Complete bipartite free graphs *

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Abstract

The study of the maximum size $ex(n; K_{t,t})$ of a graph of order n not containing the complete bipartite graph $K_{t,t}$ as a subgraph is the aim of this paper. We show an upper bound for this extremal function that is optimum for infinity related values of n and t. Moreover, we characterize the corresponding family of extremal graphs.

1 Introduction

We deal with the task of finding out solutions for the extremal problem consisting in determining the maximal number of edges in a graph of order n that does not contain a complete bipartite graph $K_{t,t}$ as a subgraph, denoted by the extremal function $ex(n; K_{t,t})$. Likewise, we characterize the corresponding family of extremal graphs $EX(n; K_{t,t})$, that is, the family of graphs of order n and size $ex(n; K_{t,t})$ not containing $K_{t,t}$ as a subgraph.

Related with this problem, we have the known extremal problem by Zarankiewicz [9, 8, 12]. Given $G_2(n)$ a bipartite graph with n vertices in each vertex class, the extremal function $z(n, t)$ denotes the maximal number of edges in $G_2(n)$, in such a way that $G_2(n)$ does not contain a copy of the complete bipartite subgraph $K_{t,t}$.

The functions $ex(n; K_{t,t})$ and $z(n; t)$ are intimately connected. For any fixed value of t, it is easy to check that

$$2ex(n; K_{t,t}) \leq z(n; t) \leq ex(2n; K_{t,t}). \quad (1)$$

Combining expression (1) with certain upperbounds for $z(n; t)$, proved in [9], we have the next assertion.

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Theorem 1 *(See [9]) Given n and t two positive integers, with $2 \leq t < n$, we have that*

$$\begin{aligned}\frac{1}{2}\left(1-(t!)^{-2}\right)n^{2-2/(t+1)} &\leq ex(n;K_{t,t})\\ &\leq \frac{1}{2}(t-1)^{1/t}(n-t+1)\,n^{1-1/t}+\frac{1}{2}(t-1)n\\ &< n^{2-1/t}+\frac{t-1}{2}n.\end{aligned}$$

Taking into account asymptotic results regarding to $z(n;2)$ by P. Kövari, V.T. Sós and P. Turán [10] and by I. Reiman [11], we have that $ex(n;K_{2,2})$ is of $n^{3/2}$. These results, joined to inequality (1), imply that $ex(n;K_{2,2}) \leq \frac{n}{4}\left(1+\sqrt{4n-3}\right)$. P. Erdös, A. Rnyi and V.T. Sós [6] noticed that certain graphs constructed by Erdös and Rényi [5] show that inequality (1) is asymptotically the best possible. This was also proved independently by W.G. Brown [2]. Finally, Z. Füredi [7] determined $ex(n;K_{2,2})$ for infinitely many values of n.

Theorem 2 *(See [7]) For every natural number q we have that*

$$ex(q^2+q+1;K_{2,2}) \leq \frac{1}{2}q(q+1)^2.$$

This bound is optimum when q is a prime power.

By applying a result of W.G. Brown [2] regarding to $z(n;3)$, we also have that $ex(n;K_{3,3})$ has order $n^{5/3}$.

In the remaining non mentioned cases, exact values of $ex(n;K_{t,t})$ and extremal graphs are unknown. Our purpose in this work is to analyze what happens when n and t are related and the difference $n-2t$ is a small value. Inequality (1) is best possible asymptotically, but however, it is very far from the exact value for the extremal function when $n-2t$ is not large. For this reason our goal is to prove another upperbound for the extremal function that approaches the exact value for infinitely many pairs of values (n,t). In fact, we will deduce that this bound is optimum for $n=2t$ and $n=2t+1$. Moreover, we will characterize the corresponding family of extremal graphs.

2 Definitions and Notations

As usual, it is said that a graph G contains the complete bipartite $K_{t,t}$ as a subgraph, if it is possible to find out two disjoint subsets of t vertices, U and V, in the set of vertices of G, in such a way that every vertex of U is adjacent to each one of V.

For a graph G, we denote by $E(G)$ the set of edges of G and by $e(G)$ the cardinality of this set; we also denote by $v(G)$ the cardinality of the set of vertices of G. We denote by $\delta_G(v)$ the degree of v in the graph G, for any vertex v of G, and by $\Delta(G)$ the maximum degree of G. We also denote by C_s the cycle with s vertices.

In order to avoid excessive repetitions, given a graph H, we will say that $\{v_1, \ldots, v_r\}$ is a decreasing sequence of vertices in H, when it is verified that

$$\delta_{H_{j-1}}(v_j) = \max_{v \in V(H_{j-1})} \{\delta_{H_{j-1}}(v)\} \text{ for each } j = 1, \ldots, r,$$

where $H_0 = H$ and H_j is the resultant graph from H by removing the set $\{v_1, \ldots, v_j\}$.

Using terminology defined in [3], we say that a graph is bisectable when its vertex set can be partitioned into two parts of equal size such that there are no edges between these two parts.

Notations and terminologies not explicitly given here can be found in [1].

3 Main results

Our purpose in this section is to show a sufficient condition to guarantee that a graph contains the complete bipartite graph $K_{t,t}$ as a subgraph. This result will take us to deduce an upperbound for the extremal function $ex(n; K_{t,t})$ that approaches the exact value when n and t are related and the number $n - 2t$ is not very large. In fact, we will check that the bound is optimum in the cases $n = 2t$ and $n = 2t + 1$. Finally, for these cases we will also characterize the corresponding family $EX(n; K_{t,t})$ of extremal graphs.

In order to get these goals, we will use this result shown in [4]

Lemma 1 *Let k be a nonnegative integer and H a graph with maximum degree 2 and at least $3k + 1$ vertices of maximum degree. Then, at least $k + 1$ of them are independent.*

For any graph G with certain order and certain size, the following two results provide an upper bound for the maximum size of every subgraph of G of order $2t$.

Lemma 2 *Let n and t be two positive integers, with $n \geq 2t$. Let H be a graph with $n - i$ vertices and, at most, $2n - 2i - 3t - 1$ edges, for $i \in \{0, \ldots, n - 2t\}$. If the maximum degree of H is, at most, 2, then there exists a subset of $n - i - 2t$ vertices of H, $\{v_1, \ldots, v_{n-i-2t}\}$ in such a way that the resultant graph of H by removing these vertices has, at most, $t - 1$ edges.*

Proof.

Let m be a nonnegative integer such that $e(H) = 2n - 2i - 3t - 1 - m$. For $i = n - 2t$, the result is evident, because $e(H) = t - 1 - m$. Suppose $i \leq n - 2t - 1$. In this case, two cases are possible:

If $2e(H) - v(H) > 0$ then $\Delta(H) = 2$ and H has, at least, $3(n - i - 2t - 1) + 1$ vertices of degree 2. So, by applying Lemma 1, at least, $n - i - 2t$ of them are independent. But in this case, denoting by H^* the resultant graph by removing these $n - i - 2t$ vertices, we deduce that

$$e(H^*) = e(H) - 2(n - i - 2t) = t - 1.$$

If $2e(H) - v(H) \leq 0$ then $2m \geq 3(n - i - 2t - 1) + 1$. Thus, if $\{v_1, \ldots, v_{n-i-2t}\}$ is a decreasing sequence of vertices of H and H^* is the resultant graph by removing these vertices, we have that

$$\begin{aligned} e(H^*) &\leq e(H) - (n - i - 2t) \\ &= n - i - t - 1 - 2m \\ &\leq -2n + 2i + 5t + 1 \\ &\leq -2n + (2n - 4t - 2) + 5t + 1 = t - 1. \end{aligned}$$

□

Lemma 3 *Let n and t be two positive integers, with $n \geq 2t$. Let H be a graph with n vertices and $2n - 3t - 1$ edges. Then there exists a subset of $n - 2t$ vertices of H, $\{v_1, \ldots, v_{n-2t}\}$ in such a way that the resultant graph of H by removing these vertices has, at most $t - 1$ edges.*

Proof.

If the maximum degree of H is, at most 2, then, by applying Lemma 2 for $i = 0$, the result is immediate. Suppose that $\Delta(H) \geq 3$.

Let $\{v_1, \ldots, v_{n-2t}\}$ a decreasing sequence of vertices of H.

If $\Delta(H_j) \geq 3$, for each $j \in \{1, \ldots, n - 2t\}$, then

$$e(H_{n-2t}) \leq e(H) - 3(n - 2t) = 3t - n - 1 \leq t - 1.$$

If there exists $j \in \{1, \ldots, n - 2t - 1\}$ such that $\Delta(H_{j-1}) \geq 3$ and $\Delta(H_j) \leq 2$, then the graph H_j has $v(H_j) = n - j$ vertices and $e(H_j) \leq 2n - 3t - 1 - 3j \leq 2n - 3t - 1 - 2j$ edges. So, by applying Lemma 2, there exists a set of vertices of H_j, $\{w_1, \ldots, w_{n-2t-j}\}$, such that $e\left((H_j)_{n-2t-j}\right) \leq t - 1$. Thus, the resultant graph of H obtained by removing the set of vertices $\{v_1, \ldots, v_j, w_1, \ldots, w_{n-2t-j}\}$ has, at most $t - 1$ edges. □

The following step consists in showing that every graph of order $2t$ and size, at most, $t - 1$, is always bisectable. This is the goal of the next result.

Lemma 4 *Given a positive integer t, every graph H of order $2t$ and size, at most, $t-1$, is bisectable.*

Proof. We denote by $H_1, \dots, H_k$, with $k \in \{1, \dots, t-1\}$, the connected components of H with size, at least, 1. We can suppose that

$$v(H_1) \leq v(H_2) \leq \dots \leq v(H_k).$$

If $k = 1$ the result is trivial, because $v(H_1) \leq e(H_1) + 1 \leq t$. Then, suppose that $k \geq 2$. We consider the following disjoint subsets of vertices of H :

$$U^* = \bigcup_{i=1}^{\lfloor \frac{k}{2} \rfloor} V(H_i) \text{ and } V^* = \bigcup_{\lfloor \frac{k}{2} \rfloor + 1}^{k} V(H_i).$$

It is clear, by construction, that $|U^*| \leq |V^*|$. So, it suffices to show that $|V^*| \leq t$.

For that, we suppose that $|V^*| \geq t + 1$. We know that

$$\begin{aligned} t-1 &\geq \sum_{i=1}^{k} e(H_i) \\ &\geq \sum_{i=1}^{k} (v(H_i) - 1) \\ &= |U^*| + |V^*| - k \\ &\geq |U^*| + t + 1 - k. \end{aligned}$$

Thus, $k - 2 \geq |U^*| \geq \left\lfloor \frac{k}{2} \right\rfloor \cdot v(H_1) \geq \frac{k-1}{2} v(H_1)$. Therefore, $v(H_1) \leq 1$ and this is not possible, because $e(H_1) \geq 1$.

Hence, $|V^*| \leq t$ and this proves the result. □

The previous results permit us to deduce the following upper bound for the function $ex(n; K_{t,t})$.

Theorem 3 *Given n and t two positive integers, with $n \geq 2t$, it is verified that*

$$ex(n; K_{t,t}) \leq \binom{n}{2} - (2n - 3t).$$

Proof. Let G be a graph of order n and size, at least, $\binom{n}{2} - (2n - 3t) + 1$ and we denote by $H = \overline{G}$ its complement graph. H has order n and size, at most, $2n - 3t - 1$. So, by applying Lemma 3, there exists a subset of $n - 2t$ vertices of H, $\{v_1, \dots, v_{n-2t}\}$, in such a way that the resultant graph H_{n-2t} of H by removing these vertices has, at most, $t - 1$ edges. And, by applying Lemma 4, the graph H_{n-2t} is bisectable.

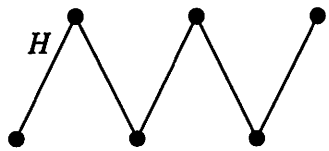

Figure 1: Structure of H for $n = 10$ and $t = 5$.

Hence, G contains $K_{t,t}$ as a subgraph and the result follows. □

Let's consider the graph G with $2t$ vertices whose complement graph H is formed by a path with $(t+1)$ vertices and $t-1$ isolated vertices (see Figure 1). It is evident that the graph G does not contain $K_{t,t}$ as a subgraph, because H is not bisectable.

Analogously, let G^* be the graph of order $2t+1$ whose complement graph H^* is formed by a cycle of length $t+2$ and $t-1$ isolated vertices (see Figure 2). It is impossible to remove one vertex v of H^* in such a way that the resultant graph $H - v$ is bisectable. So, G^* does not contain $K_{t,t}$ as a subgraph.

Hence, we may deduce that inequality shown in Theorem 3 is optimum, as we express in this corollary.

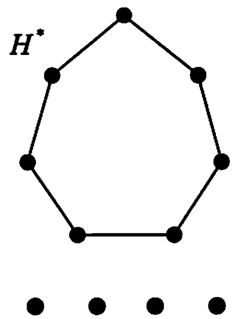

Figure 2: Structure of H^* for $n = 11$ and $t = 5$.

Corollary 1 *Let n and t two positive integers, with $2t \leq n \leq 2t+1$. Then*

$$ex\,(n; K_{t,t}) = \binom{n}{2} - (2n - 3t).$$

From now on, our goal in this work is to find out all the graphs with order $n = 2t$ and $n = 2t+1$, and extremal size, not containing the complete bipartite graph $K_{t,t}$ as a subgraph. In other words, we will characterize the families $EX(2t; K_{t,t})$ and $EX(2t+1; K_{t,t})$.

Theorem 4 *For each positive integer t we have that*
- *If t is even, then*

$$EX(2t; K_{t,t}) = \left\{\overline{T} + K^{t-1} : \; T \textit{ tree of order } t+1\right\}.$$

- *If t is odd, then*

$$EX(2t; K_{t,t}) = \left\{\overline{T} + K^{t-1} : \; T \textit{ tree of order } t+1\right\} \cup \left\{t\overline{K^2}\right\}.$$

Proof.
It sufficies to show that if $G \in EX(2t; K_{t,t})$, then its complement graph H is formed by one tree of order $t+1$ and $t-1$ isolated vertices or is formed by t disjoint edges (if t is odd), because the other contention is immediate.

Let G be a graph belonging to the family $EX(2t; K_{t,t})$ and we denote by $H = \overline{G}$ its complement graph. Applying Corollary 1, we have that H has $2t$ vertices and t edges. We denote by $H_1, \dots, H_k$, with $k \in \{1, \dots, t\}$, the connected components of H with size, at least, 1. As in the proof of Lemma 4, we can suppose that $v(H_1) \le v(H_2) \le \dots \le v(H_k)$.

Since H is not bisectable, if $k = 1$, then $v(H_1) \ge t+1$. But in this case,

$$t \le v(H_1) - 1 \le e(H_1) = t.$$

Thus, H_1 is a connected graph of order $t+1$ and size t. So, H_1 is a tree with $t+1$ vertices and, therefore, H is formed by a tree with $t+1$ vertices and $t-1$ isolated vertices.

Now, we suppose that $k \ge 2$. Let $j \ge 1$ be an integer such that $\sum_{i=1}^{j} v(H_i) \le t$ and $\sum_{i=1}^{j+1} v(H_i) \ge t+1$.

On the one hand, since H is not bisectable, we have that

$$\sum_{i=j+1}^{k} v(H_i) \ge t+1. \tag{2}$$

On the other hand,

$$j \ge \left\lfloor \frac{k}{2} \right\rfloor, \tag{3}$$

because, otherwise,

$$2t \ge \sum_{i=1}^{k} v(H_i) \ge 2\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} v(H_i) \ge 2\sum_{i=1}^{j+1} v(H_i) \ge 2(t+1) > 2t$$

and this is not possible.

So, using inequalities (2) and (3), we deduce that

$$t+1 \le \sum_{i=j+1}^{k} v(H_i) \le \sum_{i=j+1}^{k} e(H_i)+k-j = t-\sum_{i=1}^{j} e(H_i)+k-j \le t+k-2j \quad (4)$$

If k is even, then $t+k-2j \le t$ and this is not possible by applying (4).

If k is odd, then, by (3) we have that $j \ge \frac{k-1}{2}$ and, therefore, using (4) we deduce that

$$j=\frac{k-1}{2}, \ \sum_{i=1}^{j} e(H_i)=\frac{k-1}{2}=j \text{ and } \sum_{i=j+1}^{k} v(H_i)=t+1.$$

And, since $e(H_i) \ge 1$ for all $i=1,\dots,k$, then $e(H_i)=1$, for all $i=1,\dots,j$, i.e., $H_i=K^2$, for each $i \in \left\{1,\dots,\frac{k-1}{2}\right\}$.

But, since $\sum_{i=1}^{\frac{k+1}{2}} v(H_i) \ge t+1$, we have that $v\left(H_{\frac{k+1}{2}}\right) \ge t+2-k$ and therefore,

$$t=\sum_{i=1}^{k} e(H_i)=\frac{k-1}{2}+\sum_{i=\frac{k+1}{2}}^{k} e(H_i) \ge \sum_{i=\frac{k+1}{2}}^{k} v(H_i)-1 \ge \frac{k+1}{2}(t+2-k)-1 \ge t$$

because $k \in \{2,\dots,t\}$.

Thus, $\frac{k+1}{2}(t+2-k)-1=t$. Therefore, $k=t$ and $H_i=K^2$, for all $i \in \left\{\frac{k+1}{2},\dots,k\right\}$. Hence, t is odd and H is formed by t disjoint edges. □

Theorem 5 *For each positive integer t it is verified that*

$$EX(2t+1;K_{t,t})=\left\{\overline{C_{t+2}}+K^{t-1}\right\}.$$

Proof. It is sufficient to show that if $G \in EX(2t+1;K_{t,t})$, then its complement graph H is formed by one cycle of order $t+2$ and $t-1$ isolated vertices.

Let G be a graph belonging to the family $EX(2t+1; K_{t,t})$ and we denote by $H = \overline{G}$ its complement graph. Applying Corollary 1, we know that H has $2t+1$ vertices and $t+2$ edges.

It is clear that $\Delta(H) = 2$, because otherwise, we can remove one vertex of degree, at least, 3 from H and the resultant graph H^* has order $2t$ and size, at most, $t-1$. But in this case, by applying Lemma 4, H^* is bisectable and, therefore, G contains $K_{t,t}$ as a subgraph, and this is not possible.

Moreover, since G is an extremal graph and the maximum degree of H is 2, by applying Corollary 1 and Theorem 4, for each removed vertex of H with degree 2, there are only two possible resultant graphs,

$$P_{t+1} \cup (t-1)K^1 \text{ or } tK^2 \text{ if } t \text{ is odd.}$$

But in this case, the unique possible graph verifying this consequence is $H = C_{t+2} \cup (t-1)K^1$, and this proves the result. □

4 Conclusions

In this work we have studied the extremal function $ex(n; K_{t,t})$ for related values of n and t being $n - 2t$ non very large. We have found out the exact value and characterized the family of extremal graphs when $2t \leq n \leq 2t+1$. Moreover, we have proved an upper bound that can be a good approach to find out new solutions.

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The Frequency of Summands of a Particular Size in Palindromic Compositions

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Abstract

A composition of a positive integer n consists of an ordered sequence of positive integers whose sum is n. A palindromic composition is one for which the sequence is the same from left to right as from right to left. This paper shows various ways of generating all palindromic compositions, counts the number of times each integer appears as a summand among all the palindromic compositions of n, and describes several patterns among the numbers generated in the process of enumeration.

Keywords: Compositions, palindromes, tilings.
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1 Introduction

A *composition* of a positive integer n consists of an ordered sequence of positive integers whose sum is n. It is well-known that there are 2^{n-1} compositions of n (see for example [3]). A *palindromic composition* is one for which the sequence is the same from left to right as from right to left. For the remainder of this paper we will refer to such compositions by the short-hand term *palindrome*. Compositions can also be thought of as tilings of a 1 x n board, with 1 x k tiles of integer length k, $1 \leq k \leq n$. In this setting, a composition of n with j summands or parts is created by making $j - 1$ vertical cuts on the 1 x n board. This viewpoint allows for easy combinatorial proofs of certain facts and will be used when advantageous.

The question concerning the number of times a particular summand k occurs in all compositions of n has been answered by one of the authors in [3]. Furthermore, Chinn et al. showed that the number of times k appears as a summand in compositions of n is equal to the number of times $k+1$ appears in compositions of $n+1$. Alladi and Hoggatt enumerated the number of times the summands 1 and 2 occur in all compositions and palindromes containing only these two summands [1]. Grimaldi has investigated compositions with odd summands, and expressed the number of times a 1 occurs in all compositions of n with odd summands as a specific linear combination of Lucas and Fibonacci numbers [4]. Furthermore, the occurrence of the number $2k + 1$ in all compositions of n with odd summands equals the number of 1s in all compositions of $n - 2k$ with odd summands. We will show a somewhat similar result for palindromes, namely that the number of times the summand k occurs in a palindrome of a specific size can sometimes be reduced to the number of 1s in all palindromes of a certain smaller size. In addition, the sequence of values of occurences of 1s in palindromes of even and odd values of n, respectively, matches known sequences (A057711 and A001792 in [7]).

Section 2 contains notation and a few basic observations that will be used throughout the rest of the paper. In Section 3, we describe two methods of generating palindromes, and give a formula for the total number of palindromes. Section 4 contains explicit formulas for $R_n(k)$, the number of times the number k occurs as a summand among all the palindromes of n. We conclude in Sections 5 and 6 by discussing the various patterns found within the table of values for $R_n(k)$, and give combinatorial or analytical proofs for these patterns.

2 Notation and General Observations

Before deriving specific results, we will define our notation, and state a remark which will be used in later sections. Let

$$\begin{aligned} C_n &= \text{the number of compositions of } n \text{, where } C_0 := 1 \\ P_n &= \text{the number of palindromes of } n \text{, where } P_0 := 1 \\ R_n(k) &= \text{the number of repetitions of the integer } k \text{ in all} \\ &\quad\ \text{palindromes of } n. \end{aligned}$$

Remark 1 *1. A palindrome of an odd integer n always has an odd number of summands, and the middle summand must be an odd integer.*

2. A palindrome of an even integer n can have an odd number of summands with an even summand in the center or an even number of summands and no middle summand.

We will refer to a palindrome of the latter type as having an *even split*.

3 Generating Palindromes

Palindromes can be created in a number of ways, each of which is useful for some of the proofs in this section. In addition, these different creation methods illustrate the multiple ways of thinking about palindromes. The first method creates palindromes using compositions, whereas the second method creates palindromes recursively. We start by describing the explicit method of palindrome creation, which consists of combining all possible middle summands with a composition of an appropriate positive integer to the left, and with its mirror image on the right. This method will be referred to as the *Explicit Palindrome Creation Method* (EPCM):

> To create a palindrome of $n = 2k$ ($n = 2k + 1$), combine the middle summand $m = 2l$ ($m = 2l + 1$), for $l = 0, ..., k$, with a composition of $\frac{n-m}{2} = k - l$ on the left and its mirror image on the right. For those palindromes that result from $l = 0$, delete the middle summand of 0.

The second method creates palindromes recursively; to seed this method, we define a palindrome of $n = 0$, namely 0. We will refer to this method as the *Recursive Palindrome Creation Method* (RPCM):

> Before applying the algorithm, create a middle summand for palindromes with an even number of summands by replacing the "+" sign in the center of the palindrome by "+0+". (This artifice simplifies the algorithm and allows the treatment of palindromes having an odd and even number of summands, respectively, using the same instructions.)
>
> 1. **Creating palindromes of $2k + 1$ from those of $2k$:** Increase the middle summand by 1.
> 2. **Creating palindromes of $2k + 2$ from those of $2k$:** Create one palindrome by increasing the middle summand by 2, and another one by replacing the middle summand m by $\left(\frac{m}{2} + 1\right) + \left(\frac{m}{2} + 1\right)$.

Lemma 2 *Both the EPCM and the RPCM create all palindromes of n for $n \geq 1$.*

Proof: Clearly, the EPCM creates all palindromes of n, without duplicates or omissions. For the RPCM, we need to work a little harder to show that indeed no duplicates are created, and also that all possible palindromes are created by the algorithm. For easier readability we will refer to the middle summand(s) of a palindrome of n as m_n. Furthermore, we will only concentrate on the middle summands, as all other summands remain unchanged when creating the palindromes of $2k+1$ and $2k+2$, respectively, from those of $2k$.
• Palindromes of $2k + 1$: Every palindrome of $2k + 1$ with middle summand m_{2k+1} corresponds to a palindrome of $2k$ whose middle summand is $m_{2k+1} - 1$. (If $m_{2k+1} = 1$, then the corresponding palindrome of $2k$ is the one where the dummy 0 summand is deleted.)
• Palindromes of $2k + 2$: No duplicates are created as distinct palindromes of $2k$ lead to distinct palindromes of $2k + 2$ for each instruction. Furthermore, the first instruction creates palindromes with an odd number of summands, whereas the second instruction creates palindromes with an even number of summands. Thus, if a palindrome of $2k + 2$ has an odd number of summands, then it is created from the palindrome of $2k$ whose middle summand is $m_{2k+2} - 2$. If, on the other hand, the palindrome of $2k + 2$ has an even number of summands, then it is created from the palindrome of $2k$ whose middle summand is $2 \cdot (m_{2k+2} - 1)$. (If $m_{2k} = 0$, then delete the dummy 0 summand.)

• Initial conditions: This algorithm creates the one palindrome of $n = 1$, namely 1, and the two palindromes of $n = 2$, namely 2 and $1+1$, from the initial condition. □

The recursive method immediately shows some of the structure within the palindromes.

Remark 3 *1. The first rule of the RPCM demonstrates that half of the palindromes of an odd integer n have a 1 as the middle summand (since half of the palindromes of $n-1$ had a dummy zero summand).*

2. The second rule of the RPCM illustrates that half of all the palindromes of an even integer n have an even number of summands.

Using either the RPCM or the EPCM, we can easily determine the total number of palindromes of n.

Theorem 4 *For $k \geq 0$, $P_{2k} = P_{2k+1} = 2^k$, where $P_0 := 1$.*

Proof: In the RPCM, the number of palindromes stays the same when creating the palindromes of $2k+1$ from those of $2k$, and the number of palindromes doubles when creating the palindromes of $2k+2$. Thus,

$$P_{2k+1} = P_{2k} \quad \text{and} \quad P_{2k} = 2P_{2(k-1)} = 2^2 P_{2(k-2)} = \ldots = 2^{k-1} P_2 = 2^k$$

which completes the proof. □

4 The Frequency of k in Palindromes of n

The question regarding how many times the summand k appears among all the palindromes of n is motivated by the comparable question regarding compositions as explored in [3]. The following theorem is proved in that paper.

Theorem 5 *The number of repetitions of the integer k in all of the compositions of n is $(n-k+3) \cdot 2^{n-k-2}$ for $n > k$ and 1 for $n = k$.*

The following theorem states the corresponding result for palindromes. We need to consider different cases according to whether or not n and k have the same parity, and also according to the relative size of n and k. In particular, we get a different pattern when n is too small to accommodate two summands of k within a single palindrome.

Theorem 6 *For $n < k$, $R_n(k) = 0$. If n and k have different parity, then*

$$R_n(k) = \begin{cases} 0 & k < n < 2k \\ 2^{\lfloor n/2 \rfloor - k}(2 + \lfloor \frac{n}{2} \rfloor - k) & n \geq 2k \end{cases}.$$

If n and k have the same parity, then

$$R_n(k) = \begin{cases} 1 & n = k \\ 2^{(n-k)/2-1} & k < n < 2k \\ 2^{\lfloor n/2 \rfloor - k}(2 + \lfloor \frac{n}{2} \rfloor - k + 2^{\lfloor \frac{k+1}{2} - 1 \rfloor}) & n \geq 2k \end{cases}.$$

Proof: Let $n = 2i$ or $n = 2i + 1$, and $k = 2j$ or $2j + 1$, respectively. For $n < k$, the palindrome cannot contain the summand k. If $n = k$, then there is exactly one palindrome that contains the summand k, namely just k by itself. If $k < n < 2k$, then the summand k can occur at most once in any palindrome, and hence has to occur in the center. This is only possible if n and k have the same parity (by Remark 1), which implies that $R_n(k) = 0$ if n and k have different parity. If they have the same parity, then the palindromes that have the summand k in the center can be created using the explicit method. Thus, the number of repetitions of k is given by the number of compositions of size $(\frac{n-k}{2}) = i - j$, which gives $R_n(k) = 2^{i-j-1}$.

If $n \geq 2k$, then the summand k can occur in the center, or in symmetric pairs at other positions within the palindrome. To count the different cases, we will think of the palindrome as a 1 x n board as illustrated in Figure 1.

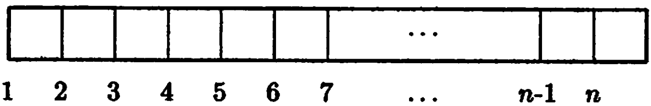

Figure 1: Palindrome as a 1 x n board

We will count according to whether a tile of length k starts at position s, for $1 \leq s \leq i - k + 1$, as we will only look at the left half of the tiling. Tilings that contain a tile of size k starting at position s can be created by combining the tile of size k with any tiling (i.e., composition) of length $s-1$ on the left, and a symmetric tiling (i.e., palindrome) of length $n-2(s-1)-2k$ on the right, and then completing the remainder of the tiling symmetrically. If n and k have the same parity, we also get occurrences of k in the center.

We look first at the case where n and k have different parity:

$$\begin{aligned} R_n(k) &= 2 \cdot \sum_{s=1}^{i-k+1} C_{s-1} \cdot P_{n-2(s+k-1)} = 2 \cdot \sum_{s=1}^{i-k+1} C_{s-1} \cdot P_{2(i-s-k+1)} \\ &= 2 \cdot C_0 \cdot P_{2(i-k)} + 2 \cdot \sum_{s=2}^{i-k+1} 2^{s-2} \cdot 2^{i-s-k+1} \\ &= 2 \cdot 1 \cdot 2^{i-k} + 2 \cdot 2^{i-k-1} \cdot (i-k) = 2^{i-k}(2+i-k) \end{aligned} \tag{1}$$

which gives the formula for $R_n(k)$ for $n \geq 2k$ where n and k have different parity.

Lastly, we consider the case where n and k have the same parity and $n \geq 2k$. In this case, the number of occurrences of k is given by off-center ones (as counted in Eq. (1)), plus those that occur in the center. The latter is given by $C_{i-j} = 2^{i-j-1}$ (see the case $k < n < 2k$). Altogether,

$$\begin{aligned} R_n(k) &= 2^{i-k}(2+i-k) + 2^{i-j-1} \\ &= \begin{cases} 2^{i-k}(2+i-k+2^{j-1}) & \text{if } k = 2j \\ 2^{i-k}(2+i-k+2^{j}) & \text{if } k = 2j+1 \end{cases} \end{aligned}$$

which proves the formula for the case $n \geq 2k$ where n and k have the same parity. These two cases can be written using a single formula by noting that $\lfloor \frac{k+1}{2} - 1 \rfloor$ gives the correct powers of $j-1$ and j, respectively. □

Table 1 displays the values of $R_n(k)$ that arise from the formulas given in Theorem 6. Examining the values in Table 1 led the authors to observe a variety of patterns. Some of these follow from combinatorial arguments while others just seem to be consequences of the formulas given in Theorem 6. In Section 5 we will present those patterns that hold across the table, and give combinatorial proofs for them. Patterns that hold only for specific columns will be discussed in Section 6. As before, we let $n = 2i$ or $n = 2i + 1$, and $k = 2j$ or $k = 2j + 1$, respectively.

5 General Patterns in the Repetitions of k in all Palindromes of n

The most striking pattern in the table is the equality of certain diagonally adjacent entries. Furthermore, diagonal sequences that start in column 1

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1									
2	2	1								
3	3	0	1							
4	6	3	0	1						
5	8	2	1	0	1					
6	16	8	2	1	0	1				
7	20	6	4	0	1	0	1			
8	40	20	6	4	0	1	0	1		
9	48	16	10	2	2	0	1	0	1	
10	96	48	16	10	2	2	0	1	0	1
11	112	40	24	6	6	0	2	0	1	0
12	224	112	40	24	6	6	0	2	0	1
13	256	96	56	16	14	2	4	0	2	0
14	512	256	96	56	16	14	2	4	0	2
15	576	224	128	40	32	6	10	0	4	0
16	1152	576	224	128	40	32	6	10	0	4
17	1280	512	288	96	72	16	22	2	8	0
18	2560	1280	512	288	96	72	16	22	2	8
19	2816	1152	640	224	160	40	48	6	18	0
20	5632	2816	1152	640	224	160	40	48	6	18
21	6144	2560	1408	512	352	96	104	16	38	2
22	12288	6144	2560	1408	512	352	96	104	16	38

Table 1: The number of occurrences of k among all palindromes of n

for $n = 2i$ are repeated on the diagonal that starts in row $2i + 2$, with two new entries inserted at the beginning of the lower diagonal. Note also that the values that occur on these diagonals are comprised of the values for even rows in column 1 (above the starting row for the diagonal), in reverse order.

Theorem 7
a) $R_{2i+1}(2j) = R_{2i+2}(2j+1)$ *for* $i \geq j \geq 1$.
b) $R_{2i}(2j-1) = R_{2i+3}(2j)$, *for* $i \geq j \geq 1$.
c) $R_{2i+2l}(2l+1) = R_{2i-2l}(1)$ *for* $l \geq 1$.

Proof: a) To show the first equality, note that a palindrome of an odd integer n must have an odd middle summand; thus, no copy of $2j$ occurs in the center. For $i \geq j$, pairs of $(2j)$s can occur. For each pair of symmetrically located occurrences of $2j$ in a palindrome of $2i + 1$, there is a corresponding palindrome of $2i + 2$ which has a pair of symmetrically located occurrences of $2j + 1$ and whose middle summand is decreased by one. Since a palindrome of an even integer n cannot have $2j + 1$ as the middle summand, the number of occurrences of $2j$ in the palindromes of $2i + 1$ equals the number of occurrences of $2j + 1$ in the palindromes of $2i + 2$.

b) To show the second equality, which together with part a) leads to the repeated diagonals, we make a similar argument. Since a palindrome of an even integer n must have an even middle summand (possibly 0), no copy of $2j - 1$ occurs in the center. For $i \geq j$, pairs of $(2j - 1)$s can occur. For each pair of symmetrically located occurrences of $2j - 1$ in a palindrome of $2i$, there is a corresponding palindrome of $2i + 3$ which has a pair of symmetrically located occurrences of $2j$ and whose middle summand is increased by 1. Since the palindrome of $2i + 3$ cannot have an even summand in the center, there is a one-to-one correspondence between the occurrences of the $(2j - 1)$s in the palindromes of $2i$ and the $(2j)$s in the palindromes of $2i + 3$.

c) Both $2i + 2l$ and $2i - 2l$ are even, and we are counting the number of occurrences of $2l + 1$ and 1, respectively. Neither of these can occur in the center of the palindromes. To make the association between the palindromes of the two sizes, we think of the palindrome as a symmetric tiling. For a tiling of length $2i - 2l$ which has at least one pair of 1 x 1 tiles, replace one pair of 1 x 1 tiles with a pair of 1 x $2l$ tiles. This increases the length of the tiling to $2i - 2l + 2(2l) = 2i + 2l$, and each pair of 1s in the shorter tiling has an associated pair of $(2l + 1)$s in the longer tiling. Thus, the number of 1s in the palindromes of $2i - 2l$ equals the number of $(2l + 1)$s in the palindromes of $2i + 2l$. Figure 2 illustrates this process for $i = 3$ and $l = 1$ to show that $R_8(3) = R_4(1)$. There are two palindromes of 4 that contain 1s: 1+1+1+1 and 1+2+1, and 3 palindromes of 8 that contain 3s: 1+3+3+1, 3+1+1+3, and 3+2+3.

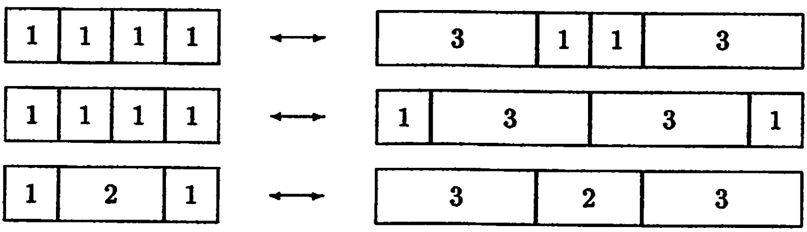

Figure 2: Replacing pairs of 1s by pairs of $(2l + 1)$s

Note that a palindrome of $2i-2l$ with j pairs of 1s will have j palindromes of $2i+2l$ associated with it. However, the correspondence of the pairs is one-to-one. □

For diagonals that start in column 1 in a row for odd n, we only get equality of adjacent pairs, but not a repetition of the whole diagonal sequence.

Theorem 8 $R_{2i+1}(2j-1) = R_{2i+2}(2j)$ *for* $i \geq j \geq 1$.

Proof: A palindrome of an odd integer must have an odd middle summand. If this middle summand is $2j-1$, increase it by 1 to get a palindrome of $2i+2$ with middle summand $2j$. For $i \geq j$, we also get symmetric pairs of $(2j-1)$s. Increase each $2j-1$ by 1 to $2j$, and decrease the middle summand by 1. Thus, there is a one-to-one correspondence between the occurrences of $2j-1$ in the palindromes of $2i+1$ and the occurrences of $2j$ in the palindromes of $2i+2$. □

The next pattern is a bit more complex.

Theorem 9 *The sum of two adjacent entries for even n in an appropriate set of two columns is equal to the sum of the two adjacent entries below them:*

$$R_{2i}(2j) + R_{2i}(2j+1) = R_{2i+1}(2j) + R_{2i+1}(2j+1) \text{ for } i \geq j \geq 1.$$

Proof: Consider any even palindrome. Using the RPCM, the palindromes of the next odd integer are generated by increasing the middle summand by 1. Note, however, that in half of the palindromes of $2i$ this middle summand is a dummy 0 and the increase therefore does not change the number of occurrences of any integer greater than 1; in particular the number of occurrences of $2j$ and $2j+1$ remains unchanged. In the other half of the palindromes of $2i$, the middle summand is even and at least 2. Increasing a middle summand of size $2j$ leads to a loss in the count of $(2j)$s, which is, however, compensated for by an increase in the number of $(2j+1)$s. □

Before stating patterns that are specific to particular columns of Table 1, we will focus on the values of $R_n(1)$ for even and odd values of n, respectively. For $k=1$, the formulas given in Theorem 6 simplify to $R_{2i}(1) = (i+1)\cdot 2^{i-1}$ and $R_{2i-1}(1) = (i+1)\cdot 2^{i+1}$ for $i \geq 1$. For even n, the sequence of values $R_{2i}(1)$, given by {2, 6, 16, 40, 96, 224, 512, 1152, 2560, 5632, 12288,....}, matches the sequence $a(i)$ defined in A057711 of [7] (with $R_{2i}(1) = a(i-1)$), which arises as the number of states in a ferry problem [5]. For odd n, the sequence of values $R_{2i-1}(1)$, given by {1, 3, 8, 20, 48, 112, 256, 576, 1280, 2816, 6144,...}, matches the sequence $a(i)$ defined in A001792 of [7] (with $R_{2i+1}(1) = a(i)$). This sequence arises in several different contexts, for example in generalizations of the Stirling number triangles [6] and as a realization of oligomorphic permutation groups [2].

Now imagine that we "color" all the values that belong to a known sequence. Due to the repeated diagonals, the sequence for $R_{2i}(1)$ occurs in all columns. If k is odd, the sequence occurs in the even rows, and if k is even, it occurs in the odd rows. The first non-zero value, 2, occurs for $n = 2k+1$ when k is even, and for $n = 2k$ when k is odd. If the preceeding zeros are included, then these values fill all the diagonals that start in an even row in column 1, giving a checker-board coloring of the table.

We consider the remaining "uncolored" sequences in each column. In the even rows of column 2, we get the sequence for odd rows of column 1, due to the equality of diagonally adjacent entries, thus column 2 is now completely "colored". Likewise, the remaining "uncolored" sequences in adjacent odd and even columns are the same. We tested these "uncolored" sequences, {4, 10, 24, 56, 128, 288, 640, 1408, 3072,...} (for columns 3 and 4), {6, 14, 32, 72, 160, 352, 768, 1664, 3584,...} (for columns 5 and 6), {10, 22, 48, 104, 224, 480, 1024, 2176, 4608,...} (for columns 7 and 8), {18, 38, 80, 168, 352, 736, 1536, 3200, 6656,...} (for columns 9 and 10), and {34, 70, 144, 296, 608, 1248, 2560, 5248, 10752,...} (for columns 11 and 12), both with and without the entries for $n < 2k$, which are described by a different formula than those for $n \geq 2k$, against the On-Line Encyclopedia of Integer Sequences [7]. (The sequences above list only the values for $n \geq 2k$). The fact that none of these sequences occurs makes it unlikely that sequences for values of $k \geq 13$ are in the encyclopedia; we are therefore in the process of submitting this family of related sequences to the encyclopedia.

6 Specific Patterns in the Repetitions of k in all Palindromes of n

The remaining patterns are specific to particular columns of Table 1. We present only analytical proofs for these, rather than combinatorial ones. The fact that the patterns hold only for specific columns seems to indicate that no general method similar to those used in the proofs in Section 5 is applicable. For each of the following theorems, the range indicated for i ensures that for all values of n and k, $n \geq 2k$ holds.

Theorem 10
a) $R_{2i}(1) = 2 \cdot R_{2i+1}(2) + 2^{i-1}$ *for* $i \geq 2$.
b) $R_{2i}(1) = R_{2i+2}(3) + R_{2i+3}(3)$ *for* $i \geq 2$.

Proof: Using the appropriate formula in Theorem 6, we get:

$$\begin{aligned} R_{2i}(1) &= 2^{i-1}(2+i-1) = 2^{i-1}(i+1), \\ 2 \cdot R_{2i+1}(2) + 2^{i-1} &= 2 \cdot (2^{i-2}(2+i-2)) + 2^{i-1} = 2^{i-1}(i+1), \text{ and} \\ R_{2i+2}(3) + R_{2i+3}(3) &= 2^{(i+1)-3}(2+(i+1)-3) \\ &\quad +2^{(i+1)-3}(2+(i+1)-3+2^1) \\ &= 2^{i-2}(i+i+2) = 2^{i-1}(i+1), \end{aligned}$$

which completes the proof. □

Theorem 11 $R_{2i+1}(1) = R_{2i+4}(3) + R_{2i+3}(3) - R_{2i+2}(3)$ *for* $i \geq 1$.

Proof: From Theorem 6 we get:

$$R_{2i+1}(1) = 2^{i-1}(2+i-1+2^0) = 2^{i-1}(i+2)$$

and

$$\begin{aligned} &R_{2i+4}(3) + R_{2i+3}(3) - R_{2i+2}(3) \\ &\quad = 2^{(i+2)-3}(2+(i+2)-3) + 2^{(i+1)-3}(2+(i+1)-3+2^1) \\ &\qquad -2^{(i+1)-3}(2+(i+1)-3) \\ &\quad = 2^{i-2}\,(2(i+1)+(2+i)-i) = 2^{i-2}(2(i+2)+i-i) \\ &\quad = 2^{i-1}(i+2), \end{aligned}$$

which proves the statement. □

Theorem 12 $R_{2i}(2) = 2 \cdot R_{2i+1}(3)$ *for* $i \geq 3$.

Proof: Again, we use the formula for $R_n(k)$ given in Theorem 6.

$$\begin{aligned} R_{2i}(2) &= 2^{i-2}(2+i-2+2^0) = 2^{i-2}(i+1) \\ &= 2\left[2^{i-3}(2+i-3+2^1)\right] = 2 \cdot R_{2i+1}(3) \end{aligned}$$

which completes the proof. □

The next three theorems seem to have a similar structure, but there is no general underlying pattern. Furthermore, these types of pattern do not seem to occur for larger values of k. The second pattern in Theorem 15 also differs somewhat from the ones of Theorems 13 and 14 in that the values are expressed as a difference rather than as a sum.

Theorem 13
a) $R_{2i+1}(2) = 4 \cdot R_{2i-1}(3)$ *for* $i \geq 4$.
b) $R_{2i+1}(2) = R_{2i+2}(4) + R_{2i+3}(4)$ *for* $i \geq 3$.

Proof: Using Theorem 6,

$$\begin{aligned} R_{2i+1}(2) &= 2^{i-2}(2+i-2) = 2^{i-2} \cdot i, \\ 4 \cdot R_{2i-1}(3) &= 4\left[2^{(i-1)-3}(2+(i-1)-3+2^1)\right] = 2^{i-2} \cdot i, \end{aligned}$$

and

$$\begin{aligned} R_{2i+2}(4) + R_{2i+3}(4) &= 2^{(i+1)-4}(2+(i+1)-4+2^1) \\ &\quad +2^{(i+1)-4}(2+(i+1)-4) \\ &= 2^{i-3}\left[(i+1)+(i-1)\right] = 2^{i-2} \cdot i, \end{aligned}$$

which proves the desired equalities. □

Theorem 14
a) $R_{2i}(3) = 4 \cdot R_{2i-2}(4)$ *for* $i \geq 5$.
b) $R_{2i}(3) = R_{2i}(4) + R_{2i+1}(4)$ *for* $i \geq 4$.

Proof: The formulas for $R_n(k)$ in Theorem 6 give

$$\begin{aligned} R_{2i}(3) &= 2^{i-3}(2+i-3) = 2^{i-3}(i-1), \\ 4 \cdot R_{2i-2}(4) &= 4 \cdot \left[2^{(i-1)-4}(2+(i-1)-4+2^1)\right] = 2^{i-3}(i-1), \end{aligned}$$

and

$$\begin{aligned} R_{2i}(4) + R_{2i+1}(4) &= 2^{i-4}(2+i-4+2^1) + 2^{i-4}(2+i-4) \\ &= 2^{i-4}(i+i-2) = 2^{i-3}(i-1). \end{aligned}$$

This completes the proof. □

Theorem 15
a) $R_{2i}(4) = 4 \cdot R_{2i-1}(5)$ *for* $i \geq 6$.
b) $R_{2i}(4) = R_{2i+3}(4) - R_{2i+2}(5)$ *for* $i \geq 4$.

Proof: Once more we use the formula for $R_n(k)$ given in Theorem 6.

$$\begin{aligned} R_{2i}(4) &= 2^{i-4}(2+i-4+2^1) = 2^{i-4}\cdot i \\ &= 4\cdot\left[2^{(i-1)-5}(2+(i-1)-5+2^2)\right] = 4\cdot R_{2i-1}(5) \end{aligned}$$

and

$$\begin{aligned} R_{2i+3}(4) - R_{2i+2}(5) &= 2^{(i+1)-4}(2+(i+1)-4) \\ &\quad -2^{(i+1)-5}(2+(i+1)-5) \\ &= 2^{i-4}\left[2(i-1)-(i-2)\right] = 2^{i-4}\cdot i, \end{aligned}$$

which completes the proof. □

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