Square-free colorings of graphs

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Abstract

Let G be a graph and let c be a coloring of its edges. If the sequence of colors along a walk of G is of the form $a_1, \ldots, a_n, a_1, \ldots, a_n$, the walk is called a square walk. We say that the coloring c is square-free if any open walk is not a square and call the minimum number of colors needed so that G has a square-free coloring a walk Thue number and denote it by $\pi_w(G)$. This concept is a variation of the Thue number introduced by Alon, Grytczuk, Hałuszczak, and Riordan in [1].

Using the walk Thue number several results of [1] are extended. The Thue number of some complete graphs is extended to Hamming graphs. This result (for the case of hypercubes) is used to show that if a graph G on n vertices and m edges is the subdivision graph of some graph, then $\pi_w(G) \leq n - \frac{m}{2}$. Graph products are also considered. An inequality for the Thue number of the Cartesian product of trees is extended to arbitrary graphs and upper bounds for the (walk) Thue number of the direct and the strong products are also given. Using the latter results the (walk) Thue number of complete multipartite graphs is bounded which in turn gives a bound for arbitrary graphs in general and for perfect graphs in particular.

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1 Introduction

A sequence $a = a_1a_2 \dots a_n$ of symbols from a set S is called *non-repetitive* if it does not contain a subsequence $a_{i+1}a_{i+2} \dots a_{i+2m}$ such that $a_{i+j} = a_{i+j+m}$ for $j = 1, \dots, m$. Such a subsequence is called a *square* because it can be written in the form xx, where x is a subsequence of consecutive symbols of a. A theorem of Thue [13] asserts that three symbols suffice to construct non-repetitive sequences of arbitrary length. Thue's theorem is a milestone for many applications and generalizations in several different areas of mathematics. For more details we refer to the introduction in [1] and references therein; see also interesting recent results in [7, 9, 10].

Suppose that in sequences we wish to forbid the existence of certain subsequences of length two. Then the problem has the following graph theoretical interpretation. Let D be a digraph with the vertex set S, does D contain arbitrarily long nonrepetitive walks? The problem was completely solved by Currie [4] by providing a certain classification scheme for digraphs. The corresponding problem for (undirected) graphs has a much simpler solution: A connected graph G contains arbitrarily long nonrepetitive walks unless G is a path on four or fewer vertices [5].

Alon, Grytczuk, Hałuszczak, and Riordan recently [1] proposed another related graph theoretic concept. Their generalization uses the concept of edge-colorings in such a way that Thue's theorem asserts that 3 colors suffice to color the edges of an arbitrary path. Since paths appear as subgraphs of arbitrary graphs, it is natural to consider sequences of colors along all paths in a graph. Thus, for a graph G, a coloring c of $E(G)$ is called *non-repetitive* if the sequence of colors on any (open) path P in G is non-repetitive. (Note that repetitive sequences on closed paths, that is on cycles, are not forbidden.) The minimum number of colors needed for a non-repetitive coloring is called the *Thue number* of a graph G, and denoted by $\pi(G)$. In this language Thue's theorem asserts that $\pi(P_n) = 3$ for $n \geq 5$. The problem of determining the Thue number of cycles has been posed by R.J. Simpson as well as in [1], and has been recently solved by Currie [6] who proved that, except for $n = 5, 7, 9, 10, 14$, and 17, $\pi(C_n) = 3$.

In this paper we introduce a variation of non-repetition in graphs that involves walks instead of paths. The concept and the corresponding graph invariant called the "walk Thue number" are introduced in the next section. We show that the walk Thue number and the Thue number coincide on trees and cycles. In Section 3 we determine the (walk) Thue number of some Hamming graphs and in particular of hypercubes, which in turn enables us to prove an upper bound for the (walk) Thue number of an arbitrary subdivision graph. Then, in Section 4, we consider the (walk) Thue number of the Cartesian, the direct, and the strong product of graphs. We extend an inequality of [1] from the Cartesian product of trees to the Cartesian product of arbitrary graphs and obtain upper bounds for the (walk) Thue number of the other two products as well. The proofs of these results are rather straightforward which indicates that the new concept is very natural. As an application of these "product results" the (walk) Thue number of complete multipartite graphs is bounded which in turn implies that for an arbitrary graph G, $\pi_w(G) \leq (2\alpha(G)-2)\cdot(2\kappa(\overline{G})-3)$, where $\alpha(G)$ denotes the independence number of G and $\kappa(G)$ the clique cover number of G. The paper is concluded with several remarks including two additional variations of nonrepetitive colorings proposed by Grytczuk and Currie.

2 Square-free walks

A walk W in a graph G is a sequence of (not necessarily different) edges $e_1,\ldots,e_k$ such that $e_i = x_{i-1}x_i$ for some sequence of vertices $x_0,\ldots,x_k$ of G. If $x_0 = x_k$ the walk is *closed*. Otherwise the walk is *open*.

Let G be a graph and c a coloring of its edges. If the sequence of colors along a walk of G forms a square then this walk will be called a *square (walk)*. Note that the walk ee, where e is an arbitrary edge of G, is a (closed) square walk. Similarly, if G contains cycles then we always have (closed) square walks just by going twice around a cycle. Hence to avoid square walks, we should restrict our concern only to open walks. Note also that eef, where f is an incident edge of e, is an open walk along which we always get a repetitive coloring.

By the above, walks inducing repetitions cannot be avoided. We thus say that the coloring c of $E(G)$ is *a square-free coloring* if any open walk is not a square. We call the minimum number of colors needed so that G has a square-free coloring the *walk Thue number* of G and denote it by $\pi_w(G)$. If a coloring of G is square-free then there are also no square paths, thus the coloring is clearly non-repetitive, and so

$$\pi(G) \leq \pi_w(G)$$

holds for any graph G. On the other hand the top coloring of the graph H on Fig. 1 is non-repetitive but contains a square walk. Nevertheless we have $\pi_w(H) = \pi(H) = 4$ as the bottom coloring from Fig. 1 shows.

The following simple lemma provides an essential simplification for our further arguments.

Lemma 2.1 *Let G be a graph and let c be a coloring of $E(G)$. If G contains an open square walk (with respect to c) then G contains an open square walk $e_1e_2\ldots e_{2n}$ such that $e_i \neq e_{i+1}$, $i = 1,2,\ldots,2n-1$.*

Proof. Let $W = e_1e_2\ldots e_{2k}$ be an open square walk with respect to c. Then $k \geq 2$. Suppose that for some i, $e_i = e_{i+1}$. We may without loss

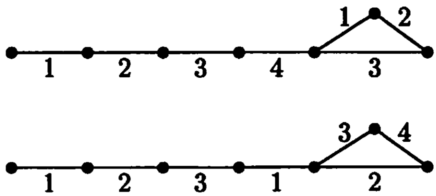

Figure 1: A non-repetitive, and a square-free coloring

of generality assume $1 \leq i \leq k$. Assume first $k = 2$. If $i = 1$, then e_3e_4 must be an open square walk and if $i = 2$, then e_1e_4 is an open square walk. So let $k > 2$. Assume $i = k$ and consider the walk $W' = e_2e_3 \ldots e_{k-1}e_{k+2}e_{k+3} \ldots e_{2k-1}$. If it is not open, then as $c(e_{2k}) = c(e_k) = c(e_{k+1}) = c(e_1)$, e_1e_{2k} forms an open square walk. Otherwise, $|W'| = |W| - 4$ and we proceed by induction. Assume finally that $i < k$. If $e_{i+k} \neq e_{i+k+1}$, then as $c(e_{i+k}) = c(e_{i+k+1})$, $e_{i+k}e_{i+k+1}$ is an open square walk. Otherwise, consider the walk $W'' = e_1 \ldots e_{i-1}e_{i+2}e_{i+3} \cdots e_{i+k-1}e_{i+k+2} \cdots e_{2k}$. It is an open square walk with $|W'| = |W| - 4$ and the induction completes the argument. □

Coloring every edge of a graph with its own color and using Lemma 2.1 we infer that π_w is well-defined:

Corollary 2.2 *For any graph G, $\pi_w(G) \leq |E(G)|$.*

For trees and cycles we can say more.

Proposition 2.3 *(i) For any tree T, $\pi_w(T) = \pi(T)$.*
(ii) For any $n \geq 3$, $\pi_w(C_n) = \pi(C_n)$.

Proof. (i) We only need to show that $\pi_w(T) \leq \pi(T)$. So let c be a non-repetitive coloring of $E(T)$ using $\pi(T)$ colors. Let a be an open square walk with respect to c. By Lemma 2.1, T contains an open square walk $a' = a_1a_2 \ldots a_{2n}$ such that $a_i \neq a_{i+1}$, $i = 1, 2, \ldots, 2n-1$. As T is a tree, a' induces a repetitive path, a contradiction.

(ii) Let c be a non-repetitive coloring of $E(C_n)$ using $\pi(C_n)$ colors, where the edges of C_n are colored with $a_1, a_2, \ldots, a_n$. Let a be an open square walk with respect to c. By Lemma 2.1 we can assume that a does not contain a subsequence of two equal colors. Also, as c is non-repetitive, a passes at least $n+1$ edges. So assume that $a = a_1a_2 \ldots a_na_1a_2 \ldots a_r$, where $r < n$ and $n + r = 2t$. As a is a square walk, $a_i = a_{t+i}$ for $i = 1, \ldots, t$. From the definition of a we also get $a_{n-t+i} = a_i$ for $i = 1, \ldots, n-t$. Consider now the sequence $a' = a_1, \ldots, a_{n-t}, a_{n-t+1}, \ldots, a_{2(n-t)}$. By the above, a' is a square. But since $t > n/2$ we have $2(n-t) < n$, which implies that a' induces a repetitive path of C_n, a contradiction. The case when $a = (a_1a_2\dots a_n)^k a_1a_2\dots a_r$, $k > 1$, is treated similarly and left to the reader. □

So the Thue number and the walk Thue number coincide on trees and cycles. On the other hand, Barát and Varjú [2] constructed examples demonstrating that these numbers are in general different.

3 Hamming graphs and subdivision graphs

The *Cartesian product* $G \square H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph with the vertex set $V(G) \times V(H)$ and (a, x) is adjacent to (b, y) in $E(G \square H)$ whenever $ab \in E(G)$ and $x = y$, or, if $a = b$ and $xy \in E(H)$. The Cartesian product is associative and commutative in a natural way, cf. [11]. *Hamming graphs* are, by definition, Cartesian products of complete graphs. We can compute the walk Thue number for the following Hamming graphs.

Theorem 3.1 *Let $k_1, k_2, \dots, k_r \geq 1$. Then*

$$\pi_w(K_{2^{k_1}} \square K_{2^{k_2}} \square \cdots \square K_{2^{k_r}}) = \sum_{i=1}^{r} 2^{k_i} - r .$$

Proof. Set $G = K_{2^{k_1}} \square K_{2^{k_2}} \square \cdots \square K_{2^{k_r}}$ and $p = \sum_{i=1}^{r} 2^{k_i} - r$. As $\Delta(G) = p$, we only need to show that $\pi_w(G) \leq p$.

Let $x \in V(G)$, then $x = (x_1, x_2, \dots, x_r)$, where $1 \leq x_i \leq 2^{k_i}$. For $i = 1, 2, \dots r$ let H_i be the group isomorphic to $\mathbb{Z}_2^{k_i}$, where we consider the groups H_i to be pairwise disjoint. Let $f_i : V(K_{2^{k_i}}) \to H_i$ be an (arbitrary) bijection.

Let $xy \in E(G)$. Then we have $x_i \neq y_i$ for some i, and $x_j = y_j$ for $j \neq i$. Denote $\ell(xy) = i$ and color the edge xy with $f_i(x_i) + f_i(y_i)$, where the computation is done in the group H_i. As $f_i(x_i) + f_i(y_i) = 0$ if and only if $x_i = y_i$ we have colored the edges of G with p colors. It remains to show that this coloring, say c, is square-free.

Let $Q = x^{(0)}, x^{(1)}, \dots, x^{(2m)}$, $m \geq 1$, be a square walk of G. Thus $c(x^{(0+i)}x^{(1+i)}) = c(x^{(m+i)}x^{(m+1+i)})$, $0 \leq i \leq m-1$. Let j be an arbitrary index $(1 \leq j \leq r)$, and consider the sequence of coordinates

$$x_j^{(0)}, x_j^{(1)}, \dots, x_j^{(2m)} .$$

Suppose that $\ell(x^{(0+i)}x^{(1+i)}) \neq j$. Since $c(x^{(0+i)}x^{(1+i)}) = c(x^{(m+i)}x^{(m+1+i)})$ we also have $\ell(x^{(m+i)}x^{(m+1+i)}) \neq j$. It follows that $x_j^{(0+i)} = x_j^{(1+i)}$ and

$x_j^{(m+i)} = x_j^{(m+1+i)}$. On the other hand, if $\ell(x^{(0+i)}x^{(1+i)}) = j$ then also $\ell(x^{(m+i)}x^{(m+1+i)}) = j$ and therefore $f_j(x_j^{(0+i)}) + f_j(x_j^{(1+i)}) = f_j(x_j^{(m+i)}) + f_j(x_j^{(m+1+i)})$. Hence

$$\sum_{i=0}^{m-1}\left(f_j(x_j^{(0+i)}) + f_j(x_j^{(1+i)})\right) = \sum_{i=0}^{m-1}\left(f_j(x_j^{(m+i)}) + f_j(x_j^{(m+1+i)})\right),$$

which in turn implies $f_j(x_j^{(0)}) + f_j(x_j^{(m)}) = f_j(x_j^{(m)}) + f_j(x_j^{(2m)})$ and so $f_j(x_j^{(0)}) = f_j(x_j^{(2m)})$. We conclude that $x^{(0)} = x^{(2m)}$, so any square walk is closed. □

As for any graph G we have $\Delta(G) \le \pi(G) \le \pi_w(G)$ we infer

Corollary 3.2 *Let* $k_1, k_2, \ldots, k_r \ge 1$. *Then*

$$\pi(K_{2^{k_1}} \Box K_{2^{k_2}} \Box \cdots \Box K_{2^{k_r}}) = \sum_{i=1}^{r} 2^{k_i} - r.$$

This result is an extension of the first part of a proposition from [1] for complete graphs. Similarly we can extend the second part of the proposition as follows.

Corollary 3.3 *Let* $n_1, n_2, \ldots, n_r \ge 2$. *Then*

$$\pi_w(K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_r}) \le 2\sum_{i=1}^{r} n_i - 3r.$$

Proof. Set $H = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_r}$ and select k_i, $1 \le i \le r$, such that $2^{k_i - 1} < n_i \le 2^{k_i}$. Let $G = K_{2^{k_1}} \Box K_{2^{k_2}} \Box \cdots \Box K_{2^{k_r}}$, then by Theorem 3.1, $\pi_w(H) \le \pi_w(G)$. It suffices to show our assertion for $n_i = 2^{k_i - 1} + 1$, $1 \le i \le r$. In this case we have:

$$\pi_w(H) \le \sum_{i=1}^{r} 2^{k_i} - r = 2\sum_{i=1}^{r}(n_i - 1) - r = 2\sum_{i=1}^{r} n_i - 3r,$$

and we are done. □

The Cartesian product of r copies of K_2 is the *r-cube* Q_r. Theorem 3.1 also immediately implies:

Corollary 3.4 *For any* $r \ge 1$, $\pi_w(Q_r) = \pi(Q_r) = r$.

The *subdivision graph* $S(G)$ of a graph G is obtained from G by subdividing every edge of G. From Corollary 3.4 we can deduce:

Corollary 3.5 *Let G be a connected graph. Then $\pi_w(S(G)) \leq |V(G)|$.*

Proof. Set $n = |V(G)|$. Using a construction from [3], see also [12], it follows that $S(G)$ is a subgraph of a hypercube. Alternatively, we can define an embedding of $S(G)$ into Q_n as follows. Label the original vertices of G by the n-tuples containing $n-1$ zeros, so that vertices receive 1's at pairwise different positions. The new vertices of $S(G)$ receive two 1's on the same places as their neighbors. This is obviously an embedding of $S(G)$ into Q_n, and since $\pi_w(Q_n) = n$, we conclude that $\pi_w(S(G)) \leq n$. □

By Corollary 3.3, $\pi_w(G) \leq 2n-3$ for any graph G on n vertices. Let H be a graph on n' vertices and m' edges and let $G = S(H)$ has n vertices and m edges. Then $n = n' + m'$ and $m = 2m'$. Hence by Corollary 3.5, $\pi_w(G) \leq n' = n - m' = n - \frac{m}{2}$. In conclusion, if G is a subdivision graph of some graph, then the general bound $\pi_w(G) \leq 2n-3$ can be improved to $\pi_w(G) \leq n - \frac{m}{2}$.

4 Thue numbers of graph products

Besides the Cartesian product introduced in the previous section we will consider two additional standard products of graphs, the direct product, and the strong product [11]. Let G and H be arbitrary graphs. As for the Cartesian product, the vertex set of any of these products is $V(G) \times V(H)$. In the *direct product* $G \times H$ the vertex (a, x) is adjacent to the vertex (b, y) whenever $ab \in E(G)$ and $xy \in E(H)$. For the *strong product* $G \boxtimes H$ of graphs G and H we have $E(G \boxtimes H) = E(G \,\Box\, H) \cup E(G \times H)$. The projections p_G, p_H of a product graph to factors are defined in a natural way. Note that projections map edges of $G \times H$ to edges in factors, while an edge of $G \,\Box\, H$ is mapped to either an edge or a vertex. Also the direct and the strong product are associative and commutative in a natural way.

Theorem 4.1 *Let G and H be connected graphs. Then*
(i) $\pi_w(G \,\Box\, H) \leq \pi_w(G) + \pi_w(H)$,
(ii) $\pi_w(G \times H) \leq \pi_w(G) \cdot \pi_w(H)$,
(iii) $\pi_w(G \boxtimes H) \leq (\pi_w(G) + 1) \cdot (\pi_w(H) + 1) - 1$.

Proof. Let g an h be disjoint square-free colorings of G and H, respectively, each with the minimum number of colors.

(i) Define a coloring c of $G \,\Box\, H$ by setting $c((a,x)(b,y)) = g(ab)$ if $x = y$ (that is, if $p_G((a,x)(b,y)) = ab$), and $c((a,x)(b,y)) = h(xy)$ if $a = b$. We claim that this is a square-free coloring of $G \,\Box\, H$. Suppose that $P = e_1, e_2, \ldots, e_{2m}$, $m \geq 1$ is a square walk in G. Then the projections of P to the factors induce square walks in the factors. Note that the projection on a factor can also be empty, that is, a walk of length zero. Since g and h are square-free colorings, these walks are all closed, which in turn implies that P is closed.

(ii) Define a coloring c of $E(G \times H)$ by $c(e) = (g(p_G(e)), h(p_H(e))$. We claim that c is a square-free coloring of $G \times H$ (which obviously uses $\pi_w(G) \cdot \pi_w(H)$ colors). Suppose that $W = e_1, e_2, \ldots, e_{2m}$, $m \geq 1$ is a square walk in $G \times H$. Then $W_G = p_G(e_1), p_G(e_2), \ldots, p_G(e_{2m})$, resp. $W_H = p_H(e_1), p_H(e_2), \ldots, p_H(e_{2m})$ is a square walk in G, resp. H. Thus W_G (W_H) must be closed in G (resp. H), and so W is closed in $G \times H$.

(iii) For the strong product we combine the coloring of $E(G \Box H)$ defined in (i) and the coloring of $E(G \times H)$ defined in (ii) in such a way that the colors for the Cartesian edges are disjoint with the colors for the direct edges. This gives a coloring of $E(G \boxtimes H)$ using $\pi_w(G) \cdot \pi_w(H) + \pi_w(G) + \pi_w(H)$ colors. Using similar arguments as above we deduce that this coloring is square-free. □

In [1] it is proved that for any trees $T_1, \ldots, T_n$,

$$\pi(T_1 \Box \cdots \Box T_n) \leq \sum_{i=1}^{n} \pi(T_i).$$

As the considered products are associative, Theorem 4.1 can be extended to a finite number of factors. For instance, if $G_1, \ldots, G_n$ are (connected) graphs then

$$\pi(G_1 \Box \cdots \Box G_n) \leq \pi_w(G_1 \Box \cdots \Box G_n) \leq \sum_{i=1}^{n} \pi_w(G_i).$$

Since by Proposition 2.3 (i), $\pi_w(T) = \pi(T)$ holds for any tree T, Theorem 4.1 (i) thus extends the above result for trees.

Another application of Theorem 4.1 is the following. Denote by K_n^r the complete multipartite graph $K_{\underbrace{n, \ldots, n}_{r\text{-times}}}$.

Corollary 4.2 $\pi_w(K_n^r) \leq (2n-2)(2r-3)$ *for all* $n \geq 2$, $r \geq 2$.

Proof. Consider the direct product $G_{n,r} = K_n \times K_r$. By Theorem 4.1 (ii) and Corollary 3.3 (for one factor), $\pi_w(G_n) \leq (2n-3)(2r-3)$. Now, K_n^r can be obtained from $G_{n,r}$ by adding those Cartesian edges of $K_n \boxtimes K_r$ whose projections to K_r are edges (that is, to $G_{n,r}$ we add edges of n disjoint copies of K_r). Color the new edges of K_n^r with new colors like in the proof of (i) of Theorem 4.1, or like in the proof of (iii) of the same theorem in the case of Cartesian edges. Thus altogether $(2n-3)(2r-3) + 2r - 3$ colors are used in this construction. Using again similar arguments as above (by projecting colors) we see that this is indeed a square-free coloring of K_n^r. □

Recall that $\kappa(G)$ denotes the clique cover number of G (that is, the smallest number of complete graphs needed to cover the vertices of G), $\alpha(G)$ the independence number of G, and $\omega(G)$ the cardinality of a largest complete subgraph of G. By $\overline{G}$ we denote the complement of G.

Corollary 4.3 *For an arbitrary connected graph G,*

$$\pi_w(G) \le (2\alpha(G) - 2) \cdot (2\kappa(\overline{G}) - 3).$$

Proof. Note that $\kappa(\overline{G})$ also means the minimum number of independent subsets of G such that each vertex of G is included in at least one of these subsets. By setting $r = \kappa(\overline{G})$, and $n = \alpha(G)$ we infer that $G \subseteq K_n^r$. As the walk Thue number of subgraphs is not greater than of their original graphs the result follows from Corollary 4.2. □

Among the most well-known graph classes are the *perfect graphs* [8]. One of their characteristic properties is that $\kappa(G) = \alpha(G)$ for any induced subgraph of G. Also note that the graph G is perfect if and only if $\overline{G}$ is perfect. Therefore from the corollary above we infer the following result for perfect graphs.

Corollary 4.4 *For a perfect graph G,*

$$\pi_w(G) \le (2\alpha(G) - 2) \cdot (2\omega(G) - 3).$$

Since any bipartite graph B is obviously perfect, and its maximum clique has two vertices, we get in this case $\pi_w(B) \le 2\alpha(G) - 2$.

5 Concluding remarks

1. We present (a sketch of) an alternative proof of Theorem 3.1. We claim first that Theorem 4.1 (iii) implies $\pi(K_{2^k}) = 2^k - 1$ for all $k \ge 1$. Indeed, the statement is clear for $k = 1$. For $k > 1$ we have

$$\pi(K_{2^k}) \le \pi_w(K_{2^k}) = \pi_w(K_2 \boxtimes K_{2^k}) \le 2(2^{k-1} - 1 + 1) - 1 = 2^k - 1.$$

As $\pi(K_{2^k}) \ge \Delta(K_{2^k}) = 2^k - 1$ the claim follows. Let $G = K_{2^{k_1}} \square \cdots \square K_{2^{k_r}}$. By Theorem 4.1 (i) (and the remarks after its proof), $\pi_w(G) \le \sum_{i=1}^{r} 2^{k_i} - r$. For the lower bound we again use the fact that $\pi_w(G) \ge \Delta(G) = \sum_{i=1}^{r} 2^{k_i} - r$, and Theorem 3.1 follows.

2. Is there an upper bound for the walk Thue number for graphs with a given maximum degree Δ? Recall that such a bound exists for the original

Thue number, defined in [1], where it is expressed as a constant times Δ^2. This result is obtained by probabilistic approach, and it is natural to ask whether such approach could be applied for bounding π_w.

3. We conclude the paper by proposing two different variations of nonrepetitive walk colorings suggested by Grytczuk and Currie (personal communication), respectively. In the first variation, the coloring is called nonrepetitive if colors on any open simple walk form a nonrepetitive sequence (by a simple walk we mean a walk in which each edge appears only once). Obviously, this is a weaker concept than the square-free coloring as presented here, yet it is stronger than the (path) nonrepetitive coloring from [1]. In the second one, coloring is called nonrepetitive, if colors on any nonrepetitive walk (that is a walk in which there is no subsequence of edges forming a square (as sequence of edges)) form a nonrepetitive sequence. Any of these concepts seems to be natural so an investigation of relations between them would be welcome. We refer again to Barát and Varjú [2] that in general all these four concepts are different.

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References

[1] N. Alon, J. Grytczuk, M. Hałuszczak, and O. Riordan, Non-repetitive colorings of graphs, *Random Structures Algorithms*, **21** (2002), 336–346.

[2] J. Barát and P.P. Varjú, Remarks and improvements on square-free coloring of graphs, manuscript, 2003.

[3] B. Brešar, W. Imrich and S. Klavžar, Tree-like isometric subgraphs of hypercubes, *Discuss. Math. Graph Theory*, **23** (2003), 227–240.

[4] J.D. Currie, *Non-repetitive Walks in Graphs and Digraphs*, Ph.D. Thesis, University of Calgary, Alberta, Canada, 1987.

[5] J.D. Currie, Which graphs allow infinite nonrepetitive walks?, *Discrete Math.*, **87** (1991), 249–260.

[6] J.D. Currie, There are ternary circular square-free words of length n for $n \geq 18$, *Electron. J. Combin.*, **9** (2002), #N10, 7 pp.

[7] J.D. Currie and J. Simpson, Non-repetitive tilings, *Electron. J. Combin.*, **9** (2002), #R28, 13 pp.

[8] M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs, Academic Press*, New York, 1980.

[9] J. Grytczuk, Pattern avoiding colorings of Euclidean spaces, *Ars Combin.*, **64** (2002), 109–116.

[10] J. Grytczuk, Thue-like sequences and rainbow arithmetic progressions, *Electron. J. Combin.*, **9** (2002), #R44, 10 pp.

[11] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition, Wiley*, New York, 2000.

[12] W. Imrich, S. Klavžar and H.M. Mulder, Median graphs and triangle-free graphs, *SIAM J. Discrete Math.*, **12** (1999), 111–118.

[13] A. Thue, Über unendliche Zeichenreichen, *Norske Vid Selsk. Skr. I. Mat. Nat. Kl. Christiana*, **7** (1906), 1–22.