Vertex-Magic Total Labelings of Complete Bipartite Graphs

I. D. Gray and J. A. MacDougall
School of Mathematical and Physical Sciences,
University of Newcastle
R. J. Simpson
School of Mathematics and Statistics,
Curtin University of Technology
W. D. Wallis
Department of Mathematics,
Southern Illinois University

Abstract

A vertex-magic total labeling on a graph G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \cdots, |V(G) \cup E(G)|$ with the property that, given any vertex x, $\lambda(x) + \sum_{y \sim x} \lambda(y) = k$ for some constant k.

In this paper we completely determine which complete bipartite graphs have vertex-magic total labelings.

1 Magic labelings

All graphs in this paper are finite, simple and undirected. The graph G has vertex-set $V(G)$ and edge-set $E(G)$, and we denote $|V(G)|$ and $|E(G)|$ by v and e respectively. A general reference for graph-theoretic ideas is [8].

A *labeling* (or *valuation*) of a graph is any map that carries some set of graph elements to numbers (usually to the positive or non-negative integers). If the domain is the vertex-set, the edge-set, or the set $V(G) \cup E(G)$, labelings are called respectively *vertex*-labelings, *edge*-labelings or *total* labelings. The most complete recent survey of graph labelings is [2].

Magic labelings are one-to-one maps onto the appropriate set of consecutive integers starting from 1, with some kind of "constant-sum" property. A labeling is *edge-magic* if the sum of all labels associated with an edge equals a constant independent of the choice of edge, and *vertex-magic* if the same property holds for vertices. For example, a vertex-magic edge-labeling has as its range the integers from 1 to $|E(G)|$, and the sum of the labels on edges adjacent to vertex x equals a constant that is independent of the choice of x. Further discussion can be found in [4, 7].

In this paper we discuss vertex-magic total labelings (VMTLs). Such a labeling on G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \cdots, |V(G) \cup E(G)|$ with the property that, given any vertex x,

$$\lambda(x) + \sum_{y \sim x} \lambda(y) = k$$

for some constant k, called the *magic constant.*

Vertex-magic total labelings are discussed in [4]. In that paper, several classes of graphs are shown to have VMTLs, while they are ruled out for other classes. One outstanding problem is the existence of VMTLs of complete bipartite graphs. In this paper we resolve that existence question.

2 Concerning vertex-magic total labeling of $K_{m,n}$

We shall take the complete bipartite graph $K_{m,n}$ to have vertex-set

$$\{x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n\}$$

and edge-set

$$\{x_i y_j : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

So a VMTL λ of $K_{m,n}$ can be represented by an $m+1 \times n+1$ array

$$A = \left[\begin{array}{c|cccc} a_{00} & a_{01} & a_{02} & \cdots & a_{0n} \\ \hline a_{10} & a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \cdots & \\ & & & \cdots & \\ a_{m0} & a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right] \qquad (1)$$

where

$$\begin{array}{ll} a_{00} = 0 & a_{i0} = \lambda(x_i) \\ a_{0j} = \lambda(y_j) & a_{ij} = \lambda(x_i y_j). \end{array}$$

The matrix A will be called the *representation matrix* of λ. The magic requirement is that all row-sums and column-sums, except for row 0 and column 0, must be equal (to k say), and that the $(m+1)(n+1)$ entries are $\{0, 1, \cdots, mn+m+n\}$ in some order.

We shall call a $K_{m,n}$ *unbalanced* if its parts differ in size by more than 1. We observe that an unbalanced $K_{m,n}$ cannot have a vertex-magic total labeling:

Theorem 1 [4] *If $K_{m,n}$ is unbalanced, then it has no VMTL.*

Proof. Without loss of generality, assume $m \leq n$. Suppose $K_{m,n}$ has a vertex-magic total labeling with magic constant k. For this graph $v = m + n$ and $e = mn$ so the label set is $\{1, 2, \cdots, mn + m + n\}$. The sum of the weights on $\{x_1, \cdots, x_m\}$ is at least the sum of all but the largest n labels, so

$$\begin{aligned} mk &\geq 1 + 2 + \cdots + (mn + m) \\ &= \frac{(mn + m)(mn + m + 1)}{2}; \\ k &\geq \frac{(n + 1)(mn + m + 1)}{2} \end{aligned} \tag{2}$$

On the other hand, the sum of the weights on $\{y_1, \cdots, y_n\}$ is at most the total of all but the m smallest labels:

$$\begin{aligned} nk &\leq (m + 1) + (m + 2) + \cdots + (mn + m + n) \\ &= \frac{(mn + m + n)(mn + m + n + 1) - m(m + 1)}{2} \\ &= \frac{(mn^2 + 2mn + n^2 + n)(m + 1)}{2}; \\ k &\leq \frac{(mn + 2m + n + 1)(m + 1)}{2}. \end{aligned} \tag{3}$$

Combining(2) and (3),

$$(n + 1)(mn + m + 1) \leq (mn + 2m + n + 1)(m + 1),$$

and on simplifying one obtains $m \geq n - 2 + \frac{2}{n+2}$, so $m \geq n - 1$. □

3 Construction of VMTLs of $\mathrm{K}_{\mathrm{m,n}}$

In this section we give constructions for vertex-magic total labelings of complete bipartite graphs in the cases not eliminated by Theorem 1. (The case of $K_{m,m}$ was solved in [4], but is repeated here for convenience.)

3.1 $\mathrm{K}_{\mathrm{m,m}}$

Theorem 2 *For every $m > 1$, $K_{m,m}$ has a vertex-magic total labeling with magic constant $\frac{1}{2}[(m + 1)^3 - (m + 1)]$.*

Proof. Labelings for $K_{2,2} = C_4$ are presented in [4] (and will be enumerated in Section 4), so we assume $m > 2$. Let $S = (s_{ij})$ be any magic square of order $m+1$ on the numbers $\{1, \cdots, (m+1)^2\}$. (For convenience, assume that the rows and columns of S are numbered $0, 1, \ldots, m$.) Each row and column sums to the magic square constant $\frac{1}{2}(m+1)(m^2+2m+2)$. Form the matrix $A = (a_{ij})$ where $a_{ij} = s_{ij} - 1$. Since S is magic, the rows and columns of A will each sum to the constant

$$k = \tfrac{1}{2}(m+1)(m^2+2m+2) - (m+1) \qquad (4)$$

and the entries of A will be the numbers in $\{0, \cdots, (m+1)^2 - 1\}$, once each. There are standard constructions ([1], [5]) for magic squares of all orders greater than 3. We shall assume that the rows and columns of A are permuted so that $a_{00} = 0$. (This may mean the diagonals no longer sum to the magic constant, but they are not required for the construction.) Then A is the representation matrix of a vertex-magic total labeling λ with k given by equation (4). The magic constant is easily checked. □

3.2 $K_{m,m+1}$, m odd

A solution for $K_{1,2}$ is easily constructed. So let us write $m = 2n-1$ where $n > 1$. The construction proceeds for a given n by defining two $2n-1 \times 2n$ matrices, $A = (a_{ij})$ and $B = (b_{ij})$, and then using them to construct a $2n \times 2n+1$ representation matrix C. For consistency with the earlier notation, C has first row and column indexed with 0.

The value of a_{ij} depends on the parity of i and j, as well as their values. The formula is

$$\begin{aligned} a_{ij} &= m+1-j \quad \text{if} \quad i+j \text{ is odd, } j+i \le m+1 \\ & \qquad\qquad\quad\; \text{or} \quad i+j \text{ is even, } j+i > m+1, \\ a_{ij} &= j-1 \qquad\quad\;\; \text{otherwise.} \end{aligned}$$

We shall need the row and column sums of this matrix. If i is even, say $i = 2t$, the sum of elements in row i is

$$\begin{aligned} \sum_{j=1}^{2n} a_{ij} &= \sum_{k=1}^{n} a_{i,2k-1} + a_{i,2k} \\ &= \sum_{k=1}^{n-t} (a_{i,2k-1} + a_{i,2k}) + \sum_{k=n-t+1}^{n} (a_{i,2k-1} + a_{i,2k}) \\ &= \sum_{k=1}^{n-t} ((2n-2k+1) + (2k-1)) + \sum_{k=n-t+1}^{n} ((2k-2) + (2n-2k)) \\ &= \sum_{k=1}^{n-t} 2n + \sum_{k=n-t+1}^{n} (2n-2) \end{aligned}$$

$$= \quad 2n^2 - i.$$

If i is odd, say $i = 2t + 1$,

$$\begin{aligned}
&\sum_{j=1}^{2n} a_{ij} \\
&= \sum_{k=1}^{n} a_{i,2k-1} + a_{i,2k} \\
&= \sum_{k=1}^{n-t} a_{i,2k-1} + \sum_{k=1}^{n-t-1} a_{i,2k} + \sum_{k=n-t+1}^{n} a_{i,2k-1} + \sum_{k=n-t}^{n} a_{i,2k} \\
&= \sum_{k=1}^{n-t}(2k-2) + \sum_{k=1}^{n-t-1} (2n-2k) + \sum_{k=n-t+1}^{n} (2n-2k+1) + \sum_{k=n-t}^{n} (2k-1) \\
&= 2n^2 - i.
\end{aligned}$$

so in either case row i has sum $2n^2 - i$. The column sum is more easily calculated: column j contains n entries equal to $j-1$ and $n-1$ equal to $2n-j$. So for each j,

$$\begin{aligned}
\sum_{i=1}^{2n-1} a_{ij} &= n(j-1) + (n-1)(2n-j) \\
&= 2n^2 - 3n + j.
\end{aligned}$$

Matrix B has first and last columns $2n-2, 2n-3, \ldots, 1, 0$:

$$b_{1j} = b_{mj} = m - i - 1.$$

The second column begins with the odd integers $2n-5, 2n-7, \ldots, 1$, then has the even integers $2n-2, 2n-4, \ldots, 0$ and ends with $2n-3$. The other columns are formed by back-circulating this column. That is,

$$b_{i,j} = b_{i+1,j-1}$$

(with subscripts reduced $\bmod(2n, 2n+1)$ where necessary).

In each row of B, columns 2 through $2n-1$ contain all the integers $0, 1, \ldots, m-2$ except

$$\begin{aligned}
x_i &= 2n - 1 - 2i \text{ is missing from row } i, i = 1, \ldots, n-1, \\
x_i &= 4n - 2i - 2 \text{ is missing from row } i, i = n, n+1, \ldots, 2n-1.
\end{aligned}$$

So the row sums are

$$\sum_{j=1}^{m} b_{ij} = 2(2n - i - 1) + \sum_{k=0}^{2n-2} k - x_i$$

$$
\begin{aligned}
&= 2n^2 + n - 1 - 2i - x_i \\
\sum_{j=1}^{m} b_{ij} &= 2n^2 - n \text{ if } i \le n-1, \\
\sum_{j=1}^{m} b_{ij} &= 2n^2 - 3n + 1 \text{ if } i \ge n.
\end{aligned}
$$

Since each column is a permutation of $\{0, 1, \ldots, 2n-2\}$, each column sum is

$$\sum_{i=1}^{2n-1} b_{ij} = 2n^2 - 3n + 1.$$

we now define a $2n \times (2n+1)$ matrix C by

$$
\begin{aligned}
c_{00} &= 0, \\
c_{0j} &= 4n^2 + 2n - j,\ 1 \le j \le 2n, \\
c_{i0} &= i,\ 1 \le i \le n-1, \\
&= 4n^2 - 2n + i,\ n \le i \le 2n-1, \\
c_{ij} &= a_{ij} + 2nb_{ij} + n, 1 \le i \le 2n-1, 1 \le j \le 2n.
\end{aligned}
$$

Theorem 3 *The matrix C is the representation matrix of a vertex-magic total labeling of $K_{2n-1,2n}$ with magic constant $4n^3 + 2n^2$.*

Proof. It is necessary to show that the sum of entries in every row and column of C (except possibly row 0 and column 0) equals $4n^3 + 2n^2$ and that every integer from 0 to $v + e = 4n^2 + 2n - 1$ occurs exactly once in C. (We know that 0 appears in the $(0,0)$ position, as required.)

The row sums of C are:

$$
\begin{aligned}
\sum_{j=0}^{2n} c_{ij} &= c_{i0} + \sum_{j=1}^{2n} a_{ij} + 2n \sum_{j=1}^{2n} b_{ij} + 2n^2 \\
&= c_{i0} + 2n^2 - i + 2n \sum_{j=1}^{2n} b_{ij} + 2n^2 \\
&= 4n^3 + 2n^2,
\end{aligned}
$$

after inserting the appropriate values of c_{i0} and $\sum_{j=1}^{2n} b_{ij}$, depending on whether or not $i \le n-1$. Similarly the column sums are:

$$
\begin{aligned}
\sum_{i=0}^{2n-1} c_{ij} &= c_{0j} + \sum_{i=1}^{2n-1} a_{ij} + 2n \sum_{i=1}^{2n-1} b_{ij} + (2n-1)n \\
&= 4n^2 + 2n - j + 2n^2 - 3n + j + 2n(2n^2 - 3n + 1) + n(2n-1) \\
&= 4n^3 + 2n^2.
\end{aligned}
$$

Thus all row and column sums (except the first) equal $4n^3 + 2n^2$, as required.

Finally, we prove that each integer from 0 to $4n^2 - 2n - 1$ appears exactly once in C. The numbers $0, 1, 2, \ldots, n-1$ and $4n^2-n, 4n^2-n+1, \ldots, 4n^2+2n-1$ appear in the first row and column. The entries in A lie in the (closed) interval $[0, 2n-1]$ and those of B lie in the interval $[0, 2n-2]$. Thus the entries in C outside the first row and column lie in the interval $[n, 3n-1+2n(2n-2)] = [n, 4n^2-n-1]$. There are $4n^2-2n$ such integers so it remains to show that the entries are distinct. To prove this we need to show that the pairs (a_{ij}, b_{ij}) are distinct. The first and last columns of A contain only the integers 0 and $2n-1$, the first and last columns of B contain the integers $0, 1, ..., 2n-2$ and it is easy to see that there are no repeated pairs. For the rest of the matrices A and B, note that the entries b_{ij} for a fixed value of $i + j \bmod (2n-1)$ are constant, while in matrix A the equivalent entries take all the values $1, 2, \ldots, 2n-2$. Thus all pairs are distinct and so each integer from 1 to $4n^2 - 2n - 1$ appears exactly once in C. □

3.3 $K_{m,m+1}$, m even

In this case we write $m = 2n$. Then $v = 4n + 1$, $e = 4n^2 + 2n$, and a total labeling requires $4n^2 + 6n + 1$ labels.

Theorem 4 *There exists a vertex-magic total labeling of $K_{2n,2n+1}$ with magic constant $(n+1)(2n+1)^2$.*

Proof. We construct a representation matrix $C = (c_{ij})$ for a VMTL of $K_{2n,2n+1}$ as follows.

(i) Row 0 of C is $0, (2n+1)^2, (2n+1)^2+1, \ldots, (2n+1)^2+2n$, that is $c_{00} = 0$ and $c_{0j} = (2n+1)^2 + j - 1$ for $1 \le j \le 2n+1$.

(ii) $c_{i0} = (2n+2)i, 1 \le i \le 2n$.

(iii) If $1 \le i < n$ and $1 \le j \le n+1$, or if $n+2 \le i \le 2n$ and $n+2 \le j \le 2n+1$, then

$$c_{ij} = 2n(2n+2) - [j + (i-1)(2n+2)].$$

(iv) If $1 \le i < n$ and $n+2 \le j \le 2n+1$, or if $n+1 < i \le n$ and $1 \le j \le n+1$, then

$$c_{ij} = j + (i-1)(2n+2).$$

(v) If $1 \le j \le n+1$, then

$$c_{nj} = n(2n+2) + 2n - 2j + 3, \; c_{n+1,j} = (n-1)(2n+2) + n + j.$$

If $n+2 \le j \le 2n+1$, then

$$c_{nj} = (n-1)(2n+2) + 4n - 2j + 4 \quad c_{n+1,j} = n(2n+2) + j - n - 1.$$

Part (v) can also be expressed as follows: (except for column 0) rows 0, n and $n+1$ of C are derived from rows of X, where

$$X = \left[\begin{array}{ccccc|cccc} 1 & 2 & 3 & \dots & n+1 & n+2 & n+3 & \dots & 2n+1 \\ 2n+1 & 2n-1 & 2n-3 & \dots & 1 & 2n & 2n-2 & \dots & 2 \\ n+1 & n+2 & n+3 & \dots & 2n+1 & 1 & 2 & \dots & n \end{array}\right],$$

by adding

$$\left[\begin{array}{c} (2n+1)^2 - 1 \\ n(2n+2) \\ (n-1)(2n+2) \end{array}\right]$$

to each of the first $n+1$ columns and

$$\left[\begin{array}{c} (2n+1)^2 - 1 \\ (n-1)(2n+2) \\ n(2n+2) \end{array}\right]$$

to the remainder. It is clear that each row of X is a permutation of $\{1, 2, \dots, 2n+1\}$, so rows n and $n+1$ between them contain each of $(n-1)(2n+2)+1, (n-1)(2n+2)+2, \dots, (n+1)(2n+2)$ exactly once. When $1 \le i < n$, rows i and $2n+1-i$ contain between them all the integers

$$\begin{array}{rcl} t + (i-1)(2n+2) & : & 1 \le t \le 2n+2, \\ t + (2n-i)(2n+2) & : & 1 \le t \le 2n+2 \end{array}$$

precisely once each ($2n+2+(i-1)(2n+2) = i(2n+2) = c_{i0}$, $2n+2+(2n-i)(2n+2) = c_{2n+1-i,0}$, and the others are given by (iii) and (iv)). Row 0 provides $0, (2n+1)^2, (2n+1)^2+1, \dots, (2n+1)^2+2n = 4n^2+6n+1$. So C contains each of $0, 1, \dots, 4n^2+6n+1$ exactly once.

From (iii) and (iv) it also follows that

$$c_{ij} + c_{2n+1-i,j} = j + (i-1)(2n+2) + 2n(2n+2) - [j + (i-1)(2n+2)] = 2n(2n+2)$$

for $1 \le i < n$. Each column of X has sum $3n+3$, so $c_{nj} + c_{n+1,j} + c_{0,j} = (n+1)(8n+1)$ for $1 \le j \le 2n+1$. Therefore the sum of column j is

$$\begin{aligned} \sum_{i=0}^{2n} c_{ij} &= (n-1)2n(2n+2) + c_{nj} + c_{n+1,j} + c_{0,j} \\ &= (n-1)2n(2n+2) + (n+1)(8n+1) \\ &= (n+1)(2n+1)^2. \end{aligned}$$

If $i \leq n$,

$$\begin{aligned}\sum_{j=0}^{2n+1} c_{ij} &= \sum_{j=1}^{2n+1} j + c_{i0} + n(i-1)(2n+2) + (n+1)(2n-i)(2n+2) \\ &= \binom{2n+2}{2} + (2n+2)i + (2n+2)[n(i-1) + (n+1)(2n-i)] \\ &= (n+1)(2n+1) + 2(n+1)n(2n+1) \\ &= (n+1)(2n+1)^2,\end{aligned}$$

and a similar calculation gives the same sum if If $i \geq n+1$. □

Remark. The array X was constructed by Kotzig and is found in the unpublished technical report [3]. He produced the array in solving a magic labeling problem. However, the problem was one of *edge*-magic total labeling, and does not even remotely involve complete bipartite graphs. This surprising connection was not noticed until after Theorem 4 had been proven, when one of us realized that Kotzig's array had the required properties.

As [3] is not readily available, the interested reader may wish to consult [6], where the original application of the array is presented.

4 The spectrum of VMTLs

In those cases where magic labelings are known to exist, it is interesting to know the set of values k such that there is a magic labeling with magic constant k. This is the *spectrum* of the labeling problem.

Suppose G has a vertex-magic total labeling λ. Write S_E for the sum of edge-labels: $S_E = \sum_{x \in E(G)} \lambda(x)$. Then, counting the sum of labels at all vertices, we have

$$S_E + \binom{v+e+1}{2} = vk. \tag{5}$$

Clearly,

$$\sum_{i=1}^{e} i \leq S_E \leq \sum_{i=v+1}^{v+e} i,$$

or

$$\binom{e+1}{2} \leq S_E \leq \binom{e+1}{2} + ve. \tag{6}$$

These equations can be used to put bounds on the spectrum of VMTLs. For $K_{m,m}$, (5) and (6) give

$$\frac{1}{2}[(m+1)^3 - m^2] \leq k \leq \frac{1}{2}[(m+1)^3 + m^2]. \tag{7}$$

As $K_{m,m}$ is regular, the duality Theorem ([4], Theorem 1) applies, so there will be a VMTL with magic constant $\frac{1}{2}[(m+1)^3+x]$ if and only if there is one with $k=\frac{1}{2}[(m+1)^3-x]$.

For $K_{2,2}$, (7) yields $12 \leq k \leq 15$, and a complete search shows that every value can be realized. In fact, there are exactly six vertex-magic total labelings of $K_{2,2}$ (up to isomorphism). The representation matrices are:

$$k=12: \left[\begin{array}{c|cc} 0 & 5 & 7 \\ \hline 8 & 1 & 3 \\ 4 & 6 & 2 \end{array}\right] \qquad k=13: \left[\begin{array}{c|cc} 0 & 7 & 3 \\ \hline 8 & 1 & 4 \\ 2 & 5 & 6 \end{array}\right] \qquad k=13: \left[\begin{array}{c|cc} 0 & 4 & 6 \\ \hline 7 & 1 & 5 \\ 3 & 8 & 2 \end{array}\right]$$

$$k=14: \left[\begin{array}{c|cc} 0 & 6 & 2 \\ \hline 7 & 3 & 4 \\ 1 & 5 & 8 \end{array}\right] \qquad k=14: \left[\begin{array}{c|cc} 0 & 5 & 3 \\ \hline 6 & 1 & 7 \\ 2 & 8 & 4 \end{array}\right] \qquad k=15: \left[\begin{array}{c|cc} 0 & 4 & 2 \\ \hline 5 & 3 & 7 \\ 1 & 8 & 6 \end{array}\right]$$

For $K_{3,3}$, the bounds are $28 \leq k \leq 36$, and all these values can be realized. There are 35 isomorphism classes with $k=28$ and with $k=36$, 70 with $k=29$ and with $k=35$, 477 with $k=30$ and with $k=34$, 250 with $k=31$ and with $k=33$, and 882 with $k=32$.

In view of the ease with which examples are found for small m, we conjecture that every value of k allowed by (7) can be realized, but this is far from established.

In the case of $K_{m,m+1}$ we can improve on (6) by an argument similar to that used in the proof of Theorem 1. Write S_1 and S_2 for the sums of the labels on the m-set and $(m+1)$-set of vertices respectively, and S_E again for the sum of the edge-labels. Then every edge is adjacent to exactly one of the vertices in each set. Adding all labels on or adjacent to all vertices in the m-set, we get

$$\begin{aligned} km &= S_1+S_E \\ &\geq 1+2+\ldots+(m+m(m+1)), \end{aligned}$$

so

$$k \geq \tfrac{1}{2}(m+1)^2(m+2) \tag{8}$$

while the larger set of vertices yields

$$\begin{aligned} k(m+1) &= S_2+S_E \\ &\leq (m+1)+(m+2)+\ldots+((2m+1)+m(m+1)), \end{aligned}$$

whence

$$k \leq \tfrac{1}{2}(m+1)(m^2+4m+2). \tag{9}$$

For $K_{1,2}$, (8) and (9) yield $6 \leq k \leq 7$, and both values can be realized. In fact there are exactly two labelings up to isomorphism, with representation matrices

$$k = 6: \quad \left[\begin{array}{c|cc} 0 & 4 & 5 \\ \hline 3 & 2 & 1 \end{array}\right] \qquad k = 7: \quad \left[\begin{array}{c|cc} 0 & 5 & 3 \\ \hline 1 & 2 & 4 \end{array}\right].$$

However, for $K_{2,3}$, the bounds are $18 \leq k \leq 21$, but only 18, 19 and 20 can be realized. There are four labelings up to isomorphism:

$$k = 18: \left[\begin{array}{c|ccc} 0 & 11 & 10 & 9 \\ \hline 7 & 4 & 6 & 1 \\ 5 & 3 & 2 & 8 \end{array}\right] \quad k = 19: \left[\begin{array}{c|ccc} 0 & 11 & 9 & 8 \\ \hline 7 & 5 & 6 & 1 \\ 2 & 3 & 4 & 10 \end{array}\right] \quad k = 19: \left[\begin{array}{c|ccc} 0 & 11 & 9 & 8 \\ \hline 5 & 6 & 7 & 1 \\ 4 & 2 & 3 & 10 \end{array}\right] \quad k = 20: \left[\begin{array}{c|ccc} 0 & 11 & 9 & 6 \\ \hline 1 & 7 & 8 & 4 \\ 5 & 2 & 3 & 10 \end{array}\right].$$

For $K_{2,3}$, labelings are easily found for each k satisfying the bounds ($40 \leq k \leq 46$.

An obvious open question is: for which k satisfying (8) and (9) do vertex-magic total labelings exist? We lean toward the view that the case $m = 2, k = 21$ is a "small numbers" anomaly, and that all other possible magic constants can be realized.

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