Edge Labelings with a Condition at Distance Two

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ABSTRACT

For graph G with non-empty edge set, a (j,k)-edge labeling of G is an integer labeling of the edges such that adjacent edges receive labels that differ by at least j, and edges which are distance two apart receive labels that differ by at least k. The $\lambda'_{j,k}$-number of G is the minimum span over the (j,k)-edge labelings of G. By establishing the equivalence of the edge labelings of G to particular vertex labelings of G and the line graph of G, we explore the properties of $\lambda'_{j,k}(G)$. In particular, we obtain bounds on $\lambda'_{j,k}(G)$, and prove that the Δ^2 conjecture of Griggs and Yeh is true for graph H if H is the line graph of some graph G. We investigate the $\lambda'_{1,1}$-numbers and $\lambda'_{2,1}$-numbers of common classes of graphs, including complete graphs, trees, n-cubes, and joins.

1. Introduction. In this paper, we introduce and consider the problem of labeling the edges of simple, loopless graph $G = (V, E)$ with integers constrained by edge distance conditions. We say that two edges e_1 and e_2 are adjacent (at distance one) if and only if there exists a vertex to which e_1 and e_2 are incident. Two edges e_1 and e_2 are at distance two if and only if they are not adjacent and there exists an edge to which e_1 and e_2 are incident. We let $d(x)$ (the degree of x) denote the number of edges incident to x if $x \in V$ or $x \in E$.

If G is a graph with non-empty edge set and if j and k are positive integers with $j \geq k$, then a *(j,k)-edge labeling* of G is a mapping L from $E(G)$ into the integers such that

: $|L(e_2) - L(e_1)| \geq j$ if e_1 and e_2 are adjacent in G, and
: $|L(e_2) - L(e_1)| \geq k$ if e_1 and e_2 are at distance two in G.

Elements of the image of L are called *labels,* and the *span* of L, $s(L)$, is the difference between the largest and smallest labels. The minimum span taken over all (j,k)-edge labelings of G, denoted $\lambda'_{j,k}(G)$, is called the *$\lambda'_{j,k}$-number* of G, and if L is a labeling with minimum span, then L is called a *$\lambda'_{j,k}$-labeling* of G. We shall assume with no loss of generality that the minimum label of (j,k)-edge labelings of G is 0.

The (j,k)-edge labeling problem defined above is analogous to the (j,k)-vertex labeling problem; i.e., the problem of labeling the vertices of a graph with a condition at distance two (called the $L(j,k)$ vertex labeling problem), on which there exists much literature ([1], [2], [5]-[13], [15]-[20]). The vertex labeling problem was first investigated in the case $j = 2$ and $k = 1$ by Griggs and Yeh [13]. There, they considered the $\lambda_{2,1}$-number (i.e. the minimum span over $L(2,1)$-labelings) of certain classes of graphs such as paths, cycles, trees, and n-cubes. They also presented bounds on $\lambda_{2,1}(G)$ in terms $\Delta(G)$, $\chi(G)$ and $|V(G)|$, and submitted the following:

Conjecture 1.1. For any graph H with $\Delta(H) \geq 2$, $\lambda_{2,1}(H) \leq \Delta^2(H)$.

Two additional results from the literature on vertex labelings, useful in this paper, are found in [10] and [5], respectively.

Theorem 1.2. Let G be a graph whose complement has path-covering number c. Then

i: $\lambda_{2,1}(G) \leq |V(G)|$ if $c = 1$.
ii: $\lambda_{2,1}(G) = |V(G)| + c - 2$ if $c > 1$. •

Theorem 1.3. For $r \geq 2$, let G be an r-regular graph. Then
i: $\lambda_{1,1}(G) \geq r$
ii: $\lambda_{2,1}(G) \geq r + 2$. •

In Section 2 of this paper, we derive the $\lambda'_{j,k}$-numbers of paths, cycles and complete bipartite graphs by noting the equivalence between the $\lambda'_{j,k}$-number of G and the $\lambda_{j,k}$-number of $L(G)$, the line graph of G. We also introduce another useful correspondence between (j,k)-edge labelings of the edges of G and a particular type of vertex labeling, defined subsequently, in which the labels are sets. In Section 3, we produce bounds on the $\lambda'_{1,1}$-number and $\lambda'_{2,1}$-number of arbitrary graph H, and show that if $H = L(G)$ for some graph G, then H satisfies Conjecture 1.1. In Section 4, we investigate the $\lambda'_{1,1}$-numbers and $\lambda'_{2,1}$-numbers of trees, complete graphs and regular graphs, and in Section 5 we similarly consider the n-cube. Finally, in Section 6, we investigate the $\lambda'_{1,1}$-numbers and $\lambda'_{2,1}$-numbers of joins and t-point suspensions, deriving $\lambda'_{1,1}$-numbers and $\lambda'_{2,1}$-numbers of the n-wheel W_n for $n \geq 3$.

2. Preliminary Definitions and Results. We begin with two definitions.

Definition 2.1. For positive integer p, a subset S of the set $\mathcal{Z}$ of integers is said to be *p-separated* if and only if the absolute difference between any two distinct elements in S is at least p.

Definition 2.2. For graph G, a function $f : V(G) \rightarrow 2^{\{0,1,2,3,\ldots m\}}$ is a

$(j,k)-$*set labeling of* G *with span* m if and only if f has the following properties: for any $v, w, w' \in V(G)$ such that w and w' are adjacent to v,

i: $|f(v)| = d(v)$;
ii: $f(v)$ is j-separated;
iii: $|f(v) \bigcap f(w)| = 1$;
iv: if $x \in f(v)$ and $y \in f(w)$, then $x = y$ or $|x - y| \geq k$;
v: if $f(v) \bigcap f(w) = f(v) \bigcap f(w')$ then $w = w'$;
vi: for some $v_1, v_2 \in V(G)$, $0 \in f(v_1)$ and $m \in f(v_2)$.

The smallest m for which a $(j,k)-$set labeling of G exists is called the $s\lambda_{j,k}$*-number* of G, denoted $s\lambda_{j,k}(G)$. Any $(j,k)-$set labeling of G with span $s\lambda_{j,k}(G)$, is called an $s\lambda_{j,k}$*-labeling* of G. •

For arbitrary graph G, it is clear that a (j,k)-edge labeling of G with span m induces a $(j,k)-$set labeling f of G with span m by setting $f(v)$ equal to the set of labels assigned to the edges incident to v. The converse is also true; a $(j,k)-$set labeling f of G with span m induces a (j,k)-edge labeling L of G with span m by setting $L(\{w,v\})$ equal to the unique integer in $f(w) \bigcap f(v)$. Since an analogous relationship exists between a (j,k)-edge labeling with span m and a (j,k)-vertex labeling with span m of the line graph $L(G)$, we have the following.

Proposition 2.3. Let G be a graph with non-empty edge set. Then the following are equivalent:

i: there is a (j,k)-edge labeling of G with span m;
ii: there is a (j,k)-vertex labeling of $L(G)$ with span m;
iii: there is a $(j,k)-$set labeling of G with span m.

Consequently, $\lambda'_{j,k}(G) = \lambda_{j,k}(L(G)) = s\lambda_{j,k}(G)$. •

In Figure 2.1a, we illustrate a (2,1)-edge labeling with span 6 of a graph G along with its induced $(2,1)-$ set labeling. In Figure 2.1b, we illustrate the corresponding $(2,1)$-vertex labeling of $L(G)$.

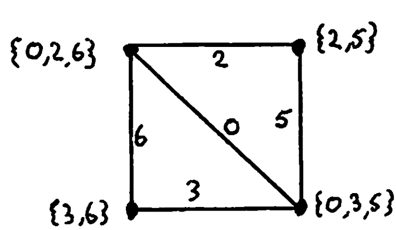

Figure 2.1a

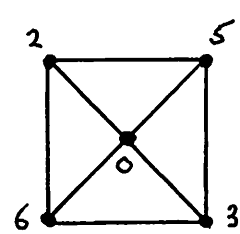

Figure 2.1b

(We may argue that $\lambda'_{2,1}(G) = 6$ as follows: noting that $\lambda'_{2,1}(G) = \lambda_{2,1}(L(G))$, we observe that $L(G)^c$ has order 5, size 2 and path covering number 3. By Theorem 1.2, $\lambda_{2,1}(L(G)) = 6$. More generally, since $L(G)$ is a 1-point suspension of C_4, we may apply results in [9] (Theorem 5.8) which imply $\lambda'_{j,k}(G) = \lambda_{j,k}(L(G)) = 2j + 2k$ for $j \geq k$.)

Definition 2.4. Let $m \geq 0, d \geq 1$ and $X = \{0, 1, 2, ..., m\}$. A collection C of non-empty subsets of X is said to be (m, d)-*feasible* if and only if the following properties hold:

(i) each element of C is d-separated, and
(ii) for each $A \in C$ and $x \in A$, there exists $B \in C$ such that $A \bigcap B = \{x\}$.
•

Theorem 2.5. If G is a graph with $(d, 1)-$set labeling L, then L induces an $(s(L), d)$-feasible set. Furthermore, if C is an (m, d)-feasible set, then there exists a (possibly infinite) graph G and a $(d, 1)-$set labeling with span at most m of G whose image is contained in C.

Proof: The first assertion follows from the fact that the image of any $(j, k)-$set labeling L with span m is an (m, j)-feasible set by definitions 2.2 and 2.4. To prove the second assertion, it suffices to produce an algorithm which generates a (possibly infinite) tree T along with a $(d, 1)-$set labeling of T whose labels are taken from C.

1. Select $V_0 \in C$. Establish 0^{th}-generation (root) vertex v_0 with $d(v_0) = |V_0|$.
2. Assign V_0 to v_0 and assign distinct elements of V_0 to the edges incident to v_0.
3. Assign to each first-generation vertex v_1^i (those incident to v_0) an element V_1^i of C which intersects V_0 at exactly the label assigned to the edge $\{v_0, v_1^i\}$.
4. Establish $|V_1^i| - 1$ unlabeled edges incident to v_1^i, and assign to those edges distinct values from $V_1^i - (V_0 \bigcap V_1^i)$. (The vertex v_1^i is a leaf if and only if $|V_1^i| = 1$.)
5. Assign to each second-generation vertex v_2^j with parent v_1^i an element V_2^j of C which intersects V_1^i at exactly the label assigned to the edge $\{w_2^j, v_1^i\}$.
6. Continue this process. •

Let $T_\infty(\Delta)$ be the infinite Δ-regular tree. As a consequence of the algorithm described in the proof of Theorem 2.5, we have

Corollary 2.6. For $\Delta \geq 2$, $T_\infty(\Delta)$ has a $(d, 1)$-edge labeling of span at most m if and only if there exists an (m, d)-feasible set each of whose elements has cardinality Δ. •

The $\lambda_{j,k}$-numbers of various graphs, which are themselves line graphs of other well-known graphs, have been studied in [9] and [12]. As a result, the $\lambda'_{j,k}$-numbers of paths, cycles and complete bipartite graphs easily follow from Proposition 2.3.

Theorem 2.7. [9] For $n \geq 2$, $\lambda'_{j,k}(P_n) = \lambda_{j,k}(L(P_n)) = \lambda_{j,k}(P_{n-1}) =$

0	if $n = 2$
j	if $n = 3$
$j + k$	if $n = 4$ or 5
$j + 2k$	if $n \geq 6$ and $\frac{j}{k} \geq 2$
$2j$	if $n \geq 6$ and $1 \leq \frac{j}{k} < 2$. •

Theorem 2.8. [9] For $n \geq 3$, $\lambda'_{j,k}(C_n) = \lambda_{j,k}(L(C_n)) = \lambda_{j,k}(C_n)$, which equals

Case 1: For $\frac{j}{k} \geq 2$

$2j$	if n is odd and $n \geq 3$
$j + 2k$	if $n = 0 \bmod 4$
$2j$	if $n = 2 \bmod 4$ and $\frac{j}{k} \leq 3$
$j + 3k$	if $n = 2 \bmod 4$ and $\frac{j}{k} \geq 3$

Case 2: For $\frac{j}{k} \leq 2$

$2j$	if $n = 0 \bmod 3$
$4k$	if $n = 5$
$j + 2k$	otherwise •

Theorem 2.9. [12] For integers $2 \leq n \leq m$, $\lambda'_{j,k}(K_{m,n}) = \lambda_{j,k}(L(K_{m,n})) = \lambda_{j,k}(K_n \times K_m) =$

$(m-1)j + (n-1)k$	if $n < m$ and $\frac{j}{k} > n$
$(mn-1)k$	if $n < m$ and $\frac{j}{k} \leq n$
$(n-1)j + (2n-2)k$	if $n = m$ and $\frac{j}{k} > n - 1$
$(n^2-1)k$	if $n = m$ and $\frac{j}{k} \leq n - 1$ •

In the remainder of this paper, we will concentrate our attention on $\lambda'_{2,1}(G)$ and $\lambda'_{1,1}(G)$.

3. General Bounds on $\lambda'_{2,1}(G)$ and $\lambda'_{1,1}(G)$. Let G be a graph with $1 \leq \delta \leq \Delta$ (where $\delta(G)$ is the minimum vertex degree over $V(G)$.) The degree of edge $\{u, v\}$ is the number of edges incident to $\{u, v\}$. Since G is assumed to be simple and loopless, $d(\{u, v\}) = d(u) + d(v) - 2$; that is, the degree of the edge $\{u, v\}$ is two fewer than the sum of the degrees of u and v. Let Ψ and ψ denote respectively the maximum and minimum edge degree of G. It follows that $2(\delta-1) \leq \psi \leq (\delta+\Delta-2) \leq \Psi \leq 2(\Delta-1)$.

We now obtain a general upper bound for $\lambda'_{2,1}(G)$. By Chang and Kuo [2], we note that for any graph H, $\lambda_{2,1}(H) \leq \Delta^2(H)+\Delta(H)$. Thus,

$$\lambda'_{2,1}(G) = \lambda_{2,1}(L(G)) \leq \Psi^2+\Psi = \Psi(\Psi+1).$$

We improve this bound to $\frac{\Psi^2}{2}+3\Psi$ by applying the strategy in Griggs and Yeh's proof [13] of $\lambda_{2,1}(H) \leq \Delta^2(H)+2\Delta(H)$ to the adjacency structure of the edges of G.

Theorem 3.1. For graph G with maximum vertex degree Δ and maximum edge degree Ψ,

$$2(\Delta-1) \leq \lambda'_{2,1}(G) \leq \Psi(\Delta+2) \leq 2(\Delta-1)(\Delta+2).$$

Furthermore, if G is Δ-regular, then $2\Delta \leq \lambda'_{2,1}(G) \leq \frac{\Psi^2}{2}+3\Psi$.

Proof: To obtain the lower bound, we note that K_Δ is a subgraph of $L(G)$, from which it follows that $\lambda'_{2,1}(G) \geq \lambda_{2,1}(K_\Delta) = 2(\Delta-1)$. If G is Δ-regular, then $L(G)$ is $2(\Delta-1)$-regular, from which the lower bound follows by Theorem 1.3.

To obtain the upper bound, arbitrarily order the edges of G, and label them in a greedy way, starting with the smallest available integer. An edge $e \in E(G)$ is adjacent to at most Ψ edges, and at distance two from at most $\Psi(\Delta-1)$ edges. So there are at most $3\Psi+\Psi(\Delta-1) = \Psi(\Delta+2)$ integers not available for assignment to e, implying that e is labelable from among the first $\Psi(\Delta+2)$ non-negative integers. Hence $\lambda'_{2,1}(G) \leq \Psi(\Delta+2) \leq 2(\Delta-1)(\Delta+2)$ since $\Psi \leq 2(\Delta-1)$, with equality if G is Δ-regular. •

As pointed out, Griggs and Yeh [13] conjectured that for all graphs H with $\Delta(H) \geq 2$, $\lambda_{2,1}(H) \leq \Delta^2(H)$. Theorem 3.1 implies the truth of the conjecture for a particular class of graphs.

Corollary 3.2. If H is a graph such that $H = L(G)$ where $\delta(G) \geq 4$, then $\lambda_{2,1}(H) \leq \Delta^2(H)$.

Proof: Noting that $\Delta(H) = \Psi(G) \geq \Delta(G)+\delta(G)-2 \geq \Delta(G)+2$, we have $\lambda_{2,1}(H) = \lambda'_{2,1}(G) \leq \Psi(G)(\Delta(G)+2) \leq \Psi^2(G) = \Delta^2(H)$. •

Since $K_{1,\Psi}$ is a subgraph of $L(G)$, we easily modify the argument of Theorem 3.1 to obtain

Theorem 3.3. Let G be a graph with $\Delta \geq 1$. Then $\Psi \leq \lambda'_{1,1}(G) \leq \Psi\Delta \leq 2\Delta(\Delta-1)$. Furthermore, if G is Δ-regular, then $\Psi \leq \lambda'_{1,1}(G) \leq \frac{\Psi^2}{2}+\Psi$ •.

For Δ-regular graph G, $\Delta \geq 3$, we note that, in the next section, the lower bound of 2Δ for $\lambda'_{2,1}(G)$ in Theorem 3.1 will be improved to $2\Delta + 1$.

4. On the $\lambda'_{1,1}$-numbers and $\lambda'_{2,1}$-numbers of Complete Graphs, Trees and Regular Graphs. For $n \geq 2$, the edges of K_n are pairwise at most distance two apart. Thus, $\binom{n}{2} - 1$ serves as a lower bound for both $\lambda'_{1,1}(K_n)$ and $\lambda'_{2,1}(K_n)$. It is clear that $\lambda'_{1,1}(K_n) = \binom{n}{2} - 1$ since any bijection from $\{0, 1, 2, ..., \binom{n}{2} - 1\}$ to $E(K_n)$ is a $(1,1)$-edge labeling.

Theorem 4.1. For $n \geq 2$, $\lambda'_{2,1}(K_n) =$

$$\begin{cases} 0 & \text{if } n = 2 \\ 4 & \text{if } n = 3 \\ 7 & \text{if } n = 4 \\ \binom{n}{2} - 1 & \text{if } n \geq 5 \end{cases}$$

Proof: The cases $n = 2$ and $n = 3$ follow immediately from Theorems 2.7 and 2.8 since K_2 and K_3 are respectively P_2 and C_3.

For $n \geq 4$, we observe that $L(K_n)$ is a $(2n-4)$-regular graph with order $\binom{n}{2}$ and diameter two. As a result, $\lambda'_{2,1}(K_n) \geq \binom{n}{2} - 1$, and each $\lambda'_{2,1}$-labeling of K_n is necessarily injective.

For $n = 4$, the complement of $L(K_4)$ is 1-regular and has path covering number 3. By Theorem 1.2, it follows that $\lambda_{2,1}(L(K_4)) = \lambda'_{2,1}(K_4) = 7$.

For $n = 5$, $L(K_5)^c$ is isomorphic to the Petersen graph which has a Hamilton path. Thus, $\lambda'_{2,1}(K_5) = 9$ by Theorem 1.2 and our established lower bound of $\binom{5}{2} - 1$.

In Figure 4.1 we exhibit a $(2,1)-$set labeling of K_6 with span 14 (inducing a $(2,1)$-edge labeling with span 14), implying (by our lower bound of $\binom{6}{2} - 1$) that $\lambda'_{2,1}(K_6) = 14$.

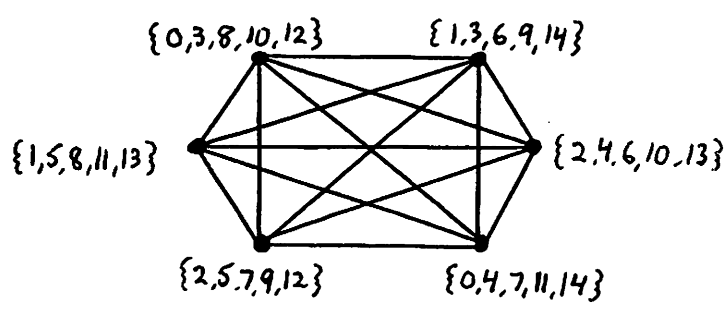

Figure 4.1

For $n \geq 7$, we observe that $L(K_n)^c$ is m-regular where $m = \binom{n}{2} - 1 - (2n-4) = \frac{n^2-5n+6}{2} \geq \frac{1}{2}(\binom{n}{2} - 1)$. By Dirac's theorem on Hamilton paths [3], $L(K_n)^c$ has a Hamilton path which, by Theorem 1.2 and our lower bound $\binom{n}{2} - 1$, implies that $\lambda'_{2,1}(K_n) = \binom{n}{2} - 1$. •

In Figures 4.2a and 4.2b, we give $\lambda'_{2,1}$-labelings of K_4 and K_5.

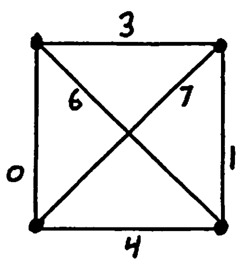

Figure 4.2a

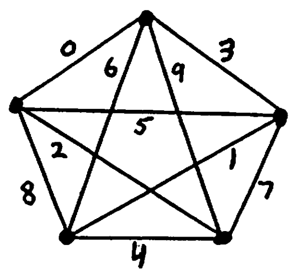

Figure 4.2b

We next turn our attention to the $\lambda'_{1,1}$-numbers and $\lambda'_{2,1}$-numbers of trees. In the case of the latter, it will be convenient to consider the edge labeling properties of infinite trees with particular attention to $T_\infty(\Delta)$, the infinite Δ-regular tree.

Theorem 4.2. Let T be a non-trivial tree. Then

i: $\lambda'_{1,1}(T) = \Psi$, and

ii: $\lambda'_{1,1}(T_\infty(\Delta)) = 2(\Delta - 1)$.

Proof: By Theorem 3.3, $\lambda'_{1,1}(T) \geq \Psi$. It thus suffices to produce a $(1,1)$-edge labeling L of T with span Ψ. We proceed by induction as follows: let $e_0 \in E(T)$ have degree Ψ, and let $L(e_0) = 0$. Then we may distribute the labels 1, 2, 3, ... Ψ over the Ψ first-generation edges adjacent to e_0. Now assume that labels from $\{0, 1, 2, ..., \Psi\}$ have been assigned to the edges of T through the k^{th}-generation in accordance with the definition of $(1,1)$-edge labelings. Let $e = \{a, b\}$ be a k^{th}-generation edge where, with no loss of generality, the $d(b)$ edges incident to b are labeled and the $d(a) - 1$ edges incident to a of the $(k+1)^{st}$ generation are unlabeled. Consider the set C of the $d(a) - 1$ unlabeled children of $\{a, b\}$. The labels unavailable for assignment to the edges in C are precisely the labels assigned to the edges incident to b. Thus, there are $\Psi + 1 - d(b)$ labels available for assignment to the edges in C. However, we have seen that $\Psi \geq d(b) + d(a) - 2$, implying $\Psi + 1 - d(b) \geq d(a) - 1$. Hence, $\lambda'_{1,1}(T) = \Psi$.

To see that $\lambda'_{1,1}(T_\infty(\Delta)) = 2(\Delta - 1)$, we merely note that the maximum edge degree in $T_\infty(\Delta)$ is $\Psi = 2(\Delta - 1)$. •

We now turn to the $\lambda'_{2,1}$-numbers of tree T with maximum degree Δ and $T_\infty(\Delta)$. Since the case $\Delta = 1, 2$ is addressed by Theorem 2.8, our main theorems are:

Theorem 4.3. Let T be a tree with maximum degree $\Delta \geq 3$. Then $2\Delta+2 \leq \lambda'_{2,1}(T) \leq \lambda'_{2,1}(T\infty(\Delta)) \leq 2\Delta + 3$.

Theorem 4.4. $\lambda'_{2,1}(T\infty(\Delta)) =$

$2\Delta + 1$ if $\Delta = 3, 4$
$2\Delta + 2$ if $\Delta = 5$
$2\Delta + 3$ if $\Delta \geq 6$

Proof of Theorem 4.3: Since $K_{1,\Delta}$ is a subgraph of T, $\lambda'_{2,1}(K_{1,\Delta}) = 2\Delta - 2$ is a lower bound for $\lambda'_{2,1}(T)$.

To show that $2\Delta + 3$ is an upper bound, it suffices to produce a $(2,1)$-edge labeling of $T_\infty(\Delta)$ with span $2\Delta + 3$. To that end, let $X_0 = \{0, 2, 4, ..., 2\Delta + 2\}$ and $X_1 = \{1, 3, 5, ..., 2\Delta + 3\}$. We note that $|X_0| = |X_1| = \Delta + 2$. Let $\{u, v\}$ be the 0^{th}-generation edge, to which we assign the label 0. Then each first-generation edge is incident to either u or v. Assign distinct labels to the $\Delta - 1$ first-generation edges incident to u from the set $X_0 - \{0\}$ and assign distinct labels to the $\Delta - 1$ first-generation edges incident to v from $X_1 - \{1\}$. Now assume that, for $1 \leq h \leq k$, the h^{th}-generation edges descended from u are labeled entirely from $X_{h+1 mod 2}$. Without loss of generality, let e be a k^{th}-generation edge with label $L(e) \in X_{k+1 mod 2}$ and let e' be the father of e with label $L(e') \in X_{k mod 2}$. We assign labels to the $\Delta - 1$ children of e from the set $W = X_{k+2 mod 2} - \{L(e) - 1, L(e) + 1, L(e')\}$. Since $|W| \geq \Delta - 1$, such a labeling can be achieved. Since no two $k+1^{st}$-generation edges (cousins) descended from u with distinct parents are within distance two, all of the $k + 1^{st}$-generation edges may be labeled in this manner. As similar argument may be used to label the edges descended from v. •

Proof of Theorem 4.4: We begin with an improvement of the lower bound given in Theorem 3.1.

Lemma 4.5. For $\Delta \geq 3$, $2\Delta + 1 \leq \lambda'_{2,1}(T_\infty(\Delta))$.

Proof: The line graph of $T_\infty(\Delta)$ is $2\Delta - 2$-regular, implying that $\lambda'_{2,1}(T_\infty(\Delta)) \geq 2\Delta$ by Theorem 3.1. Suppose that $\lambda'_{2,1}(T_\infty(\Delta)) = 2\Delta$. We first show that for any $\lambda'_{2,1}$-labeling L of $T_\infty(\Delta)$, there is an edge e_1 such that $L(e_1) = 1$.

Let L be a $\lambda'_{2,1}$-labeling of $T_\infty(\Delta)$ and let e_0 be an edge with $L(e_0) = 0$. Then the $2\Delta - 2$ edges incident to e_0 receive labels from $\{2, 3, 4, ..., 2\Delta\}$, implying that at least one edge e incident to e_0 receives a label from

$\{3, 4, 5, .., 2\Delta - 1\}$. Hence, the $2\Delta - 2$ edges incident to e receive distinct labels from the $2\Delta - 2$ labels in $\{0, 1, 2, ..., 2\Delta\} - \{L(e_1) - 1, L(e_1), L(e_1) + 1\}$. Since this set has cardinality $2\Delta - 2$ and contains 1, there is an edge e_1 which receives that label.

Let $e_1 = \{u, v\}$. With no loss of generality, the edges incident to u receive labels 3, 5, 7, ..., $2\Delta - 1$ and the edges incident to v receive labels 4, 6, 8, ..., 2Δ. Let $\{v, v_1\}$ be the edge which receives label 4. Then the $\Delta - 1$ remaining edges incident to v_1 must receive labels 0, 2, and the odd integers from 7 to $2\Delta - 1$. (If $\Delta = 3$, there are no such odd integers.) Let $\{v_1, v_2\}$ be the edge which receives label 2. Then the remaining $\Delta - 1$ edges incident to v_2 must receive labels $5, 6, 8, 10, ...2\Delta$, implying the contradiction that two adjacent edges receive consecutive labels 5 and 6. •

Lemma 4.6. Let $\Delta \geq 2$ and let G be a Δ-regular graph. Then every (j, k)-edge labeling L of G induces a (j, k)-edge labeling of $T_\infty(\Delta)$ with span at most $s(L)$, implying $\lambda'_{j,k}(G) \geq \lambda'_{j,k}(T_\infty(\Delta))$.

Proof: Let L be a (j, k)-edge labeling of G with span $s(L)$. Then by Theorem 2.3, L induces a $(j, k)-$set labeling L^* of G with span $s(L)$. Let v_{n_0} be an arbitrarily selected vertex in $V(G)$ and let the neighbors of v_{n_0} be $v_{n_1}, v_{n_2}, .., v_{n_\Delta}$. We assign the label $L^*(v_{n_0})$ to the root w_0 of $T_\infty(\Delta)$, and we assign the labels $L^*(v_{n_1}), L^*(v_{n_2}) ...., L^*(v_{n_\Delta})$ to the children $w_1, w_2, ..., w_\Delta$ of w_0, respectively. The $\Delta - 1$ children of w_i may then be assigned the labels of the neighbors of v_{n_i} which have not already been assigned to the parent of w_i. By induction, there exists a $(j, k)-$set labeling of $T_\infty(\Delta)$ with span at most $s(L)$ which, by Theorem 2.2, induces a (j, k)-edge labeling of $T_\infty(\Delta)$ with span at most $s(L)$. Hence $\lambda'_{2,1}(G) \geq \lambda'_{2,1}(T_\infty(\Delta))$.
•

By Lemma 4.6, $\lambda'_{2,1}(T_\infty(3)) \leq \lambda'_{2,1}(K_4) = 7$, and by Lemma 4.5, $\lambda'_{2,1}(T_\infty(3)) \geq 7$. Theorem 4.4 is thus proved in the case $\Delta = 3$. The case $\Delta = 4$ is handled identically.

Lemma 4.7. For $\Delta \geq 5$, $\lambda'_{2,1}(T_\infty(\Delta)) \geq 2\Delta + 2$.

Proof: Suppose to the contrary that L is a $(2, 1)$-edge labeling with span $2\Delta + 1$. Then L assigns the label 4 or it does not.

Suppose L assigns the label 4 to the edge $\{x, y\}$. Let $X = \{x_1, x_2, ..., x_{\Delta-1}\}$ be the set of edges incident to x not labeled 4 under L, and similarly let $Y = \{y_1, y_2, ..., y_{\Delta-1}\}$ be the set of edges incident to y not labeled 4 under L. Let $B = \{0, 1, 2\}$ and $C = \{6, 7, ...2\Delta + 1\}$. We first observe that X and Y contain at least one edge labeled from B since C does not contain a 2-separated subset of size $\Delta - 1$. We next show that X and Y cannot both contain exactly one edge with labels from B.

Suppose the contrary that X and Y contain exactly one edge labeled from B. Then X and Y have exactly $\Delta - 2$ edges labeled from C. So, since the edges of X (resp. Y) must have labels which differ pairwise by at least 2, the edges in Y must have labels which differ pairwise by at least 2, and the labels assigned to the edges in X and Y pairwise distinct, then with no loss of generality, the labels in C assigned to the edges of X must be 6, 8, 10, ... 2Δ and the labels in C assigned to the edges of Y must be $7, 9, 11, ..., 2\Delta + 1$. Let $\{y, w\}$ be the edge in Y which receives label $2\Delta - 1$ under L and let $W = \{w_1, w_2, ..., w_{\Delta-1}\}$ be the set of edges incident to W without label $2\Delta - 1$. Then the $\Delta - 1$ labels of the edges in W must be in $[0, 2\Delta - 4] - \{4\}$, an impossibility due to the unavailability of $\Delta - 1$ 2-separated integers in that set. Thus, it follows from the distance conditions that, with no loss of generality, exactly 2 edges in X receive labels from B and exactly one edge in Y receives a label from B. Furthermore, the two edges in X which receive labels from B must be assigned 0 and 2 due to the adjacency of those edges. Let $\{x, z\}$ be the edge on X which receives label 2 and let Z be the set of $\Delta - 1$ edges incident to z not labeled 2. Then the edges of Z must receive 2-separated labels in $\{5, 6, 7, ..., 2\Delta + 1\}$, and hence those labels must be 5, 7, 9, .. , $2\Delta + 1$. Let $\{z, u\}$ be the edge in Z with label $2\Delta - 1$ and let U be the set of $\Delta - 1$ edges incident to u not labeled $2\Delta - 1$. Then the edges in U must receive labels in $\{0, 1, 2, .., 2\Delta - 4\} - \{2, 5\}$, an impossibility due to the unavailability of $\Delta - 1$ 2-separated integers in that set. Thus L assigns 4 to no edge, implying that L assigns $2\Delta - 3$ to an edge since $2\Delta + 1 - L$ is a $(2, 1)$-edge labeling.

We next show that L assigns 2 to no edge. Let e be an edge labeled 2 under L. Then $2\Delta - 2$ edges incident to e must receive distinct labels from the $2\Delta - 3$ integers in $\{0\} \bigcap \{5, 6, 7, ...2\Delta + 1\} - \{2\Delta - 3\}$, an impossibility. Consequently, L assigns $2\Delta - 1$ to no edge as well.

We have thus shown that L assigns labels from the set

$$R = \{0, 1, 2, ..., 2\Delta + 1\} - \{2, 4, 2\Delta - 3, 2\Delta - 1\}.$$

However, any edge e along with the $2\Delta - 2$ edges to which e is adjacent require $2\Delta - 1$ distinct labels. Since $|R| = 2\Delta - 2$, L is not a $(2, 1)$-edge labeling. •

Now consider the claim $\lambda'_{2,1}(T_\infty(\Delta)) = 2\Delta + 2$ for $\Delta = 5$. By Lemma 4.7 and Corollary 2.6, it suffices to show the existence of a $(12, 2)$-feasibility set C in which every element in C has cardinality 5. But such a set is

$\{\{0, 2, 5, 8, 11\}, \{0, 3, 5, 8, 12\}, \{0, 3, 6, 8, 11\}, \{0, 3, 6, 9, 12\}, \{0, 4, 7, 9, 11\},$
$\{0, 4, 7, 10, 12\}, \{1, 3, 5, 7, 10\}, \{1, 3, 5, 8, 10\}, \{1, 3, 5, 9, 12\}, \{1, 3, 6, 8, 11\},$
$\{1, 3, 6, 9, 11\}, \{1, 4, 6, 8, 11\}, \{1, 4, 7, 9, 11\}, \{1, 5, 7, 10, 12\}, \{2, 4, 6, 8, 10\},$
$\{2, 4, 6, 8, 12\}, \{2, 4, 6, 10, 12\}, \{2, 4, 7, 9, 12\}, \{2, 4, 8, 10, 12\},$

$\{2,5,7,9,11\}, \{3,5,7,10,12\}, \{4,6,8,10,12\}\}$.

Hence, Theorem 4.4 is proved for $\Delta = 5$.

We next establish that for $\Delta = 6$, $\lambda'_{2,1}(T_\infty(6)) > 2\Delta + 2 = 14$. By Lemma 4.7, $\lambda'_{2,1}(T_\infty(6)) \geq 14$. If equality holds, then there exists a $(14,2)$-feasible set C each of whose elements has cardinality 6. We note that C is a subset of the collection of all 2-separated 6-subsets of the set $X = \{0,1,2,3,...14\}$, of which there are $\binom{10}{4} = 210$. We also note, however, that many of the 2-separated 6-subsets of the set X fail to meet property 2 of Definition 2.4; for example, the reader can verify that there is no 2-separated 6-subset in X which intersects $\{2,4,6,8,11,14\}$ at only the element 4. A computer search reveals that the maximal $(14,2)$-feasible set C is empty. This, along with Theorem 4.3 for $\Delta = 6$, proves $15 = \lambda'_{2,1}(T_\infty(\Delta)) = 2\Delta+3$.

Lemma 4.8. For $\Delta \geq 2$, if $T_\infty(\Delta+1)$ has a $(d,1)$-edge labeling of span m, then $T_\infty(\Delta)$ has a $(d,1)$-edge labeling of span at most $m - d$.

Proof: If $T_\infty(\Delta+1)$ has a $(d,1)$-edge labeling of span m, then there exists an (m,d)-feasible set each of whose members has cardinality $\Delta + 1$ by Corollary 2.6. Let $C = \{A_1, A_2, ...A_n\}$ be such a set and consider the set $C' = \{B_1, B_2, ..., B_n\}$ where $B_i = A_i - \{m-d+1, m-d+2, ..., m\}$. Since the elements of A_i are d-separated, we note that $|B_i| \geq |A_i| - 1 = \Delta$, implying that C' is a (y,d)-feasible set, $y \leq m - d$. By Corollary 2.6, C' induces a $(d,1)$-labeling of an infinite tree T each of whose vertices has degree at least Δ. Since $T_\infty(\Delta)$ is a subgraph of T, the result now follows. •

To complete the proof of Theorem 4.4, we note that $\lambda'_{2,1}(T_\infty(\Delta)) \geq 2\Delta + 2$ for $\Delta \geq 6$ by Lemma 4.7. If, for some $\Delta_0 \geq 7$, $\lambda'_{2,1}(T_\infty(\Delta_0)) = 2\Delta_0 + 2$, then by Lemma 4.8 and an inductive argument, $\lambda'_{2,1}(T_\infty(6)) = 14$, a contradiction of our demonstration that $\lambda'_{2,1}(T_\infty(6)) = 15$. Hence, $\lambda'_{2,1}(T_\infty(\Delta)) \geq 2\Delta + 3$ for $\Delta > 6$. But Theorem 4.3 indicates that $\lambda'_{2,1}(T_\infty(\Delta)) \leq 2\Delta + 3$, concluding the proof. •

5. On the $\lambda'_{1,1}$ and $\lambda'_{2,1}$ numbers of Q_n. We denote the vertices of Q_n by n-tuples each of whose components is 0 or 1, and we note that $|E(Q_n)| = n2^{n-1}$. It is clear that $\lambda'_{j,k}(Q_1)) = 0$ for all $j \geq k$. Hence, we consider the case $n \geq 2$, proving the following two theorems:

Theorem 5.1. For $n \geq 2$, $\lambda'_{1,1}(Q_n) = 2n - 1$.

Theorem 5.2

i. $\lambda'_{2,1}(Q_2) = 4$
ii. $\lambda'_{2,1}(Q_3) = 7$

iii. $\lambda'_{2,1}(Q_4) = 10$
iv. $\lambda'_{2,1}(Q_5) = 12$ or 13
v. $\lambda'_{2,1}(Q_6) = 15$ or 16.

Before the proofs, however, we make a few observations.

For $0 \leq i \leq n-1$, let E_i denote the set of edges $\{\vec{u}, \vec{v}\}$ such that $\vec{u}$ and $\vec{v}$ differ only in the i^{th} component. Also, for $h = 0, 1$, let $E_i^h = \{\{\vec{u}, \vec{v}\} \in E_i | \sum_{j=1, j\neq i}^{n} v_i = h \mod 2\}$. Then:

1. Each E_i is a perfect matching in Q_n; hence, $|E_i| = 2^{n-1}$ and no two edges in E_i are adjacent

2. The set $\{E_0, E_1, ..., E_{n-1}\}$ is a partition of $E(Q_n)$;

3. The set $\{E_i^0, E_i^1\}$ is a partition of E_i, and for fixed h, $|E_i^h| = 2^{n-2}$ and the edges in E_i^h are pairwise distance at least three apart

4. For $n \geq 2$ and fixed i, $Q_n - E_i$ is isomorphic to the sum of two copies of Q_{n-1}.

5. For fixed $h \in \{0, 1\}$ and fixed i, every edge in $Q_n - E_i$ is adjacent to some edge in E_i^h.

6. For $n \geq 2$, if $X \subseteq E(Q_n)$ with $|X| = 2^{n-2}$ such that elements of X are pairwise distance at least three apart, then $X = E_i^h$ for some fixed i, h.

Proof of Theorem 5.1: By Theorem 3.3, $\lambda'_{1,1}(Q_n) \geq 2n-2$. Suppose L is a $(1,1)$-edge labeling with span $2n-2$. Then by the pigeon-hole principle, some fixed label l is assigned by L to x edges, where $x \geq \lceil \frac{n2^{n-1}}{2n-1} \rceil \geq 2^{n-2}+1$. These edges are incident to $2x \geq 2^{n-1}+2$ distinct vertices. Since two of these vertices must be adjacent in Q_n, there exist two edges at distance two each of which receives label l under L, a contradiction. It thus suffices to demonstrate a $(1,1)$-labeling with span $2n-1$. To that end, let L be the edge labeling such that $L(e) = 2i + h$ for $e \in E_i^h$. By Property 3, edges which receive the same label under L are at least distance three apart. It thus follows that L is a $(1,1)$-edge labeling. •

Proof of Theorem 5.2. We begin by establishing that $\lambda'_{2,1}(T_\infty(n)) \leq \lambda'_{2,1}(Q_n) \leq 3n-2$ for $n \geq 2$. Since Q_n is n-regular, the lower bound follows from Lemma 4.6. The upper bound follows from the construction of an edge-labeling with span $3n-2$. If $L(e) = 3i + h$ for $e \in E_i^h$, then by Property 3, edges which receive the same label under L are at least distance three apart. Additionally, two edges which receive consecutive labels are necessarily in E_i for some fixed i, and are thus not adjacent by Property 1.

Now, since $\lambda'_{2,1}(T_\infty(2)) = 4$, $\lambda'_{2,1}(T_\infty(3)) = 7$, $\lambda'_{2,1}(T_\infty(4)) = 9$, $\lambda'_{2,1}(T_\infty(5)) = 12$, and $\lambda'_{2,1}(T_\infty(6)) = 15$, these bounds imply

i. $\lambda'_{2,1}(Q_2) = 4$
ii. $\lambda'_{2,1}(Q_3) = 7$
iii. $\lambda'_{2,1}(Q_4) = 9$ or 10
iv. $\lambda'_{2,1}(Q_5) = 12$ or 13
v. $\lambda'_{2,1}(Q_6) = 15$ or 16.

We close the proof by showing that $\lambda'_{2,1}(Q_4) = 10$. Assume there exists a $(2,1)$-edge labeling L with span 9. Then the 32 edges of Q_4 can be placed into 10 labeling classes $M_0, M_1, .., M_9$ such that M_j contains precisely those edges labeled j under L. We observe that no 5 edges can receive the same label under L, for if such were the case, then Q_4 would need to have at least $7 \times 5 = 35$ edges. As a result, there must exist at least 2 labeling classes with order 4. We argue that

a. if $|M_j| = 4$ for some j, $1 \le j \le 8$, then $|M_{j-1}| + |M_{j+1}| \le 4$

b. if there exist 3 labeling classes representing labels j, $j+1$ and $j+2$ such that $|M_j| = |M_{j+2}| = 4$, then $|M_{j+1}| = 0$.

c. no two labeling classes M_j and M_{j+1} representing consecutive labels can each have order 4.

To show a, we appeal to Properties 5 and 6 thus: for some i, h, $M_j = E_i^h$. Since no edge in $M_{j-1} \bigcup M_{j+1}$ is adjacent to any edge in M_j, then by Property 5, $M_{j-1} \bigcup M_{j+1} \subseteq E_i - E_i^h$, implying the result.

To show b, we suppose that $M_{j+1} > 0$. By Property 6, $M_j = E_i^h$ and $M_{j+2} = E_{i'}^{h'}$ for some i, h, i', h'. By Property 5, $M_{j+1} \subseteq E_i - E_i^h$ and $M_{j+1} \subseteq E_{i'} - E_{i'}^{h'}$. Regardless of whether or not $i = i'$, we have that $E_i - E_i^h$ and $E_{i'} - E_{i'}^{h'}$ are disjoint, which implies $M_{j+1} = \phi$.

To show c, we first suppose $|M_0| = |M_1| = 4$. Since no edge in M_0 is adjacent to any edge in M_1, we have by properties 3, 5, and 6 that $M_0 = E_i^h$ and $M_1 = E_i^{h'}$ for some i, implying $M_0 \bigcup M_1 = E_i$. From b, we have $|M_2| = 0$. Thus, the remaining edges in $Q_4 - E_i$, the sum of two disjoint copies of Q_3 by Property 4, have labels in $\{3, 4, 5, ...9\}$, implying that Q_3 can be $(2,1)$-edge labeled with span 6, a contradiction of the result $\lambda'_{2,1}(Q_3) = 7$. (A symmetric argument can be given if $j = 8$.) Now suppose $1 \le j \le 7$. Then arguing as above, $|M_{j-1}| = |M_{j+2}| = 0$ by a (implying $j > 0$ since $|M_0| \ne 0$ and $j < 7$ since $|M_9| \ne 0$.) Noting again that $M_j \bigcup M_{j+1} = E_i$ for some i, we can label the edges of $Q_4 - E_i$ with labels in $\{0, 1, 2, ...j-2\} \bigcup \{j+3, j+4, ...9\}$. We may thus produce a $(2,1)$-edge labeling L' of $Q_4 - E_i$ where

$$L'(e) = \begin{cases} L(e) & \text{if } L(e) \leq j-2 \\ L(e)-3 & \text{if } L(e) \geq j+3 \end{cases}$$

But by property 4, L' is $(2,1)$-edge labeling of Q_3 with span 6, a contradiction.

We next show that for $1 \leq j \leq 8$, $|M_j| \leq 3$. Suppose to the contrary that there exists j, $1 \leq j \leq 8$, such that $|M_j| = 4$ and let j^* be the smallest such j. From a, $|M_{j^*-1}| + |M_{j^*}| + |M_{j^*+1}| \leq 8$, which implies that at least 24 edges of Q_n have labels from the remaining 7 labeling classes. Therefore, at least 3 of the remaining 7 labeling classes have order 4, and hence there exists q, $j^*+2 \leq q \leq 8$, such that $|M_q| = 4$. If $q \neq j^*+2$, then the labeling classes $M_{j^*-1}, M_{j^*}, M_{j^*+1}, M_{q-1}, M_q, M_{q+1}$ are distinct and, by a, contain at most 16 edges. The remaining four labeling classes must each have order at exactly 4 (since, as noted, no 5 edges can receive the same label). This forces the existence of two labeling classes representing consecutive labels each with order 4, contradicting c. If, on the other hand, $q = j^*+2$, then by b, $|M_{j^*+1}| = 0$. Furthermore, from a, we observe that $|M_{j^*-1}| \leq 3$ and $|M_{j^*+3} \leq 3$. Hence the five labeling classes $M_{j^*-1}, M_{j^*}, M_{j^*+1}, M_{j^*+2}$, and M_{j^*+3} contain at most 14 edges, implying that the remaining 5 labeling classes contain at least 18 edges. At least three of these classes must have order 4, implying the existence of q', $j^*+4 \leq q' \leq 8$ such that $|M_{q'}| = 4$. By the preceding argument, this forces the existence of two labeling classes representing consecutive labels each with order 4, contradicting c.

Since $1 \leq j \leq 8$, $|M_j| \leq 3$, it follows that $|M_0| = |M_9| = 4$ and that for $1 \leq j \leq 8$, $|M_j| = 3$. From Property 6, $M_0 = E_i^h$ for some i, h, and since the edges in M_1 are not adjacent to the edges in M_0, from Property 5 it follows that $M_1 \subseteq E_i - E_i^h$. But no edge labeled 2 can be among the 18 edges adjacent to the edges in M_1, nor can it be an edge among the 7 edges in $M_0 \bigcup M_1$. Hence, the remaining 7 edges in Q_n which may receive the label 2 are precisely the single edge in $E_i - M_0 - M_1$ and its 6 adjacent edges. However, it is impossible to find three edges among this collection which are pairwise distance three apart, thus concluding the proof. •

6. On the $e\lambda_1^1$-numbers and $e\lambda_1^2$-numbers of Joins. Let G_1 and G_2 be graphs with orders m_1 and m_2. Recall that the join of G_1 and G_2, denoted $G_1 \vee G_2$, is the graph whose vertex set is $V(G_1) \bigcup V(G_2)$ and whose edge set is $E(G_1) \bigcup E(G_2) \bigcup Z$, where $Z = \{\{u,v\} | u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Theorem 6.1.

i: if G_1 and G_2 have non-empty edge sets, then $\lambda'_{1,1}(G_1 \vee G_2) = \lambda'_{1,1}(G_1) + \lambda'_{1,1}(G_2) + m_1 m_2 + 1$

ii: if (without loss of generality) $E(G_1)$ is non-empty and $E(G_2)$ is empty, then $\lambda'_{1,1}(G_1 \vee G_2) = \lambda'_{1,1}(G_1) + m_1m_2$

iii: if $E(G_1)$ and $E(G_2)$ are each empty, then $\lambda'_{1,1}(G_1 \vee G_2) = \lambda'_{1,1}(K_{m_1,m_2}) = m_1m_2 - 1$.

Proof: i: We first show that $\lambda'_{1,1}(G_1 \vee G_2) \leq \lambda'_{1,1}(G_1) + \lambda'_{1,1}(G_2) + m_1m_2 + 1$, and then show that no $(1,1)$-edge labeling of $G_1 \vee G_2$ has smaller span.

Let $Z = \{z_1, z_2, ...z_{m_1m_2}\}$ be the edges joining the vertices of G_1 to the vertices of G_2, and for $i = 1,2$, let L_i be a $\lambda'_{1,1}$-labeling of G_i. Noting that every edge in Z is at most distance two away from each edge in $G_1 \vee G_2$, and that every edge in G_1 is exactly distance two away from each edge in G_2, we produce a $(1,1)$-edge labeling L of $G_1 \vee G_2$ as follows: $L(e) =$

$L_1(e)$	if $e \in E(G_1)$
$\lambda'_{1,1}(G_1) + L_2(e) + 1$	if $e \in E(G_2)$
$\lambda'_{1,1}(G_1) + \lambda'_{1,1}(G_2) + 1 + p$	if $e = z_p$.

Now suppose that $\lambda'_{1,1}(G_1 \vee G_2) \leq \lambda'_{1,1}(G_1) + \lambda'_{1,1}(G_2) + m_1m_2$ and let L^* be a $(1,1)$-edge labeling of $G_1 \vee G_2$ with span at most $\lambda'_{1,1}(G_1) + \lambda'_{1,1}(G_2) + m_1m_2$. For $i = 1,2$, let $X_i = \{x | L^*(e) = x, e \in G_i\}$ and let $Y = \{y | L^*(e) = y, y \in Z\}$. We observe that $|X_i| \geq \lambda'_{1,1}(G_i) + 1$, and that $|Y| = m_1m_2$. So $|X_1| + |X_2| + |Y| \geq \lambda'_{1,1}(G_1) + \lambda'_{1,1}(G_2) + m_1m_2 + 2$, which implies either $X_1 \bigcap Y$ or $X_2 \bigcap Y$ or $X_1 \bigcap X_2$ is non-empty, contradicting the distance constraints.

Part ii is similar to part i, and part iii follows from Theorem 2.9. •

Recalling that a t-point suspension of G (t-$spn(G)$) is the join of G with the sum of t copies of K_1, we have the following:

Corollary 6.2.

i: for $n \geq 2$, $\lambda'_{1,1}(t\text{-}spn(P_n)) = \lambda_{1,1}(P_{n-1}) + tn$.
ii: for $n \geq 3$, $\lambda'_{1,1}(t\text{-}spn(C_n)) = \lambda_{1,1}(C_n) + tn$. •

For positive integers $j > k$ and general graphs G_1 and G_2, arguments analogous to those used in Theorem 6.1 above yield

(1) $\lambda'_{j,k}(G_1 \vee G_2) \leq \lambda'_{j,k}(G_1) + \lambda'_{j,k}(G_2) + \lambda'_{j,k}(K_{m_1,m_2}) + j + k$ if $E(G_1)$ and $E(G_2)$ are each non-empty;

(2) $\lambda'_{j,k}(G_1 \vee G_2) \leq \lambda'_{j,k}(G_1) + \lambda'_{j,k}(K_{m_1,m_2}) + j$ if (without loss of generality) $E(G_1)$ is non-empty and $E(G_2)$ is empty;

(3) $\lambda'_{j,k}(G_1 \vee G_2) = \lambda'_{j,k}(K_{m_1,m_2})$ if both $E(G_1)$ and $E(G_2)$ are empty.

We note that for $j = 2$ and $k = 1$, the above bound in (1) is met when $G_1 = G_2 = P_2$ (since the bound is 7 and $\lambda'_{2,1}(K_4) = 7$), but is not met when $G_1 = P_2$ and $G_2 = C_3$ (since the bound is 12 and $\lambda'_{2,1}(K_5) = 9$). We also note a given graph G may have more than one representation as a join of graphs, in turn giving rise to distinct upper bounds on $\lambda'_{2,1}(G)$. For example, K_4 also equals $C_3 \vee K_1$, which by (2) above implies $\lambda'_{2,1}(K_4) \leq 11$.

For the remainder of this section, we investigate the $\lambda'_{2,1}$-number of the n-wheel $W_n = C_n \vee K_1$, for $n \geq 3$. Since $K_{1,n}$ is a subgraph of W_n, we have from (2) above $2n-2 = \lambda'_{2,1}(K_{1,n}) \leq \lambda'_{2,1}(W_n) \leq \lambda'_{2,1}(C_n) + \lambda'_{2,1}(K_{1,n}) + 2 = 2n+4$. Below, we show $\lambda'_{2,1}(W_n) = 2n-2$ if and only if $n \geq 6$.

Theorem 6.3. For $n \geq 3$, $\lambda'_{2,1}(W_n) =$

$$\begin{cases} 7 & \text{if } n = 3 \text{ or } 4 \\ 9 & \text{if } n = 5 \\ 2n-2 & \text{if } n \geq 6 \end{cases}$$

Proof: If $n = 3$, then $W_3 = K_4$, and the result follows by Theorem 4.1.

Observing that W_4 and W_5 have edge diameter 2, we have that $\lambda'_{2,1}(W_4) \geq 7$ and $\lambda'_{2,1}(W_5) \geq 9$. In Figures 6.1a,b, we provide $(2,1)$-edge labelings of W_4 and W_5 with these respective spans.

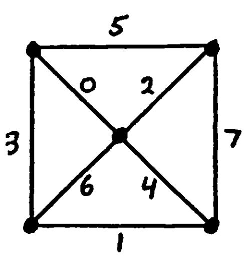

Figure 6.1a

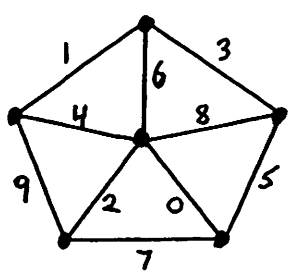

Figure 6.1b

Let $n \geq 6$. We denote the vertices of C_n by $u_0, u_1, u_2, \ldots, u_{n-1}$ and the edge $\{u_i, u_{i+1 \bmod n}\}$ by e_i, $0 \leq i \leq n-1$. Additionally, we denote the vertex of K_1 by w and the spoke $\{w, u_i\}$ by s_i, $0 \leq i \leq n-1$. We produce a $(2,1)$-edge labeling L of W_n which meets the lower bound of $2n-2$ as follows: $L(e_i) =$

$$\begin{cases} 2i+1 & \text{if } i = 0,1,2 \\ 1 & \text{if } i = 3 \\ 2i-1 & \text{if } 4 \leq i \leq n-1 \end{cases}$$

Also, $L(s_i) =$

$$\begin{cases} 2n-8 & \text{if } i = 0 \\ 2n-6 & \text{if } i = 1 \end{cases}$$

0 if $i = 2$
$2n - 4$ if $i = 3$
$2n - 2$ if $i = 4$
$2i - 8$ if $5 \leq i \leq n - 1$

Although the reader can easily verify that L is a $(2,1)$-edge labeling, we observe that the spokes of W_n are labeled with distinct even integers, $L(e_0) = L(e_3) = 1$, and all other edges along C_n are given distinct odd labels not equal to 1. For each spoke s_i, $L(s_i)$ differs from its incident edges in C_n by at least 3.

In Figure 6.2 below, we give an $\lambda'_{2,1}$-edge labeling of W_6.

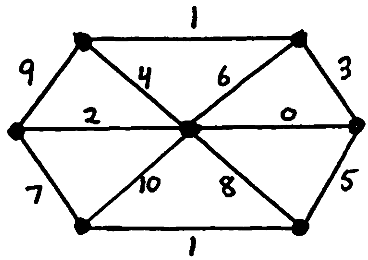

Figure 6.2

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7. Concluding Remarks. Our results on $T_\infty(\Delta)$ show that for any tree T with maximum degree Δ, $\lambda'_{2,1}(T_\infty(\Delta))$ is in

$S_1 = \{0\}$ if $\Delta = 1$
$S_2 = \{2, 3, 4\}$ if $\Delta = 2$
$S_3 = \{4, 5, 6, 7\}$ if $\Delta = 3$
$S_4 = \{6, 7, 8, 9\}$ if $\Delta = 4$
$S_5 = \{8, 9, 10, 11, 12\}$ if $\Delta = 5$
$S_\Delta = \{2\Delta - 2, 2\Delta - 1, ..., 2\Delta + 3\}$ if $\Delta \geq 6$

Thus, for $\Delta \geq 2$, the set $\mathcal{T}_\Delta$, the collection of all finite trees with maximum degree Δ, can be classified according to their $\lambda'_{2,1}$-number. It is clear that $K_{1,\Delta}$ is the smallest possible tree in the class of trees with $\lambda'_{2,1}$-number equal to $2\Delta - 2$, and is the only tree in this class for $\Delta = 2$ alone. We conjecture that for each $s \in S_\Delta$, the class of finite trees with λ'-number s is non-empty.

By Lemma 4.6, $\lambda'_{j,k}(T_\infty(\Delta)) \leq \lambda'_{j,k}(G)$ where G is a Δ-regular graph. For $j = 2$ and $k = 1$, we have seen that K_3 and K_4 meet the lower bound for $\Delta = 3$ and $\Delta = 4$. For $j = k = 1$ and $m \geq 2$, the odd graph O_m, that is, the graph whose vertices are precisely the $m - 1$-subsets of $\{0, 1, 2, .., 2m - 2\}$ (see [4]) and whose edges join vertices which are disjoint, can be $(1,1)$-set labeled by assigning to vertex v the m-set $\{0,1,2,..,2m-2\}-v$. Since $\lambda'_{1,1}(O_m) \geq 2m-2$ by Theorem 4.2, the odd graphs represent a class of graphs which attain the minimum $(1,1)$-edge number.

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