On Motzkin words and noncrossing partitions

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Abstract

In this paper the notions of c-Motzkin and d-Motzkin words are introduced, studied and the cardinal numbers of their sets are evaluated. Finally bijections between the sets of the introduced Motzkin words and certain sets of noncrossing partitions are exhibited.

1. Introduction

This work deals with certain types of Motzkin words. Most of the work of [8] for Dyck words is here extended to Motzkin words.

A word w is called Motzkin if the word w_D obtained by deleting every letter except two given letters α, $\overline{\alpha}$ from w is a Dyck word. In the literature most common are the Motzkin words containing three different letters [1], [7], though there is some work concerning Motzkin words with four different letters [11].

Throughout this paper, for a nonempty set T we denote by T* the set of all the words with letters in T. In sections 1 and 2, Motzkin words in $\{\alpha, \overline{\alpha}, v\}^*$ are considered, whereas Motzkin words in $\{\alpha, \overline{\alpha}, v, \mu\}^*$ are studied in section 3.

A *Motzkin path* $(s_0,s_1,\ldots,s_n)$ of length n is a path of N×N where $s_0=(0,0)$, $s_n=(n,0)$ and every edge (s_i,s_{i+1}), $0<i\leq n-1$, is elementary North-East (i.e. $s_i=(x,y)$, $s_{i+1}=(x+1,y+1)$) or elementary South-East (i.e. $s_i=(x,y)$, $s_{i+1}=(x+1,y-1)$) or elementary East (i.e. $s_i=(x,y)$, $s_{i+1}=(x+1,y)$) [1].

It is well known that the Motzkin paths of length n are coded by the Motzkin words $w=w_1w_2\ldots w_n$ so that every elementary North-East (resp. South-East) edge corresponds to the letter $w_i=\alpha$ (resp. $w_i=\overline{\alpha}$) and every elementary East edge corresponds to the letter $w_i=v$.

It is also known (for instance see [1]) that the number of Motzkin words w with $|w|=n$ and $|w_D|=2m$ is equal to

$$\frac{1}{n+1}\binom{n+1}{m+1,\, m,\, n-2m}.$$

Let $w=w_1w_2\ldots w_n$ be a Motzkin word of $\{\alpha, \overline{\alpha}, v\}^*$ and w_D the Dyck word obtained by deleting the letter v from w. Two indices $i,j\in[n]=\{1,2,\ldots,n\}$

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are called *conjugates,* with respect to w, if i,j are conjugates with respect to w_D [8], i.e., j is the smallest number in $[i+1,|w|]$ for which the subword $w_i w_{i+1} \dots w_j$ is a Motzkin word.

If $S=\{\{i,j\}: i<j$ and i,j conjugates$\}$, we have the following proposition.

Proposition 1.1 *For any Motzkin word w, the set of the intervals* $\hat{S}=\{[i,j]: \{i,j\} \in S\}$ *is nested.*

Proof. Given two distinct intervals $[i_1,j_1]$ and $[i_2,j_2] \in \hat{S}$ we show that either these are disjoint or the one is contained in the other. Indeed if $[i_1,j_1]$, $[i_2,j_2] \in \hat{S}$ we assume without loss of the generality that $i_1<i_2<j_1$. Clearly since the words $w_{i_1} \dots w_{j_1}$ and $w_{i_2} \dots w_{j_2}$ are Motzkin words, it follows easily that the word $w_{i_2} \dots w_{j_1}$ is also a Motzkin word. Thus by the minimality of j_2 it follows that $j_2 < j_1$ and $[i_2,j_2] \subset [i_1,j_1]$.

A partition $\pi=B_1/B_2/\dots/B_m$ of [n] is called noncrossing if there do not exist four numbers $a<b<c<d$ such that $a,c \in B_i$ and $b,d \in B_j$ and $i \neq j$.

The set of all noncrossing partitions of [n] ordered by refinement, is a lattice and was first studied in [6], [9], and later by a number of authors (for instance see [2], [3], [4], [5] and [10]).

It is well known that the number of all noncrossing partitions of [n] is equal to C_n, the Catalan number, while the number of all noncrossing partitions of [n] which contain k-blocks is equal to the number of Narayana

$$\frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

In section 2 the c-Motzkin words are introduced and it is shown how they are generated by a context-free grammar. Further, a result similar to the decompositon of Dyck words [8] is shown for c-Motzkin words and it is used to show that the number of all c-Motzkin words w of length n with $|w|=n$ and $|w_D|=2m$ equals to

$$\frac{1}{n-m+1}\binom{n}{m}\binom{n-m-1}{m-1}.$$

Finally a bijection between the set of all c-Motzkin words w with $|w|=n$ and $|w_D|=2m$, and the set of all noncrossing partitions π of [n] with m blocks and without singletons is constructed.

In section 3 the d-Motzkin words are introduced and it is shown that the number of all d-Motzkin words $w \in \{\alpha, \bar{\alpha}, v, \mu\}^*$ of length n with $|w|_\alpha + |w|_\mu = k$ is equal to the Narayana number.

2. c-Motzkin words

A Motzkin word $w \in \{\alpha, \overline{\alpha}, v\}^*$ is called *c-Motzkin* word if $\bigcup_{\{i,j\} \in S} [i,j] = [n]$

i.e. every $\rho \in [n]$ with $z_\rho = v$ lies between two conjugate indices.

For example the word $w = \alpha\ v\ v\ \alpha\ v\ \overline{\alpha}\ \ \overline{\alpha}\ \ \alpha\ \overline{\alpha}\ \ \alpha\ \ v\ \overline{\alpha}$ is a c-Motzkin word with $S=\{\{1,7\},\{4,6\},\{8,9\},\{10,12\}\}$.

It is easy to prove the following result.

Proposition 2.1 *The language M of c-Motzkin words is generated by the context-free grammar*

$$M = N^*,$$

$$N = \alpha\,(v+M)^*\,\overline{\alpha}\,.$$

The c-Motzkin words are generated more quickly by the language L which contains the elementary words: $\omega_r = \alpha v^r \overline{\alpha}$, $r \in N$. This is obtained by concatenation or insertion.

So, for the previous example we have

$\alpha\, v\, v\, \overline{\alpha}$

$\alpha\, v\, v\, \alpha\, v\, \overline{\alpha}\ \ \overline{\alpha}$

$\alpha\, v\, v\, \alpha\, v\, \overline{\alpha}\ \ \overline{\alpha}\ \ \alpha\, \overline{\alpha}$

$\alpha\, v\, v\, \alpha\, v\, \overline{\alpha}\ \ \overline{\alpha}\ \ \alpha\, \overline{\alpha}\ \ \alpha\, v\, \overline{\alpha}\,.$

A c-Motzkin word that is not a product of two nonempty c-Motzkin words is called *prime*. It is clear that the c-Motzkin prime words are these that the only intersection of their corresponding Motzkin paths with the x-axis are the initial and final points of the paths.

It is evident that a c-Motzkin word $w = w_1 w_2 \ldots w_n$ is prime iff 1,n are conjugates with respect to w.

The following result extends proposition 1.1 of [8] for Dyck words to c-Motzkin words. The proof is similar and it is omitted.

Proposition 2.2 *Every c-Motzkin word is uniquely decomposed into a product of c-Motzkin prime words.*

The c-Motzkin words which decompose into k c-Motzkin prime words (components) are those whose corresponding Motzkin paths meet the x-axis at exactly k-1 points apart from the points (0,0) and (n,0).

In [8] it is proved that

$$d(m,k) = \frac{k}{m}\binom{2m-k-1}{m-1}$$

where d(m,k) is the number of all Dyck words of length 2m which are decomposed into k Dyck primes.

This result will be used for the evaluation of the cardinal number of the set of c-Motzkin words.

Proposition 2.3 *The number of all c-Motzkin words w with* $|w|=n$ *and* $|w_D|=2m$ *is equal to*

$$\frac{1}{n-m+1}\binom{n}{m}\binom{n-m-1}{m-1}.$$

Proof. Let M be the set of all c-Motzkin words w with $|w|=n$ and $|w_D|=2m$. For each $k\in[m]$ we denote by M_k the set of all $u\in M$ which decompose into k components. For each element $w\in M_k$ the corresponding Dyck word w_D is selected in d(m,k) ways. The word w is constructed from w_D by inserting the n-2m v's of w in 2m-k intervals. Thus the number of c-Motzkin words of M which we obtain from a given w is equal to the number of combinations with repetitions of 2m-k by n-2m i.e., to

$$\begin{bmatrix}2m-k\\ n-2m\end{bmatrix}=\binom{2m-k+n-2m-1}{n-2m}=\binom{n-k-1}{n-2m}.$$

It follows that

$$|M_k|=d(m,k)\binom{n-k-1}{n-2m}=\frac{k}{m}\binom{2m-k-1}{m-1}\binom{n-k-1}{n-2m}.$$

Further, using the identities

$$n\binom{n}{m}=(n-m+1)\binom{n}{m-1} \text{ and } \sum_{k=0}^{m}\binom{n-m+k+1}{k}=\binom{n}{m},$$

easily obtained from [12], we have

$$|M|=\sum_{k=1}^{m}|M_k|=\sum_{k=1}^{m}\frac{k}{m}\binom{2m-k-1}{m-1}\binom{n-k-1}{n-2m}$$

$$=\frac{(n-m-1)!}{m!(n-2m)!}\sum_{k=0}^{m}k\binom{n-k-1}{m-k}$$

$$=\frac{(n-m-1)!}{m!(n-2m)!}\sum_{k=0}^{m}(m-k)\binom{n-m+k-1}{k}$$

$$=\binom{n-m-1}{m-1}\frac{1}{m}\left[m\sum_{k=0}^{m}\binom{n-m+k-1}{k}-\sum_{k=1}^{m}k\binom{n-m+k-1}{k}\right]$$

$$=\binom{n-m-1}{m-1}\left[\sum_{k=0}^{m}\binom{n-m+k-1}{k}-\frac{1}{m}\sum_{k=1}^{m}(n-m)\binom{n-m+k-1}{k-1}\right]$$

$$=\binom{n-m-1}{m-1}\left[\sum_{k=0}^{m}\binom{n-m+k-1}{k}-\frac{n-m}{m}\sum_{k=0}^{m-1}\binom{n-(m-1)+k-1}{k}\right]$$

$$=\binom{n-m-1}{m-1}\left[\binom{n}{m}-\frac{n-m}{m}\binom{n}{m-1}\right]$$

$$=\binom{n-m-1}{m-1}\left[\binom{n}{m}-\frac{n-m}{m}\frac{n!}{(m-1)!(n-m+1)!}\right]$$

$$=\frac{1}{n-m+1}\binom{n}{m}\binom{n-m-1}{m-1}.$$

The number of c-Motzkin words w with $|w|=n$ and $|w_D|=2m$, counts also the noncrossing partitions of $[n]$ with m-blocks and without singletons (see [6]). The aim of the following result is to exhibit a bijection between these two combinatorial objects.

Proposition 2.4 *There exists a bijection between the set of all c-Motzkin words w with $|w|=n$ and $|w_D|=2m$ and the set of all noncrossing partitions π of $[n]$ with m blocks and without singletons such that the restriction of w on each block is a word of L.*

Proof. Given a c-Motzkin word $w=w_1w_2\dots w_n$ with $|w_D|=2m$ and $I\in\hat{S}$ we set

$$B_I=I\setminus\cup\{J\in\hat{S}:J\subset I\}$$

Let $\pi(w)=\{B_I: I\in\hat{S}\}$. Clearly, $|\pi(w)|=|S|=\frac{w_D}{2}=m$ and $|B_I|\geq 2$ for each $I\in\hat{S}$.

By proposition 1.1 it follows easily that $B_I\cap B_J=\emptyset$, whenever $I\neq J$. Further we have that $[n]=\cup\{B_I: I\in\hat{S}\}$. Indeed if $\rho\in[n]$ we set $I_\rho=\cap\{I\in\hat{S}:\rho\in I\}$. Then $I_\rho\in\hat{S}$ and $\rho\notin J$ for each J in $\hat{S}$ with $J\subset I_\rho$. This shows that $\rho\in B_I$ and $[n]=\cup\{B_I: I\in\hat{S}\}$. Thus the family $\pi(w)$ is a partition of $[n]$.

Further, given $i\neq j$ $i,j\in B_I$ and $l,m\in B_J$ with $i<l<j<m$ we obtain easily that $l,j\in I\cap J$. It follows by proposition 1.1 that either $I\subset J$ or $J\subset I$. If $I\subset J$ (resp. $J\subset I$), since $l\in I$ (resp. $j\in J$) we obtain that $l\notin B_J$ (resp. $j\notin B_I$) which is a contradiction. Thus the partition $\pi(w)$ is noncrossing.

Finally, we show that the restriction of w on each B_I is a word of L. For this it is enough to show that $w_i=\alpha$, $w_j=\overline{\alpha}$ where $i=\min B_I$ and $j=\max B_I$, and $w_t=v$ for every $t\in B_I$ with $i<t<j$.

Indeed if $I=[i,j]$ we have $w_i=\alpha$, $w_j=\overline{\alpha}$, $i=\min B_I$ and $j=\max B_I$. Further if $t\in B_I$ with $i<t<j$ we show that $w_t\neq\alpha$ and $w_t\neq\overline{\alpha}$. Indeed if $w_t=\alpha$ (resp. $w_t=\overline{\alpha}$) let $J\in\hat{S}$ such that $t=\min J$ (resp. $t=\max J$). Then by proposition 1.1 follows that $J\subset I$. Thus $t\notin B_I$ which is a contradiction. Thus $w_t=v$ and the restriction of w on B_I is a word of L.

Conversely, given a noncrossing partition π on [n] with m blocks and without singletons we construct a c-Motzkin word w with $|w|=n$, $|w_D|=2m$ and $\pi(w)=\pi$.

Indeed if $i\in[n]$ and B is the (unique) block of π which contains i, we set

$$w_i=\begin{cases}\alpha, & \text{if} \quad i=\min B\\ \overline{\alpha} & \text{if} \quad i=\max B\\ v & \text{if} \quad \min B<i<\max B.\end{cases}$$

Let $w=w_1w_2\ldots w_n$. Then $|w|=n$ and $|w_D|=2m$. Moreover we consider the sets A, $\overline{A}$ and the function $f:A\to\overline{A}$ defined by $A=\{i\in[n]: w_i=\alpha\}$, $\overline{A}=\{j\in[n]: w_j=\overline{\alpha}\}$ and $f(i)=j$ iff i,j belong to the same block of π.

It is clear that f is a bijection so that

$$|w|_\alpha=|A|=|\overline{A}|=|w|_{\overline{\alpha}}.$$

Further we can easily check that $\overline{A}\cap[i]\subseteq f(A\cap[i])$ for each $i\in[n]$.

Thus,

$$|w_1w_2\ldots w_i|_\alpha=|A\cap[i]|=|f(A\cap[i])|\geq|\overline{A}\cap[i]|=|w_1w_2\ldots w_i|_{\overline{\alpha}}$$

for each $i\in[n]$.

This shows that w is a Motzkin word. Moreover we can easily check

$$\overline{A}\cap[i,f(i)]=f(A\cap[i,f(i)]),\ \text{for each } i\in A \text{ and}$$

$$\overline{A}\cap[i,j]\subset f(A\cap[i,j]),\ \text{for each } i\in A \text{ and } i<j<f(i).$$

So, we have

$$|w_iw_{i+1}\ldots w_{f(i)}|_\alpha=|A\cap[i.,f(i)]|=|\overline{A}\cap[i,f(j)]|=|w_iw_{i+1}\ldots w_{f(i)}|_{\overline{\alpha}}$$

and

$$|w_iw_{i+1}\ldots w_j|_\alpha=|A\cap[i,j]|\geq|\overline{A}\cap[i,f(j)]|=|w_iw_{i+1}\ldots w_j|_{\overline{\alpha}}.$$

This shows that f(i) is the smallest number of $[i+1,|w|]$ such that the subword $w_1w_2\ldots w_{f(i)}$ is a Motzkin word, i.e. f(i) is the conjugate of i for each $i\in A$.

In other words two positive integers $i\in A$ and $j\in\overline{A}$ are conjugates iff i,j belong to the same block of π. Finally, since we can easily check that $[n]=\cup\hat{S}$ we deduce that w is a c-Motzkin word.

It remains to show that $\pi=\pi(w)$. Indeed, there exists a 1-1 correspondence between π and $\hat{S}$ such that every $B\in\pi$ it corresponds to $I=[i,j]\in\hat{S}$, where $i=\min B$ and $j=\max B$. Since π is a noncrossing partition it is easy to check that $B=B_I$. Thus, $\pi=\pi(w)$.

3. d-Motzkin words

A Motzkin word $w \in \{\alpha, \bar{\alpha}, \nu, \mu\}^*$ is called *d-Motzkin* if the word w_M obtained by deleting the letter μ from w is a c-Motzkin word.

Proposition 3.1 *The number of all d-Motzkin words w with $|w|=n$ and $|w|_\alpha+|w|_\mu=k$ is equal to the Narayana number*

$$\frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Proof. We first find the number of all d-Motzkin words w with $|w|_\mu=\lambda$, where $\lambda=0,1,\ldots,k-1$. Clearly since $|w_M|=n-\lambda$ and $|w_M|_\alpha=k-\lambda$ from proposition 2.3 follows that the c-Motzkin word w_M is selected in

$$\frac{1}{n-k+1}\binom{n-\lambda}{k-\lambda}\binom{n-k-1}{k-\lambda-1}$$

ways.

Every such word w is generated by the corresponding c-Motzkin word w_M by

letting the μ' s in $n-\lambda+1$ intervals. Thus the number of all d-Motzkin words w that are obtained by a certain c-Motzkin word is equal to the number of all combinations with repetitions of $n-\lambda+1$ by λ, i.e.

$$\left[\begin{matrix} n-\lambda+1 \\ \lambda \end{matrix}\right]=\binom{n}{\lambda}.$$

Thus the number of all d-Motzkin words w with $|w|_\mu=\lambda$ is equal to

$$\frac{1}{n-k+1}\binom{n-\lambda}{k-\lambda}\binom{n-k-1}{k-\lambda-1}\binom{n}{\lambda}=\frac{1}{n-k+1}\binom{n}{k}\binom{k}{\lambda}\binom{n-k-1}{k-\lambda-1}.$$

So, using a well known identity we obtain the number of all d-Motzkin words:

$$\frac{1}{n-k+1}\binom{n}{k}\sum_{\lambda=0}^{k-1}\binom{k}{\lambda}\binom{n-k-1}{k-\lambda-1}=\frac{1}{n-k+1}\binom{n}{k}\binom{k+n-k-1}{k-1}=\frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Given a d-Motzkin word $w=w_1w_2\ldots w_n$ we may modify the method described in proposition 2.4 for the construction of a noncrossing partition π of $[n]$ with m blocks. This is done by adding to the non-crossing partition generated by w_M the singleton blocks $\{i\}$, where $i\in[n]$ with $w_i=\mu$. This suggests the following result.

Proposition 3.2 *There exists a bijection between the set of all d-Motzkin words w with $|w|=n$ and $|w|_\alpha+|w|_\mu=k$ and the set of all noncrossing partitions π of $[n]$ with k blocks such that the restriction of w on each non-singleton block in π is a word of L.*

To illustrate the above proposition we conclude with an example of a d-Motzkin word w and the corresponding non-crossing partition π.

Let $w=\mu\ \mu\ \alpha\ \mu\ \nu\ \alpha\ \nu\ \bar{\alpha}\ \mu\ \bar{\alpha}\ \mu\ \alpha\ \alpha\ \mu\ \nu\ \nu\ \bar{\alpha}\ \nu\ \mu\ \bar{\alpha}\ \mu$ then $\hat{S}=\{I_1,I_2,I_3,I_4\}$ where $I_1=\{3,5,6,7,8,10\}$, $I_2=\{6,7,8\}$, $I_3=\{12,13,15,16,18,20\}$ and $I_4=\{13,15,16,17\}$. It follows that π = 1/2/3 5 10/4/6 7 8/9/11/12/8 20/13 15 16 17/14/19/21.

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