\cup\{s \,|\, 0\le\Re s\le\log^2 n, \Im s=\pm 1\},$$

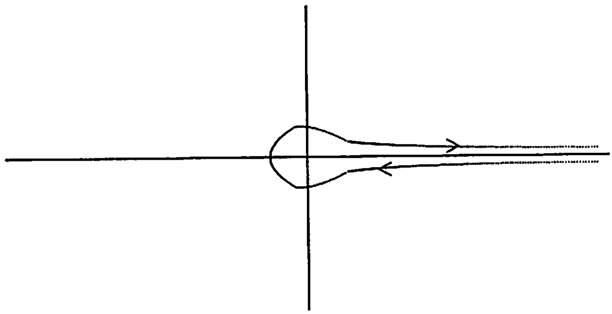

FIGURE 4. The Hankel contour γ.

we obtain:

$$I_{n,\rho,m} = \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_{11}\cup\Gamma_{12}\cup\Gamma_{13}} \int\limits_{\Gamma_{21}\cup\Gamma_{22}\cup\Gamma_{23}} \frac{\phi_2(z,w)(\phi_1(z,w))^{m-1}}{z^{(1-\rho)n+2}w^{\rho n}} dzdw$$

$$= \left(\tfrac{27}{4}\right)^n \frac{1}{9\rho(1-\rho)n^2} \frac{1}{(2\pi i)^2} \int_{C_1}\int_{C_2} e^{-t-\frac{\sqrt{3}m}{2\sqrt{n(1-\rho)}}\sqrt{-t}} e^{-s-\frac{\sqrt{3}m}{2\sqrt{n\rho}}\sqrt{-s}} \times$$

$$\times \left(1+\mathcal{O}\left(m\tfrac{|s|}{n} + m\tfrac{|t|}{n} + \sqrt{\tfrac{|s|}{n}} + \sqrt{\tfrac{|t|}{n}}\right)\right) dsdt. \qquad (9)$$

Together with the estimation

$$\int_{C_1}\int_{C_2} |e^{-t-\frac{\sqrt{3}m}{2\sqrt{n(1-\rho)}}\sqrt{-t}} e^{-s-\frac{\sqrt{3}m}{2\sqrt{n\rho}}\sqrt{-s}}||t^L||s^M||ds||dt|$$

$$\leq \int_{C_1}\int_{C_2} |e^{-s-t}||t^L||s^M||ds||dt| = \mathcal{O}_{L,M}(1)$$

for $L, M \geq 0$ we obtain

$$I_{n,\rho,m} = \left(\tfrac{27}{4}\right)^n \frac{1}{9\rho(1-\rho)n^2} \frac{1}{(2\pi i)^2} \int_{C_1}\int_{C_2} e^{-t-\frac{\sqrt{3}m}{2\sqrt{n(1-\rho)}}\sqrt{-t}} e^{-s-\frac{\sqrt{3}m}{2\sqrt{n\rho}}\sqrt{-s}} dsdt\cdot$$
$$\cdot\left(1+\mathcal{O}\left(\tfrac{|m|}{n}\right)\right). \qquad (10)$$

To evaluate this integral, we apply the following lemma, which can e. g. be found in [2]:

Lemma 1. *Let γ be a Hankel contour starting from $+e^{2\pi i}\infty$, passing around 0 and terminating at $+\infty$. Then we have for $\lambda, \mu > 0$*

$$\frac{1}{2\pi i}\int_\gamma e^{-\lambda\sqrt{-t}-\mu t} dt = \frac{\lambda\mu^{-\frac{3}{2}}}{2\sqrt{\pi}} e^{-\frac{\lambda^2}{4\mu}}.$$

To estimate the error, when using the Hankel contour γ instead of C_1 resp. C_2 we have to keep in mind, that

$$\int_{\gamma\cap\{t|\Re t\geq K\}} e^{-\lambda\sqrt{-t}-\mu t}dt = \mathcal{O}(\mu^{-1}e^{-K\mu})$$

uniformly for $K > 0$, which leads in our case to an exponentially small error term $\mathcal{O}(e^{-\log^2 n})$.

With lemma 1 we immediately get the following proposition:

Proposition 2. *For $\epsilon \leq \rho \leq 1-\epsilon$ and $m \leq C\sqrt{n}$ with arbitrary but fixed $\epsilon > 0$ and $C > 0$ we obtain*

$$I_{n,\rho,m} = \frac{m^2}{48\pi n^3(\rho(1-\rho))^{\frac{3}{2}}}\left(\frac{27}{4}\right)^n e^{-\frac{3m^2}{16n\rho(1-\rho)}}\cdot\left(1+\mathcal{O}\left(\frac{|m|}{n}\right)\right). \quad (11)$$

5. The remainder integral

In this section we will show, that the remainder integral $R_{n,\rho,m}$, where z lies in $\Gamma_{14}\cup\Gamma_{15}\cup\Gamma_{16}$ or w lies in $\Gamma_{24}\cup\Gamma_{25}\cup\Gamma_{26}$ is asymptotically neglectable. More precisely we get, that the remainder integral $R_{n,\rho,m}$ is exponentially small compared to $I_{n,\rho,m}$ ($|R_{n,\rho,m}| = o(\frac{|I_{n,\rho,m}|}{n^L})$ for every fixed $L > 0$).

5.1. **The lager circle.** First we will consider the cases $z \in \Gamma_{16}$ or $w \in \Gamma_{26}$. The integrand in (5) consists of $\phi_1(z,w)$ and $\phi_2(z,w)$ and both functions are analytic inside the integration domain, and thus bounded, let us say by a $K > 0$. With $m \leq C\sqrt{n}$ for a fixed $C > 0$ we get

$$\begin{aligned}\phi_2(z,w) &\leq K = \mathcal{O}(1),\\ \phi_1(z,w)^{m-1} &\leq K^{m-1} \leq K^{C\sqrt{n}-1} = \mathcal{O}\left(e^{\hat{C}\sqrt{n}}\right),\end{aligned} \quad (12)$$

with a real $\hat{C}$.

We consider now $w \in \Gamma_{26}$, where we have the estimation

$$|w| \geq \frac{4}{27}\left(1+\frac{\sqrt{2}}{2\rho}\frac{n^{\frac{1}{2}+\epsilon}}{n}\right)$$

and

$$\begin{aligned}|w|^{n\rho} &\geq \left(\frac{4}{27}\right)^{n\rho}\left(1+\frac{\sqrt{2}}{2\rho}\frac{n^{\frac{1}{2}+\epsilon}}{n}\right)^{n\rho}\\ &= \left(\frac{4}{27}\right)^{n\rho} e^{(n\rho)\log\left(1+\frac{\sqrt{2}n^{\frac{1}{2}+\epsilon}}{2\rho n}\right)} \geq \left(\frac{4}{27}\right)^{n\rho} e^{\frac{n^{\frac{1}{2}+\epsilon}}{2}}\end{aligned}$$

Thus we obtain

$$|w|^{-n\rho} \leq \left(\frac{27}{4}\right)^{n\rho} e^{\frac{n^{\frac{1}{2}+\epsilon}}{2}} = \mathcal{O}\left(\left(\frac{27}{4}\right)^{n\rho} e^{-\bar{C}n^{\frac{1}{2}+\epsilon}}\right), \quad (13)$$

for a positive $\bar{C} > 0$.

For an arbitrary $z \in \Gamma_1$ we have

$$|z| \geq \frac{4}{27}\left(1 - \frac{1}{(1-\rho)n}\right)$$

and therefore the estimate

$$|z|^{-n(1-\rho)-2} \leq 2e\left(\frac{27}{4}\right)^{(1-\rho)n+2} = \mathcal{O}\left(\left(\frac{27}{4}\right)^{(1-\rho)n}\right). \tag{14}$$

Thus we get, that the integral (5) is for the path $w \in \Gamma_{26}$ bounded by

$$\mathcal{O}\left((\frac{27}{4})^n e^{\hat{C}\sqrt{n}-\bar{C}n^{\frac{1}{2}+\epsilon}}\right), \tag{15}$$

with a positive $\bar{C} > 0$ and a real $\hat{C}$.

For $z \in \Gamma_{16}$ and an arbitrary w one can show in an analogous way, that the integral is here also bounded by $\mathcal{O}\left((\frac{27}{4})^n e^{\hat{C}\sqrt{n}-\bar{C}n^{\frac{1}{2}+\epsilon}}\right)$.

5.2. **Near the singularity.** Now we consider the remaining cases where $z \notin \Gamma_{16}$ and $w \notin \Gamma_{26}$ but $z \in \Gamma_{14} \cup \Gamma_{15}$ or $w \in \Gamma_{24} \cup \Gamma_{25}$. Without loss of generality we consider the case $w \in \Gamma_{24} \cup \Gamma_{25}$ and $z \notin \Gamma_{16}$.

For an arbitrary $z \in \Gamma_1$ we have the estimate (14) and for $w \in \Gamma_{24} \cup \Gamma_{25}$ we have

$$|w| \geq \frac{4}{27}\left(1 + \frac{\log^2 n}{\rho n}\right)$$

and thus we get

$$|w|^{-\rho n} \leq \left(\frac{27}{4}\right)^{\rho n} e^{-\frac{\log^2 n}{2}} = \mathcal{O}\left(\left(\frac{27}{4}\right)^{\rho n} e^{-\bar{C}\log^2 n}\right) \tag{16}$$

or

$$|z|^{-n(1-\rho)-2}|w|^{-\rho n} = \mathcal{O}\left(\left(\frac{27}{4}\right)^{n} e^{-\bar{C}\log^2 n}\right), \tag{17}$$

with a positive $\bar{C} > 0$.

In order to prove, that the remainder integral is asymptotically neglectable, it suffices to show, that

$$\phi_2(z,w)(\phi_1(z,w))^{m-1}$$

is bounded for $w \in \Gamma_{24} \cup \Gamma_{25}$ and an arbitrary $z \notin \Gamma_{16}$. Due to $\phi_2(z,w) = \mathcal{O}(1)$, we only have to consider $\phi_1(z,w)$. We want to show, that for these integration paths we have

$$\phi_1(z,w) \leq 1.$$

Here we are in the neighborhood of the singularities $z = \frac{4}{27}$ and $w = \frac{4}{27}$, where the asymptotic expansions (8) are valid. The substitution (7) together with the asymptotic expansion for $\phi_1(z,w)$ suggests to study the parameter

$$A = 1 - \frac{a}{\sqrt{n}}\sqrt{-t},$$

with an $a > 0$ and n sufficiently large.

- Consider first the path $z \in \Gamma_{11}$, which gives

$$-t = e^{i\psi} \quad \text{with} \quad -\frac{\pi}{2} \le \psi \le \frac{\pi}{2},$$

  and from this we get $|A| \le 1$.
- Next we consider $z \in \Gamma_{12} \cup \Gamma_{13}$, which leads with the substitution (7) to $t = r \pm i$ with $0 \le r \le \log^2 n$. We obtain

$$\sqrt{-t} = (1+r^2)^{\frac{1}{4}} e^{i(\frac{\pi}{2} \pm \frac{1}{2}\arctan\frac{1}{r})} = \frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{1+r^2}-r} \pm i\sqrt{\sqrt{1+r^2}+r}\right)$$

  and thus

$$|A|^2 = \begin{cases} 1 - \frac{a}{\sqrt{2n}} + \mathcal{O}(\frac{r}{\sqrt{n}}) + \mathcal{O}(\frac{1}{n}) & \text{for } r \text{ small } (r = \mathcal{O}(1)), \\ 1 - \frac{a}{2\sqrt{rn}} + \mathcal{O}(\frac{1}{\sqrt{r^5 n}}) + \mathcal{O}(\frac{\log^2 n}{n}) & \text{for } r \text{ large } (1 = o(r)). \end{cases}$$

- It remains $z \in \Gamma_{14} \cup \Gamma_{15}$, which gives $t = \log^2 + \frac{\sqrt{2}}{2}r \pm i(1 + \frac{\sqrt{2}}{2}r)$ with $0 \le r \le n^{\frac{1}{2}+\epsilon}$. We obtain

$$\sqrt{-t} = \frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{p^2+q^2}-q} \pm i\sqrt{\sqrt{p^2+q^2}+q}\right),$$

  with $p = 1 + \frac{\sqrt{2}}{2}r$ and $q = \log^2 n + \frac{\sqrt{2}}{2}r$. We get

$$|A|^2 = \begin{cases} 1 - \frac{a}{\sqrt{n}}\frac{p}{\sqrt{q}}(1 + \mathcal{O}((\frac{p}{q})^2)) + \mathcal{O}(\frac{\log^2 n}{n}) & \text{for } r \text{ small } (r = o(\log^2 n)), \\ 1 - \frac{a(\sqrt{1+\sqrt{2}c+c^2}-1-\frac{\sqrt{2}}{2}c)}{\sqrt{2}}\frac{\log^2 n}{\sqrt{n}} + \mathcal{O}(\frac{1}{\sqrt{n}}) + \mathcal{O}(\frac{\log^2 n}{n}) & \text{for } r \sim c\log^2 n, \\ 1 - \sqrt{2-\sqrt{2}}a\frac{\sqrt{r}}{\sqrt{n}} + \mathcal{O}(\frac{\log^2 n}{\sqrt{rn}}) + \mathcal{O}(\frac{n^{\frac{1}{2}+\epsilon}}{n}) & \text{for } r \text{ large } (\log^2 n = o(r)). \end{cases}$$

Thus for n sufficiently large we have $|A| \le 1$ for all paths $z \notin \Gamma_{16}$. These considerations are still valid if we study the parameter $1 - \frac{a}{\sqrt{n}}\sqrt{-t} - \frac{b}{\sqrt{n}}\sqrt{-s}$ (with positive constants $a > 0$, $b > 0$), so we get $|\phi_1(z,w)| \le 1$ for the paths $z \notin \Gamma_{16}$ and $w \in \Gamma_{24} \cup \Gamma_{25}$.

Analogous considerations lead also to the result $|\phi_1(z,w)| \le 1$ for the paths $w \notin \Gamma_{26}$ and $z \in \Gamma_{14} \cup \Gamma_{15}$.

That means, that in all these cases the integral (5) is bounded by

$$|z|^{-n(1-\rho)-2}|w|^{-\rho n} = \mathcal{O}\left(\left(\frac{27}{4}\right)^n e^{-\tilde{C}\log^2 n}\right).$$

6. The height distribution of the nodes

With the now shown equation (6) and the exponentially small remainder integral we finally get

Theorem 3. *The number $F_{n,\rho n,m}$ of non-crossing trees of size n, where the node labelled with ρn has height m is given by*

$$F_{n,\rho n,m} = \frac{m^2}{48\pi n^3(\rho(1-\rho))^{\frac{3}{2}}} \left(\frac{27}{4}\right)^n e^{-\frac{3m^2}{16n\rho(1-\rho)}} \cdot \left(1 + \mathcal{O}\left(\frac{|m|}{n}\right)\right)$$

uniformly for $\epsilon \le \rho \le 1-\epsilon$ and $m \le C\sqrt{n}$, where $\epsilon > 0$, $C > 0$ are arbitrary but fixed.

Either with Stirling's approximation of $T_n = \frac{1}{2n-1}\binom{3n-3}{n-1}$ or directly by means of singularity analysis (compare to [6]) we get from (2) the expansion

$$T_n = \frac{\sqrt{3}n^{-\frac{3}{2}}}{27\sqrt{\pi}} \left(\frac{27}{4}\right)^n \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \tag{18}$$

which leads to

Corollary 4. *Let $j = \rho n$ with $0 < \rho < 1$ and $m = x\sqrt{n} + o(\sqrt{n})$ for $n \to \infty$. Then we have for $x > 0$*

$$\sqrt{n}\frac{F_{n,j,m}}{T_n} = \frac{3\sqrt{3}x^2}{16\sqrt{\pi}(\rho(1-\rho))^{\frac{3}{2}}} e^{-\frac{3x^2}{16\rho(1-\rho)}} + o(1).$$

That means, that the limiting distribution of the normalized height $\frac{X_{n,j}}{\sqrt{n}}$ is for a fixed ratio $\rho = \frac{j}{n}$ with $0 < \rho < 1$ a Maxwell distribution with parameter $\sigma = \sqrt{\frac{8}{3}\rho(1-\rho)}$. The density function $f_\rho(x)$ of the limiting distribution is given by

$$\frac{3\sqrt{3}x^2}{16\sqrt{\pi}(\rho(1-\rho))^{\frac{3}{2}}} e^{-\frac{3x^2}{16\rho(1-\rho)}}.$$

The Maxwell distribution with parameter σ is defined as the distribution $Y = \sqrt{X_1^2 + X_2^2 + X_3^2}$, where the X_i are independently normally distributed random variables $\mathcal{N}(0, \sigma^2)$ with mean 0 and variance σ^2. It has the following density function $f(x)$ and moments $M_s = \mathbb{E}(Y^s)$

$$f(x) = \frac{\sqrt{2}x^2}{\sqrt{\pi}\sigma^3} e^{-\frac{x^2}{2\sigma^2}} \quad \text{for } x > 0; \quad M_s = \frac{2}{\sqrt{\pi}} 2^{\frac{s}{2}} \sigma^s \Gamma\left(\frac{s+3}{2}\right).$$

7. The expectation of the node-heights

From the generating function $F(z,u,v)$, we can also get the expectations $\mathbb{E}(X_{n,j})$ of the height of node j in a non-crossing tree of size n:

$$\mathbb{E}(X_{n,j}) = \frac{[z^n u^j] \frac{\partial}{\partial v} F(z,u,v)\big|_{v=1}}{T_n}. \tag{19}$$

We will use the abbreviation $G_{n,j} = [z^n u^j] \frac{\partial}{\partial v} F(z,u,v)\big|_{v=1}$. Especially we are interested in the expected values for a fixed ration $\rho = \frac{j}{n}$. Differentiating (4) with respect to v and evaluating at $v = 1$ leads to

$$\frac{\partial}{\partial v} F(z,u,v)\Big|_{v=1} = \frac{u\left(T(z) - \frac{2T^2(z)}{(1-B(z))(1-B(zu))} + \frac{T^2(z)(T(z)+T(zu))}{((1-B(z))(1-B(zu))^2}\right)}{\left(1 - \frac{T(z)+T(zu)}{(1-B(z))(1-B(zu))}\right)^2}. \tag{20}$$

With the substitution $w = zu$, we get

$$G_{n,\rho n} = [z^{(1-\rho)n+1} w^{\rho n - 1}] \frac{\phi_3(z,w)}{(1-\phi_1(z,w))^2}.$$

with

$$\phi_3(z,w) = T(z) - \frac{2T^2(z)}{(1-B(z))(1-B(w))} + \frac{T^2(z)(T(z)+T(w))}{((1-B(z))(1-B(w))^2}.$$

To extract coefficients, we use again Cauchy's integration formula, with contours Γ_1 and Γ_2 as described in section 3:

$$G_{n,\rho n} = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{z^{n(1-\rho)+2} w^{\rho n}} \frac{\phi_3(z,w)}{(1-\phi_1(z,w))^2} dz dw, \tag{21}$$

where $\phi_1(z,w)$ is defined as in section 4.

Analogous considerations as in section 5 leads to the fact, that the we get the main term of the integral for the paths $z \in \Gamma_{11} \cup \Gamma_{12} \cup \Gamma_{13}$ and $w \in \Gamma_{21} \cup \Gamma_{22} \cup \Gamma_{23}$ (the details are here omitted).

The substitutions (7) together with the local expansion for $\phi_1(z,w)$ and the expansion

$$\phi_3(z,w) = \frac{1}{9} + \mathcal{O}\left(\sqrt{\frac{|t|}{n}} + \sqrt{\frac{|s|}{n}}\right)$$

leads to the integral

$$G_{n,\rho n} = \frac{1}{9\rho(1-\rho)n^2} \left(\frac{27}{4}\right)^n \frac{1}{(2\pi i)^2} \int_\gamma \int_\gamma e^{-s-t} \frac{dsdt}{\left(\frac{\sqrt{3}}{2\sqrt{1-\rho}}\sqrt{-\frac{t}{n}} + \frac{\sqrt{3}}{2\sqrt{\rho}}\sqrt{-\frac{s}{n}}\right)^2} \cdot \left(1 + \mathcal{O}(\tfrac{1}{\sqrt{n}})\right). \tag{22}$$

The evaluation of the integral follows from the following lemma, which can e. g. be found in [2]:

Lemma 5. *Let γ be a Hankel contour. Then we have for $\alpha, \beta > 0$*

$$\frac{1}{(2\pi i)^2} \int_\gamma \int_\gamma e^{-t-s} \frac{dsdt}{(\alpha\sqrt{-t} + \beta\sqrt{-s})^2} = \frac{2}{\pi} \frac{\alpha\beta}{(\alpha^2+\beta^2)^2}.$$

This gives

$$G_{n,\rho n} = \left(\frac{27}{4}\right)^n \frac{8\sqrt{\rho(1-\rho)}}{27\pi n} \cdot \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right),$$

and with the expansion (18) we finally obtain from (19)

Theorem 6. *The expected height $\mathbb{E}(X_{n,j})$ of the node $j = \rho n$ for $0 < \rho < 1$ in a non-crossing tree of size n is asymptotically given by*

$$\mathbb{E}(X_{n,j}) \sim \frac{8\sqrt{\rho(1-\rho)}}{\sqrt{3\pi}}\sqrt{n}.$$

It should be remarked, that by simplifying expression (20) and extracting coefficients one can also find an explicit formula for the expectations:

Theorem 7. *The expected height $\mathbb{E}(X_{n,j})$ is for $1 \le j \le n$ given by*

$$\mathbb{E}(X_{n,j}) = 2j - 1 + \tfrac{2}{n} - \tfrac{2j}{n} - \\ - \frac{2(2n-1)}{\binom{3(n-1)}{n-1}} \sum_{k=2}^{j} \frac{(3k-5)(3n-3k+1)(j-k)}{(k-1)(2k-3)(n-k+1)(2n-2k+1)} \binom{3(k-2)}{k-2}\binom{3(n-k)}{n-k}.$$

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