Integral Sum Graphs from a Class of Trees

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ABSTRACT. A graph $G = (V, E)$ is said to be an *integral sum graph* (respectively, *sum graph*) if there is a labeling f of its vertices with distinct integers (respectively, positive integers) , so that for any two vertices u and v, uv is an edge of G if and only if $f(u) + f(v) = f(w)$ for some other vertex w. For a given graph G, the *integral sum number* $\zeta = \zeta(G)$ (respectively, *sum number* $\sigma = \sigma(G)$) is defined to be the smallest number of isolated vertices which when added to G result in an integral sum graph (respectively, sum graph). In a graph G, a vertex $v \in V(G)$ is said to a *hanging vertex* if the degree of it $d(v) = 1$. A path $P \subseteq G$, $P = x_0x_1x_2\cdots x_t$, is said to be a *hanging path* if its two end vertices are respectively a hanging vertex x_0 and a vertex x_t whose degree $d(x_t) \neq 2$ where $d(x_j) = 2$ $(j = 1, 2, \cdots, t-1)$ for every other vertex of P. A hanging path P is said to be a *tail* of G, denoted by $t(G)$, if its length $|t(G)|$ is a maximum among all hanging paths of G. In this paper, we prove $\zeta(T_3) = 0$, where T_3 is any tree with $|t(T_3)| \geq 3$. The result improves a previous result for integral sum trees from identification of Chen(1998).

1. Introduction

All graphs in this paper are finite and have no loops or multiple edges. We follow in general the graph-theoretic notation and terminology of [1] unless otherwise specified.

F.Harary introduced the idea of sum graphs and integral sum graphs [2] [3]. At first, let N denote the set of positive integers. *The sum graph* $G^+(S)$ of a finite subset $S \subset N$ is the graph (S, E) with $uv \in E$ if and only if $u+v \in S$. A graph G is said to be a *sum graph* if it is isomorphic to the sum graph of some $S \subset N$. *The sum number* $\sigma(G)$ is the smallest nonnegative m such that $G \cup mK_1$, the union of G and m isolated vertices , is a sum graph. In the above definition by using the set Z of all integers instead of N we obtain the definition of the integral sum graph. Analogously, *the integral sum number* $\zeta(G)$ is the smallest nonnegative m such that $G \cup mK_1$ is an integral sum graph. It is easy to see that the graph G is an *integral sum graph* if and only if $\zeta(G) = 0$. It is obvious that $\zeta(G) \leq \sigma(G)$. Although some results on sum graphs and integral sum graphs were presented [2-12], but a considerable number of unsolved problems were remained. One of them is the conjecture: "Every tree is an integral sum graph", which was proposed by Zhibo Chen in 1998 [10]. In order to discuss this problem, here and now, we briefly summarize some results on tree graph. F.Harary [2] has conjectured that any tree can be made into a sum graph with the addition of a single isolated vertex in 1990. This conjecture was proved by Ellingham [5] in 1993. F.Harary [3] found that all paths and stars are integral sum graphs and conjectured that every integral sum tree is a caterpillar in 1994. This conjecture was disproved by Zhibo Chen [4] in 1996. Zhibo Chen [10] has also shown that every generalized star and tree with all forks at least distance 4 apart are integral sum graphs in 1998.

In a graph G, a vertex $v \in V(G)$ is said to be a *hanging vertex* if its vertex degree $d(v) = 1$. A path $P \subseteq G$, $P = x_0x_1x_2\cdots x_t$, is said to be a *hanging path* if its two end vertices are respectively a hanging vertex x_0 and a vertex x_t with vertex degree $d(x_t) \neq 2$ and $d(x_j) = 2 (j = 1, 2, \cdots, t-1)$ for every other vertex of P. A hanging path P is said to be a *tail* of G, denoted by $t(G)$, if its length $|t(G)|$ is maximum one among all hanging paths of G. In this paper, we shall prove $\zeta(T_3) = 0$, where T_3 is any tree with $|t(T_3)| \geq 3$. A tree is said to be a *caterpillar* C, if it consists of a path $s_0s_1\cdots s_l$, called the *spine* of C, with some hanging vertices known as *feet* attached to the inner vertices (an ***inner vertex*** is a vertex with at least two adjacent vertices which are not the hanging vertices) of the spine by edges known as *legs* . Then $s_i(i = 1, 2, \cdots, l-1)$ was called as the *spine vertex* of C, s_0 as ***tail*** , s_l as ***head*** and s_{l-2} as ***heart*** and s_{l-1} as ***neck*** (see Figure 1). The result improves the previous result of integral sum trees from identification [10].

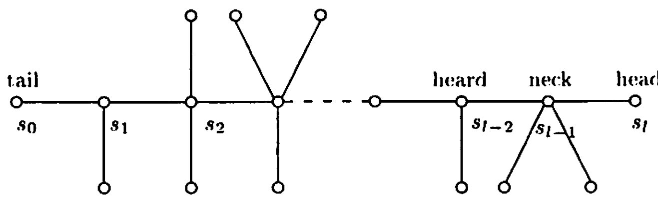

Figure 1. A caterpillar C

To prove that $\zeta(T_3) = 0$ for any tree whose tail length is not less than 3, we use a labelling algorithm. The labelling algorithm has two stages. And the first stage has to depend on the Ellingham's labelling algorithm [5] . Therefore, in the next section, we shall briefly introduce it.

Since it can easily be shown that $\zeta(T) = 0$ for $|T| < 6$, from now on, we assume $|T| \geq 6$.

2. Ellingham labelling algorithm [5]

Suppose that T is a tree with $|T| = n$ and z is an isolated vertex. And define a *shrub* S which is a special class of trees with at most one inner vertex. Then, using the Ellingham labelling algorithm, we can construct a sequence of caterpillars $C_1, C_2, \cdots, C_m$ and obtain two different types of decomposition of T.

Type 1. T is completely decomposed into some caterpillars, that is $T = C_1 \cup C_2 \cup \cdots \cup C_m$.

Type 2. T is decomposed into some caterpillars and a shrub S . Thus $T = C_1 \cup C_2 \cup \cdots \cup C_m \cup S$ $(m \geq 1)$.

Applying the Ellingham labelling algorithm, we can give a sum labelling f for the graph $T \cup \{z\}$, no matter what happens. We suppose the vertices $V(T) = \{v_1, v_2, \cdots v_n\}$ to be ordered such that $0 < f(v_1) < f(v_2) < \cdots < f(v_n) < f(z)$.

For type 1, the vertex v_n is the head of the last caterpillar C_m. If v_r and v_{n-k} are the heart and the neck of C_m, respectively, then we have that $f(v_{n-i}) = f(v_r) + (k-i)f(v_{n-k})(i = 0, 1, \cdots, k-1)$ and $f(z) = f(v_r) + (k+1)f(v_{n-k})$.

For type 2, without loss of generality, we assume that the shrub S has k hanging paths with length 2 and its root is v_r, of course v_r is also the heart of the last caterpillar C_m. Then we have that $f(z) = f(v_{n-k-i+1}) + f(v_{n-k+i})(i = 1, 2, \cdots, k)$, $f(v_r) < f(v_{n-2k})$ and $f(v_{n-k+i}) = f(v_r) + f(v_{n-2k+i})$ $(i = 1, 2, \cdots, k)$. If $v_d v_{n-2k} \in E(T)$, then we have that

$$f(v_{n-k+i}) = f(v_d) + f(v_{n-2k}) + (i-1) = f(v_{n-k+1}) + (i-1)$$

$(i = 1, 2, \cdots, k)$, $f(v_1) \geq k$ and $|f(v_i) - f(v_j)| \geq k$ for any $v_i, v_j \in V(T), i \neq j$ and $i \leq n - 2k$.

3.The integral sum labelling of T_3

Now we denote the tree whose tail length is at least 3 by T_3. In this section, we will give an integral sum labelling φ of T_3. Suppose that $|t(T_3)| = b = 3$. Therefore we can decompose T_3 into $T \cup P$, where P is a tail of T_3, $P = x_0x_1x_2x_3$, x_3 is a hanging vertex of T_3 and $V(T) = \{v_1, v_2, \cdots, v_{n-3}\}$. In order to give an integral sum labelling of T_3, using the Ellingham labelling algorithm above, we give first a sum labelling f of $T \cup \{z\}$, where z is a vertex which is not in $V(T)$. Then, we consider two cases according to the location of x_0 as follows.

Case 1. $x_0 = v_r$, in other words x_0 is the heart of the last caterpillar C_m.

In this case, let $z = x_2$ in the Ellingham labelling f above. Then we extend from sum labelling f of $T \cup \{x_2\}$ to a labelling φ of T_3 by the following algorithm. Let

$$\begin{aligned} \varphi(v) &= f(v) \text{ for } v \in V(T \cup \{x_2\}) \\ \varphi(x_1) &= -\varphi(x_2) + \varphi(v_q) \\ \varphi(x_3) &= -\varphi(x_2) + \varphi(v_q) + \varphi(v_r), \end{aligned}$$

where $v_q \in V(T)$, $v_qv_r \in E(T)$ and $\varphi(v_q) < \varphi(v_r)$ (see Figure 2).

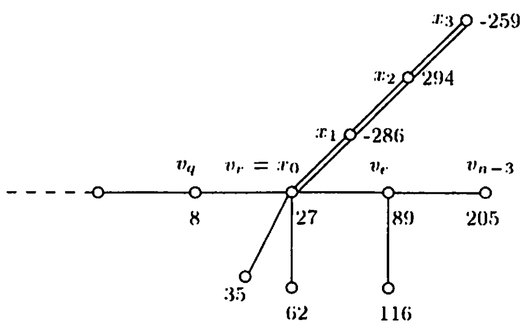

Figure 2. Illustration of Type 1 in Case 1

It is obvious that $\varphi(x_1) < \varphi(x_3) < 0 < \varphi(v_1) < \varphi(v_2) < \cdots < \varphi(v_{n-3}) < \varphi(x_2)$ and $\varphi(v_q) + \varphi(v_r) \leq \varphi(v_e)$, where v_e satisfies $v_ev_{n-3} \in E(T)$(The v_e is the neck of the last caterpillar C_m in type 1 and the v_e is equal to $v_{n-3-2k+1}$ in type 2). Now, we shall prove the labelling φ is an integral sum labelling of T_3. At first, for any $u, v \in V(T \cup \{x_2\})$, if $uv \in E(T)$, then we have $\varphi(u) + \varphi(v) = \varphi(w)$ for some $w \in V(T \cup \{x_2\})$ because $\varphi = f$ in $T \cup \{x_2\}$ and f is a sum labelling of $T \cup \{x_2\}$. In addition, we have that

$$\begin{aligned} \varphi(x_0) + \varphi(x_1) &= \varphi(v_r) - \varphi(x_2) + \varphi(v_q) = \varphi(x_3) \\ \varphi(x_1) + \varphi(x_2) &= \varphi(v_q) \text{ and} \\ \varphi(x_2) + \varphi(x_3) &= \varphi(v_q) + \varphi(v_r) = \varphi(y), \end{aligned}$$

for some $y \in V(T \cup \{x_2\})$ by $v_qv_r \in E(T)$. Therefore we have that for any $u, v \in V(T_3)$ and $uv \in E(T_3)$, $\varphi(u) + \varphi(v) = \varphi(w)$ for some $w \in V(T_3)$. Now, we just need to show that if any $u, v \in V(T_3)$, $u \neq v$ and $uv \notin E(T_3)$, then $\varphi(u)+\varphi(v)\neq\varphi(w)$ for any $w\in V(T_3)$. We may assume without loss of generality that $\varphi(u)<\varphi(v)$.

(1) For any $u,v\in V(T\cup\{x_2\})$, $u\neq v$ and $uv\notin E(T_3)$.

It is obvious that $\varphi(u)+\varphi(v)\neq\varphi(w)$ for any $w\in V(T\cup\{x_2\})$ because $\varphi=f$ in $T\cup\{x_2\}$. In addition, by the construction of the labeling φ, we know that $\varphi(x_1)<\varphi(x_3)<0$. Thus $\varphi(u)+\varphi(v)\neq\varphi(w)$ for any $w\in V(T_3)$.

(2) For any $v\in V(T_3)$ and $x_1v\notin E(T_3)$.

It is obvious that $v\neq x_0,x_2$, therefore

$$\begin{aligned}\varphi(x_1)+\varphi(v) &\leq -\varphi(x_2)+\varphi(v_q)+\varphi(v_{n-3})\\ &= -\varphi(v_{n-3})-\varphi(v_e)+\varphi(v_q)+\varphi(v_{n-3})\\ &= -\varphi(v_e)+\varphi(v_q)\\ &< -\varphi(v_r)+\varphi(v_q)<0.\end{aligned}$$

In addition, for $w=x_3$ we have only that $\varphi(x_0)+\varphi(x_1)=\varphi(x_3)$, but it is not in this case. Thus $\varphi(x_1)+\varphi(v)\neq\varphi(w)$ for any $w\in V(T_3)$ and $x_1v\notin E(T_3)$.

(3) For any $v\in V(T_3)\setminus\{x_1\}$ and $x_3v\notin E(T_3)$.

It is obvious that $v\neq x_2$, therefore

$$\begin{aligned}\varphi(x_1) &< \varphi(x_3)+\varphi(v)\leq\varphi(x_3)+\varphi(v_{n-3})\\ &= -\varphi(x_2)+\varphi(v_r)+\varphi(v_q)+\varphi(v_{n-3})\\ &= -\varphi(v_{n-3})-\varphi(v_e)+\varphi(v_r)+\varphi(v_q)+\varphi(v_{n-3})\\ &= -\varphi(v_e)+\varphi(v_r)+\varphi(v_q)\\ &\leq -\varphi(v_e)+\varphi(v_e)=0.\end{aligned}$$

Thus $\varphi(x_3)+\varphi(v)\neq\varphi(w)$ for any $w\in V(T_3)$ and $x_3v\notin E(T_3)$.

Summarizing the above mentions in Case 1, we obtain that labelling φ satisfies the condition of an integral sum labelling of T_3 when $x_0=v_r$.

Case 2. $x_0\neq v_r$.

At first, let $z=x_1$ in the Ellingham labelling f above. Then we can extend from the sum labelling f of $T\cup\{x_1\}$ to a labelling φ of T_3 by the following algorithm. Let

$$\begin{aligned}\varphi(v) &= f(v)\ \text{ for }\ v\in V(T\cup\{x_1\})\\ \varphi(x_3) &= \varphi(x_1)+\varphi(x_0)\\ \varphi(x_2) &= -\varphi(x_1)+\varphi(v_q),\end{aligned}$$

where $v_q \in V(T)$, $v_q x_0 \in E(T)$.

It is easy to see that $\varphi(x_2) < 0 < \varphi(v_1) < \varphi(v_2) < \cdots < \varphi(v_{n-3}) < \varphi(x_1) < \varphi(x_3)$.

It is easy to verify that if $u, v \in V(T_3), u \neq v$ and $uv \in E(T_3)$, then $\varphi(u) + \varphi(v) = \varphi(w)$ for some $w \in V(T_3)$. So we just need to show that for any $u, v \in V(T_3)$, if $uv \notin E(T_3)$, then there is no $w \in V(T_3)$ such that $\varphi(u) + \varphi(v) = \varphi(w)$. We may assume without loss of generality that $\varphi(u) < \varphi(v)$.

At first, if $u, v \in V(T \cup \{x_1\})$, and $uv \notin E(T_3)$, then $\varphi(u) + \varphi(v) \neq \varphi(w)$ for any $w \in V(T \cup \{x_1\})$ because $\varphi = f$ in $T \cup \{x_1\}$ and f is a sum labelling of $T \cup \{x_1\}$. Next, for $\varphi(u) + \varphi(v) = \varphi(x_3)$, $v = x_1$ if and only if $u = x_0$ according to the labeling φ. Therefore if $\varphi(u) + \varphi(v) = \varphi(x_3)$, then there must be that $\varphi(v) \leq \varphi(v_{n-3})$. We consider two subcases as follows.

Subcase 1. For $\varphi(u) \leq \varphi(v_e)$, where $v_e v_{n-3} \in E(T)$, we have that

$$\varphi(u) + \varphi(v) \leq \varphi(v_e) + \varphi(v_{n-3}) = \varphi(x_1) < \varphi(x_3)$$

Subcase 2. For $\varphi(u) > \varphi(v_e)$, where $v_e v_{n-3} \in E(T)$.

(1) If T is decomposed into type 1, namely $T = C_1 \cup C_2 \cup \cdots \cup C_m$, then

$$\begin{aligned}
\varphi(u) + \varphi(v) &= \varphi(v_{n-3-j}) + \varphi(v_{n-3-i}) \quad (i < j; i, j = 0, 1, 2, \cdots, k-1) \\
&= 2\varphi(v_{n-3}) - (i + j)\varphi(v_e) \\
&= \varphi(v_{n-3}) + \varphi(x_1) - (i + j + 1)\varphi(v_e) \\
&= \varphi(v_{n-3}) + \varphi(x_3) - \varphi(x_0) - (i + j + 1)\varphi(v_e) \\
&= \varphi(v_r) + \varphi(x_3) - \varphi(x_0) + (k - i - j - 1)\varphi(v_e).
\end{aligned}$$

Therefore now if $\varphi(u) + \varphi(v) = \varphi(x_3)$, then $\varphi(x_0) - \varphi(v_r) = (k - i - j - 1)\varphi(v_e)$. But since $x_0 \neq v_r$, we have $(k - i - j - 1) \neq 0$. Hence if $(k - i - j - 1) < 0$, then we obtain $\varphi(v_r) < \varphi(v_e)$, which is a contradiction with the Ellingham labelling of the last caterpillar C_m in type 1. If $(k - i - j - 1) > 0$, then we obtain $\varphi(x_0) \geq \varphi(v_r) + \varphi(v_e) = \varphi(v_{n-3-k+1})$, which is a contradiction with the supposition of $|t(T_3)| = 3$.

(2) If T is decomposed into type 2, namely $T = C_1 \cup C_2 \cup \cdots \cup C_m \cup S$, then

when $\varphi(u) \leq \varphi(v_{n-3-k})$,

$$\begin{aligned}\varphi(u)+\varphi(v) &\leq \varphi(v_{n-3-k})+\varphi(v_{n-3}) \\ &= \varphi(v_{n-3-k})+\varphi(v_{n-3-k+1})+(k-1) \\ &= \varphi(x_1)+(k-1) < \varphi(x_1)+\varphi(x_0)=\varphi(x_3);\end{aligned}$$

when $\varphi(u) > \varphi(v_{n-3-k})$,

$$\begin{aligned}\varphi(u)+\varphi(v) &= \varphi(v_{n-3-j})+\varphi(v_{n-3-i}) \\ &= 2\varphi(v_{n-3})-(i+j) \\ &= (\varphi(v_{n-3-k+1})+(k-1))+(\varphi(x_1)-\varphi(v_e))-(i+j) \\ &= \varphi(v_e)+\varphi(v_r)+\varphi(x_3)-\varphi(x_0)-\varphi(v_e)+(k-i-j-1) \\ &= \varphi(v_r)+\varphi(x_3)-\varphi(x_0)+(k-i-j-1).\end{aligned}$$

$(i<j; i,j=0,1,2,\cdots,k-1)$. Therefore now if $\varphi(u)+\varphi(v)=\varphi(x_3)$, then we have that $\varphi(x_0)=\varphi(v_r)+(k-i-j-1)$, that is $|\varphi(x_0)-\varphi(v_r)| = |k-i-j-1| \leq k-2$, which is a contradiction with the Ellingham labelling of the shrub S in type 2.

Summarizing the above mentions in Case 2, we obtain that the labelling φ satisfies the condition of an integral sum labelling of T_3 when $x_0 \neq v_r$.

Finally, since $\varphi(x_3)$ is the maximum label of T_3, for any $v \in V(T)$, we have that $\varphi(v)+\varphi(x_3) > \varphi(x_3)$. Hence, there is not $w \in V(T_3)$ such that $\varphi(v)+\varphi(x_3)=\varphi(w)$ for any $u,v \in V(T_3)$, $u \neq v$ and $uv \notin E(T_3)$.

Summarizing the above mentions, we can conclude that the labelling φ is an integral sum labelling of the tree T_3 with $|t(T_3)|=3$.

When $|t(T_3)|=b>3$, namely $P=x_0x_1\cdots x_b$ is a tail of T_3 , it need only to take $T=T_3 \setminus \{x_b, x_{b-1}, x_{b-2}\}$ and $P^*=x_{b-3}x_{b-2}x_{b-1}x_b$. And let $T_3=T \cup P^*$ and $\{x_{b-3}\}=T \cap P^*$. We choose the hanging vertex x_{b-3} to be the head of the first caterpillar C_1 of T (take the other end of the most longest path in T started from here as the tail of C_1) and then decompose T outright or partially into a sequence of caterpillars $C_1, C_2, \cdots, C_m$. Finally, using the above complete labeling algorithm, we can obtain an integral sum labeling φ of T_3 with the tail length more than 3. Thus we obtain the following result.

Theorem 1 *If T_3 is a tree with tail length at least 3, then $\zeta(T_3)=0$.*

4. Remarks

Recently, Zhibo Chen [10] proved that any tree T with all forks at least 4 apart is an integral sum graph. Although our result is not the final solution on integral sum trees, it improves the previous result and is very close to completion. We try hard to explore a method into the study of integral sum graphs in this paper. This method can connect the sum graph with the integral sum graph. That is, we extend from a sum labelling to the integral sum labelling. We believe that this method can be applied in elsewhere with similar problems, such as general graphs with tail.

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