Application of Upper and Lower Bounds for the Domination Number to Vizing's Conjecture

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Abstract. Vizing conjectured that $\gamma(G)\gamma(H) \leq \gamma(G\Box H)$ for all graphs G and H, where $\gamma(G)$ denotes the domination number of G and $G\Box H$ is the Cartesian product of G and H. We prove that if G and H are δ-regular then with only a few possible exceptions Vizing's conjecture holds. We also prove if $\delta(G)$, $\Delta(G)$, $\delta(H)$ and $\Delta(H)$ are in a certain range then Vizing's conjecture holds. In particular, we show that for graphs of order at most n with minimum degrees at least $\sqrt{n}\ln n$, the conjecture holds.

Keywords: graph, domination number, upper and lower bounds, Vizing's conjecture, asymptotics.

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Short Title: Vizing's Conjecture

1 Introduction

In 1963 Vizing [9] conjectured that

(1.1) $$\gamma(G_1)\gamma(G_2) \leq \gamma(G_1\Box G_2)$$

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for all graphs G_1 and G_2 where $\gamma(G)$ denotes the domination number of a graph G and $G_1 \Box G_2$ is the graph with vertex set

$$V(G_1 \Box G_2) = \{(x_1, x_2) \,|\, x_1 \in G_1, x_2 \in G_2\}$$

and adjacency relation $\sim$ defined by $(x_1, x_2) \sim (y_1, y_2)$ if either $x_1 = y_1$ and $\{x_2, y_2\} \in E(G_2)$ or $x_2 = y_2$ and $\{x_1, y_1\} \in E(G_1)$. See [6] for a recent survey of results on this conjecture.

We shall say that a graph G is an (n, δ, Δ)-graph if G has n vertices, minimum degree δ and maximum degree Δ. We say that G is an (n, δ)-graph if it has n vertices and minimum degree δ. If G_1 is an $(n_1, \delta_1, \Delta_1)$ - graph and G_2 is an $(n_2, \delta_2, \Delta_2)$ - graph, then $G_1 \Box G_2$ is an $(n_1 n_2, \delta_1 + \delta_2, \Delta_1 + \Delta_2)$-graph. This is the only property of the graph $G_1 \Box G_2$ our proofs require. Our method is quite simple: It is easy to see that

$$\left\lceil \frac{n_1 n_2}{\Delta_1 + \Delta_2 + 1} \right\rceil \leq \gamma(G_1 \Box G_2).$$

Hence, if we have an upper bound, say, $U(n, \delta, \Delta)$ for the domination number of an arbitrary (n, δ, Δ)-graph, and if

$$(1.2) \qquad U(n_1, \delta_1, \Delta_1) U(n_2, \delta_2, \Delta_2) \leq \left\lceil \frac{n_1 n_2}{\Delta_1 + \Delta_2 + 1} \right\rceil$$

then (1.1) holds whenever G_1 is an $(n_1, \delta_1, \Delta_1)$ - graph and G_2 is an $(n_2, \delta_2, \Delta_2)$ - graph. Surprisingly, (1.2) holds for many values of the parameters for known bounds.

In Section 2 we briefly review a number of different upper bounds that we will apply in the following sections. In Section 3 we prove that Vizing's conjecture holds for all pairs of δ-regular graphs G_1, G_2 if $\delta \geq 27$, $\delta \leq 3$, or both graphs have order at most 15. In Section 5 we show that for all n, if G_1 and G_2 have orders at most n and minimum degrees at least $\sqrt{n} \ln n$ then the pair of graphs G_1, G_2 satisfies Vizing's conjecture. In Section 5 we describe certain pairs of triples $(n_1, \delta_1, \Delta_1)$, $(n_2, \delta_2, \Delta_2)$ for which $\gamma(G_1)\gamma(G_2) \leq \gamma(G_1 \Box G_2)$ holds whenever G_1 is an $(n_1, \delta_1, \Delta_1)$ - graph and G_2 is an $(n_2, \delta_2, \Delta_2)$ - graph. In Section 6 we study the asymptotic behavior of two roots of a transcendental equation which arises from the use of the lower bounds in (2.1). This transcendental equation and the asymptotics of its roots may be of independent interest.

2 Upper bounds for the domination number

We recall two upper bounds that depend only on the order n and the minimum degree δ of a graph G. The first is the Arnautov bound [1]:

$$(2.1) \qquad \gamma(G) \leq \left(\frac{1 + \ln(\delta + 1)}{\delta + 1} \right) n$$

The second is an improvement on (2.1) from [5]:

$$\gamma(G) \leq \left(1 - \prod_{k=1}^{\delta+1} \frac{k}{k+1/\delta}\right) n \tag{2.2}$$

As in [2] we say that a graph is an (n, δ)-graph if it has order n and minimum degree δ, and we define

$$\gamma(n, \delta) = \max\{\gamma(G) \mid G \text{ is an } (n, \delta)\text{-graph}\}.$$

We use the following table from [2] which gives the exact values of $\gamma(n, \delta)$ for $n \leq 16$ with the three exceptions marked by an asterisk. Actually there were 6 undecided entries in [2], but three of them were decided in [3]. In any case, the table gives the best known upper bounds for $\gamma(n, \delta)$-graphs when $n \leq 16$. [Added in Proof: From [4] we now know the correct values of the other three, namely, $\gamma(15, 6) = 4$, $\gamma(16, 5) = 4$ and $\gamma(16, 7) = 3$.]

Table of Values of $\gamma(n, \delta)$ for $1 \leq n \leq 16$															
n/δ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1															
2	1														
3	1	1													
4	2	2	1												
5	2	2	1	1											
6	3	2	2	2	1										
7	3	3	2	2	1	1									
8	4	4	3	2	2	2	1								
9	4	4	3	3	2	2	1	1							
10	5	4	3	3	2	2	2	2	1						
11	5	5	4	3	3	3	2	2	1	1					
12	6	6	4	4	3	3	2	2	2	2	1				
13	6	6	4	4	3	3	3	2	2	2	1	1			
14	7	6	5	4	4	3	3	3	2	2	2	2	1		
15	7	7	5	5	4	*4	3	3	3	2	2	2	1	1	
16	8	8	6	5	*5	4	*4	3	3	3	2	2	2	2	1

3 When $\delta_1 = \delta_2 = \Delta_1 = \Delta_2$

In this section G_1 and G_2 will be δ-regular graphs of orders n_1 and n_2, respectively. We emphasize that the orders of G_1 and G_2 may be different, but both are δ-regular for the same δ. In Section 5 we will show that there are many cases when Vizing's conjecture holds when G_1 is δ_1-regular and G_2 is δ_2-regular and $\delta_1 \neq \delta_2$.

Theorem 1 *If G_1 and G_2 are δ-regular graphs with $\delta \geq 27$ or $\delta \leq 3$ then Vizing's conjecture holds for the pair G_1, G_2.*

Proof Here we use (1.2) and the two bounds (2.1) and (2.2). Since both of these upper bounds have the form

$$\gamma(G) \leq (C_\delta)n$$

where G is an (n, δ)-graph, and C_δ depends only on δ, it suffices to verify that

$$(C_\delta)^2 \leq \frac{1}{2\delta + 1} \tag{3.1}$$

for $\delta \geq 27$ for one or the other of the two bounds.

For $\delta = 27, 28, \ldots, 100$ one may use the sharper bound in (2.2) and verify by computer calculation that (3.1) holds for

$$C_\delta = 1 - \prod_{k=1}^{\delta+1} \frac{k}{k + 1/\delta}.$$

For $\delta > 100$ we use Arnautov's bound (2.1). Then we must verify that

$$\left(\frac{1 + \ln(\delta + 1)}{\delta + 1}\right)^2 \leq \frac{1}{2\delta + 1}, \quad \text{when } \delta > 100. \tag{3.2}$$

To verify this let $x = \delta + 1$, then (3.2) is equivalent to

$$\left(\frac{1 + \ln(x)}{x}\right)^2 \leq \frac{1}{2x - 1}, \quad \text{when } x > 101,$$

which holds if

$$\left(\frac{1 + \ln(x)}{x}\right)^2 \leq \frac{1}{2x}, \quad \text{when } x > 101.$$

This is equivalent to the obvious inequality

$$(1 + \ln(x))^2 \leq \frac{x}{2}, \quad \text{when } x > 101,$$

and this completes the proof for the case $\delta \geq 27$.

The cases $\delta = 1$ and $\delta = 2$ are well-known ([6]). If $\delta = 3$, then Reed [8] has proved that

$$\gamma(G) \leq \frac{3}{8}n.$$

So it suffices in this case to observe that

$$\left(\frac{3}{8}\right)^2 \leq \frac{1}{7}.$$

This completes the proof. ■

We now consider what happens for small values of n_1 and n_2.

Theorem 2 *If G_1 and G_2 are δ-regular graphs of orders n_1 and n_2, respectively, and $n_1, n_2 \leq 16$ then Vizing's conjecture holds for the pair G_1, G_2 with the possible exception $n_1 = n_2 = 16$ and $\delta = 5$.*

Proof By the previous theorem we need not consider the case when $\delta \leq 3$. We let $\hat{\gamma}(n, d)$ be the upper bound for $\gamma(n, \delta)$ given in the table in Section 2. For the cases $\delta \geq 4$ a straightforward Maple calculation yields the desired inequality

$$\hat{\gamma}(n_1, \delta)\hat{\gamma}(n_2, \delta) \leq \left\lceil \frac{n_1 n_2}{2\delta + 1} \right\rceil$$

for $n_1 \leq n_2 \leq 16$ and $4 \leq \delta \leq n_1 - 1$, except when $n_1 = n_2 = 16$ and $\delta = 5$. ■

Remark The upper bound $\gamma(16, 5) \leq 5$ given in the table in Section 2 is not known to be tight. If, in fact, $\gamma(16, 5) = 4$ then the exception in the above theorem may be removed. [Added in Proof: We now know that $\gamma(16, 5) = 4$ [4].] We also note that if the conjecture $\gamma(n, 4) \leq n/3$ from [2] is true, then Vizing's conjecture will hold for all pairs of 4-regular graphs.

4 The effect of minimum degrees

In this section, G_1 and G_2 are graphs of orders n_1, n_2 and minimum degrees δ_1 and δ_2, respectively. If the minimum degrees are large enough then the domination numbers will be 2 or less and so Vizing's conjecture will hold (see [6]). However, we can do much better than this:

Theorem 3 *Let G_1 and G_2 be graphs of orders n_1, n_2 and minimum degrees δ_1 and δ_2, respectively. If $n_1, n_2 \leq n$ and $\delta_1,\ \delta_2\ \geq\ \sqrt{n}\ln n$ then Vizing's conjecture holds for the pair G_1, G_2.*

Proof. Let $n = \max(n_1, n_2)$ and $m = \min(n_1, n_2)$. It is not difficult to show (see also [6]) that

$$\gamma(G_1 \square G_2) \geq m.$$

Using Arnautov's bound (2.1) for $\gamma(G_1)$ and $\gamma(G_2)$ and arguing as in (1.2), we have that the pair G_1, G_2 satisfies Vizing's conjecture if

$$\frac{1+\ln(\delta_1+1)}{\delta_1+1}\frac{1+\ln(\delta_2+1)}{\delta_2+1} \le \frac{1}{n}. \tag{4.1}$$

Assume next that both δ_1 and δ_2 are at least $\sqrt{n}\ln n$. Observe that the function

$$q(x) = \frac{1+\ln x}{x}$$

is a decreasing function for $x > 1$. Hence, we have

$$\begin{aligned} & \frac{1+\ln(\delta_1+1)}{\delta_1+1}\frac{1+\ln(\delta_2+1)}{\delta_2+1} \\ \le & \left(\frac{1+\ln(\sqrt{n}\ln n)}{\sqrt{n}\ln n}\right)^2 \\ \le & \frac{1}{n}, \end{aligned}$$

where the last inequality holds for sufficiently large n. In fact, the function $f(x) = q(\sqrt{x}\ln x)^2 x$ is a decreasing function for $x > e$. Maple shows that $f(213) < 1$. It therefore remains to handle the case when $n \le 212$. For this, we replace the function

$$\frac{1+\ln(\delta+1)}{\delta+1}$$

in (4.1) with

$$C(\delta) = 1 - \prod_{k=1}^{\delta+1} \frac{k\delta}{k\delta+1}$$

from (2.2). Direct calculations show that if $\delta = \lceil\sqrt{n}\ln n\rceil$ then

$$C(\delta)^2 \le \frac{1}{n} \tag{4.2}$$

for all $n \le 212$. Since $C(\delta)$ is decreasing in δ the result follows. ∎

Note that it is feasible to obtain a more accurate asymptotic lower bound for the minimum degrees. Please refer to Section 6.

Let $\xi(n)$ be the least positive integer δ such that (4.2) holds. Then $\xi(n)$ can replace $\sqrt{n}\ln n$ in Theorem 3. The following table compare $\xi(n)$ and $\lceil\sqrt{n}\ln n\rceil$ for a few values of n.

n	10	50	100	1,000	10,000	100,000
$\xi(n)$	7	26	41	180	711	2674
$\lceil\sqrt{n}\ln n\rceil$	8	28	47	219	922	3641

5 When Δ_1 And Δ_2 Are Known

We observe that when $\Delta_1+\Delta_2 < \max(n_1, n_2)$, the lower bound $n_1 n_2/(\Delta_1+\Delta_2+1)$ on $\gamma(G_1 \Box G_2)$ is better than $\min(n_1, n_2)$ used in the previous section. It is therefore useful to consider the case when Δ_1 and Δ_2 are known.

We say that the pair of triples

$$(n_1, \delta_1, \Delta_1),\ (n_2, \delta_2, \Delta_2) \tag{5.1}$$

satisfies Vizing's conjecture if $\gamma(G_1)\gamma(G_2) \leq \gamma(G_1 \Box G_2)$ holds whenever G_1 is an $(n_1, \delta_1, \Delta_1)$ - graph and G_2 is an $(n_2, \delta_2, \Delta_2)$ - graph.

Now given an upper bound of the form

$$\gamma(G) \leq (C_\delta)n$$

for all (n, δ)-graphs G, it follows that the pair of triples (5.1) satisfies Vizing's conjecture whenever

$$C_{\delta_1} C_{\delta_2} \leq \frac{1}{\Delta_1 + \Delta_2 + 1}. \tag{5.2}$$

If we use Arnautov's bound (2.1) then this inequality becomes

$$\left(\frac{1 + \ln(\delta_1 + 1)}{\delta_1 + 1}\right) \left(\frac{1 + \ln(\delta_2 + 1)}{\delta_2 + 1}\right) \leq \frac{1}{\Delta_1 + \Delta_2 + 1}. \tag{5.3}$$

Theorem 4 *There exist functions φ_1, φ_2 such that for $\delta_1 \geq 48$, the inequality*

$$\delta_1 + \delta_2 + 1 \leq \frac{(\delta_1 + 1)(\delta_2 + 1)}{(1 + \ln(\delta_1 + 1))(1 + \ln(\delta_2 + 1))}$$

is satisfied if and only if

$$\varphi_1(\delta_1) \leq \delta_2 \leq \varphi_2(\delta_1). \tag{5.4}$$

Furthermore, if $\delta_1 \geq 48$ and δ_2 satisfies (5.4), then for all Δ_1 and Δ_2 satisfying

$$\delta_1 + \delta_2 \leq \Delta_1 + \Delta_2 \leq \frac{(\delta_1 + 1)(\delta_2 + 1)}{(1 + \ln(\delta_1 + 1))(1 + \ln(\delta_2 + 1))} - 1, \tag{5.5}$$

the pair of triples $(n_1, \delta_1, \Delta_1)$, $(n_2, \delta_2, \Delta_2)$ satisfies Vizing's conjecture. The functions $\varphi_1(\delta)$ and $\varphi_2(\delta)$ are given asymptotically by $\Phi_1(\delta)$ and $\Phi_2(\delta)$, respectively, where

$$\Phi_1(\delta) = (1 + \ln(\delta + 1)) \left[1 + \ln(1 + \ln(\delta + 1)) + \ln(\ln(1 + \ln(\delta + 1)))\right],$$

and

$$\Phi_2(\delta) = \exp\left[\frac{\delta + 1}{1 + \ln(\delta + 1)} - 1\right].$$

Proof The existence of φ_1 and φ_2 will be developed in Section 6. The second part of the theorem follows directly from (5.3). ■

Here we give a table of values of $\varphi_1(\delta)$ and $\varphi_2(\delta)$ for a sampling of small values of δ.

δ	$\varphi_1(\delta)$	$\varphi_2(\delta)$
50	42	10,867
60	38	55,404
70	36	265,046
80	34	1,220,107
90	33	5,457,919
100	33	23,844,683

This table shows, for example, that if G_1 is 50-regular and G_2 is δ_2-regular where $42 \leq \delta_2 \leq 10,867$ then Vizing's conjecture holds for the pair G_1, G_2. To indicate how much Δ_1 and Δ_2 can deviate from δ_1 and δ_2, we give below a short table of upper bounds of

$$E_{1,2} = \Delta_1 + \Delta_2 - \delta_1 - \delta_2$$

computed using (5.5) for $\delta_1 = 100$. Although the range of values of δ_2 for $\delta_1 = 100$ is from 33 to 23,844,683, we only consider values of δ_2 between 33 and 1007 at intervals of 50.

δ_1	δ_2	$E_{1,2}$
100	33	1
100	83	94
100	133	174
100	183	248
100	233	318
100	283	384
100	333	448
100	383	509
100	433	569
100	483	628
100	533	685
100	583	741
100	633	796
100	683	850
100	733	903
100	783	955
100	833	1007

For example, according to the above table a pair of triples $(n_1, \delta_1, \Delta_1)$, $(n_2, \delta_2, \Delta_2)$ with $\delta_1 = 100$ and $\delta_2 = 133$ satisfies Vizing's conjecture if $\Delta_1 + \Delta_2 \leq 100 + 133 + 174 = 407$. So if, say, $n_1 = n_2 = 200$ then all graphs with these parameters will satisfy Vizing's conjecture.

Remark The number of pairs of triples satisfying Vizing's conjecture may be extended considerably using the upper bound (2.2) instead of Arnautov's bound (2.1), but qualitatively the results are the same.

6 Asymptotics

In this section we discuss the curves satisfying the transcendental equation

$$F(x,y) = 0, \tag{6.1}$$

where

$$F(x,y) = \frac{1}{1+x+y} - \frac{[1+\ln(1+x)][1+\ln(1+y)]}{(1+x)(1+y)}, \tag{6.2}$$

for $x > 0, y > 0$.

Theorem 5 *For fixed y, $y \geq 14$, the function f,*

$$f(x) = (1+x)F(x,y) = \frac{1+x}{1+x+y} - \frac{[1+\ln(1+x)][1+\ln(1+y)]}{1+y} \tag{6.3}$$

has two distinct positive critical points. The smallest is a local minimum and the largest is a local maximum.

Proof. Clearly

$$f'(x) = \frac{-y/\Omega}{(1+x)(1+x+y)^2}\left[(1+x+y)^2 - \Omega(1+x)\right]. \tag{6.4}$$

where

$$\Omega := \frac{y(1+y)}{1+\ln(1+y)}.$$

Thus the equation $f'(x) = 0$ is equivalent to the quadratic equation

$$(x+1)^2 + (x+1)(2y-\Omega) + y^2 = 0. \tag{6.5}$$

The discriminant of (6.5), as an equation in the unknown $x+1$, is $\Omega(\Omega-4y)$ which is positive if and only if $\Omega - 4y > 0$. Now

$$\Omega - 4y = \frac{y[y-3-4\ln(1+y)]}{1+\ln(1+y)},$$

which is positive for $y \geq 14$, since $y-3-4\ln(1+y)$, is an increasing function of y for $y \geq 3$ and $y-3-4\ln(1+y) > 0$ at $y = 14$. Thus f has two critical points. At the critical points $x+1$ will be positive if $\Omega - 2y > 0$. But

$\Omega - 2y > \Omega - 4y > 0$, for $y \geq 14$. The smallest critical point is positive if and only if

$$(\Omega - 2y - 2)^2 > (\Omega - 2y)^2 - 4y^2, \quad \text{that is} \quad y^2 + 2y + 1 > \Omega,$$

which holds since $y^2 + y > \Omega$. Observe that (6.4) shows that f' changes sign from negative to positive as we go through the smallest critical point and from positive to negative as we go through the second critical point. Therefore the smallest critical point must be a minimum while the largest is a local maximum.

Theorem 6 *For fixed y, $y \geq 48$, equation (6.1) has two positive solutions $x = x(y)$.*

Proof. It is clear that $f(0) < 0$, $f'(0) < 0$, and $f(x) \to -\infty$ as $x \to +\infty$. In view of the previous theorem it suffices to show that $f(x) \geq 0$ for some $x > 0$. Now $f(y) \geq 0$ if $g(y) > 0$, where g is

$$g(y) = \frac{1+y}{\sqrt{1+2y}} - 1 - \ln(1+y)$$

It is easy to see that $g'(y) \geq 0$ if $y^4 - 6y^3 - 11y^2 - 6y - 1 \geq 0$. The left-hand side of latter expression can be written as

$$y^4 - 14y^3 + 8y^3 - 112y^2 + 101y^2 - 6y - 1$$

which is positive for $y \geq 14$. Thus g increases with y when $y \geq 14$ and $g(48) = .083375912 > 0$, hence $g(y)$ takes positive values and $f(x) > 0$ when $x = y \geq 48$. This completes the proof.

Throughout the rest of this section we shall assume that y is fixed and $y \geq 48$. Let $x_1(y)$ and $x_2(y)$ be the two roots of (6.1), and assume $x_1(y) < x_2(y)$.

We now derive asymptotic estimates for $x_1(y)$ and $x_2(y)$. Consider first the equation

$$q(z) = \frac{1 + \ln z}{z} = \frac{1}{w}, \tag{6.6}$$

where $w > 1$ is a constant. Standard method shows there are two solutions for z. We are interested in the largest solution z_1 (because $z > z_1$ implies $q(z) < 1/w$), which can be found by the following iterative method. Define

$$\begin{aligned} \beta_0 &= 1 + \ln w \\ \beta_{i+1} &= 1 + \ln(w\beta_i), \qquad i \geq 0. \end{aligned}$$

Then β_i increases, as $i \to \infty$, to a limit β and β satisfies

$$\beta = 1 + \ln(w\beta), \tag{6.7}$$

and $z_1 = w\beta$ is the largest solution to (6.6). Observe also that $\beta/\ln w \to 1$ as $w \to \infty$. Hence $z_1/(w\ln w) \to 1$ as $w \to \infty$. In particular, if we put $w = \sqrt{n}$, we see that the asymptotic lower bound for the minimum degrees in Theorem 3 is $\frac{1}{2}\sqrt{n}\ln n$.

We next give estimates for x_1 and x_2. These estimates lead to the asymptotic forms of Φ_1 and Φ_2 stated in Theorem 4.

Theorem 7 *Suppose that $w = w(y) = 1 + \ln(1+y)$. Let β be defined as in (6.7). Then for sufficiently large y,*

$$w\beta \le 1 + x_1(y) \le (1 + (\ln y)^2/y)w\beta$$

$$\exp\left(\frac{1+y}{w}\right) - \frac{(y+1)y}{w} - 1 \le 1 + x_2(y) \le \exp\left(\frac{1+y}{w}\right) - \frac{(y+1)y}{w}.$$

Proof. Let

$$f_1(x) = \frac{(1+y)f(x)}{1+\ln(y+1)} = \frac{1+y}{1+x+y}\frac{1+x}{w} - (1+\ln(1+x)).$$

In our proof of the previous two theorems, we see that $f(x) \ge 0$ (and $f_1(x) \ge 0$) if and only if $x_1 \le x \le x_2$. It therefore suffices to show that for sufficiently large y, the function f_1 is of the appropriate signs at the lower and upper bounds of x_1, x_2. This can be done for the bounds on x_1 by noting the properties of β observed above. For the bounds on x_2, we write

$$\alpha = \exp\left(\frac{1+y}{w}\right), \qquad \eta = \frac{(y+1)y}{w},$$

For the lower bound on x_2, we note that for $1 + x = \alpha - \eta - 1$,

$$\frac{1+y}{w}\frac{1+x}{1+x+y} = \frac{1+y}{w} - \frac{\eta}{\alpha} + O(\eta^2/\alpha^2),$$

and

$$1 + \ln(1+x) = \frac{1+y}{w} - \frac{\eta+1}{\alpha} + O(\eta^2/\alpha^2).$$

This show that

$$f_1(x) = \frac{1+y}{w}\frac{1+x}{1+x+y} - (1+\ln(1+x)) = \frac{1}{\alpha} + O(\eta^2/\alpha^2) \ge 0,$$

where the last inequality holds for all large enough y. This implies $f(x) \ge 0$ and hence the lower bound on x_2. The proof of the upper bound for x_2 follows similarly. ■

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