Cartesian Products and Edge Domination

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Abstract

A set of edges D in a graph G is a dominating set of edges if every edge not in D is adjacent to at least one edge in D. The minimum cardinality of an edge dominating set of G is the edge domination number of G, denoted $D_E(G)$. In this paper we investigate the edge domination number for the cartesian product of an n-colorable graph G and the complete graph K_n.

We consider only finite, undirected simple graphs $G(V, E)$ where V is the vertex set and E is the edge set. $v(G)$ will denote the number of vertices of G. A matching in a graph G is a subset of disjoint edges and a **full matching** in G is a matching such that at most one vertex is not on some edge of the matching. A graph G is n-colorable if there is a map $\alpha : V \mapsto \{1, 2, ..., n\}$ such that if u and v are adjacent then $\alpha(u) \neq \alpha(v)$. The chromatic number of a graph G, denoted $\chi(G)$, is the smallest n for which G is n-colorable. For other terminology used in this paper please see [5].

A subset of edges D of a graph $G(V, E)$ is called an **edge dominating set** of G if each edge in $E - D$ is adjacent to at least one edge in D. The **edge domination number** of G, denoted $D_E(G)$, is the cardinality of a minimum edge dominating set of G. The edge domination number of a graph was first discussed in [4] and in [9]. Yannakakis and Gavril show that the problem of determining the edge domination number of bipartite graphs with degree at most three is NP-complete. They also note that any graph has a minimum edge dominating set that consists of disjoint edges. Mitchell and Hedetniemi [7] show that a minimum edge dominating set can be found for a tree in linear time. Forcade [2] discusses the edge domination of the n-cube. Forcade's work is generalized by Cutler [1] where he considers the edge domination number of the cartesian product of a graph with the n-cube Q_n (which can also be viewed as a repeat cartesian product of K_2 with itself). Georges et. al. [3] compute the edge domination number for the cartesian product of two complete graphs. As a corollary to the theorems in this paper we compute the edge domination number for the cartesian product of an arbitrary number of complete graphs.

Definition 1 *Let $G(U, E)$ and $H(V, F)$ be graphs. The* **cartesian product** *of G and H, denoted $G \times H$, is the graph with vertex set $U \times V$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $\{u_1, u_2\}$ is in E and $v_1 = v_2$ or if $\{v_1, v_2\}$ is in F and $u_1 = u_2$. See Figure 1 for an example of the cartesian product of two graphs.*

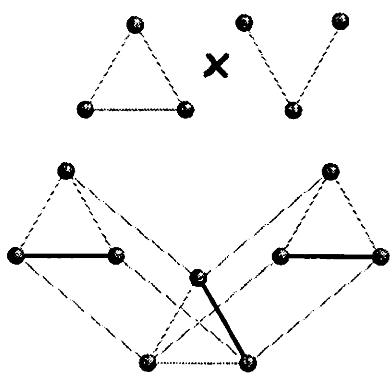

Figure 1: C_3, P_3, and $C_3 \times P_3$

We will also need the following idea.

Definition 2 *Let G be a graph and let D be a subset of edges of G. We say a vertex v in G is* **saturated with respect to D** *if v is contained in some edge of D. Otherwise, v is* **unsaturated with respect to D**.

We note that for a graph G a subset of edges D is an edge dominating set if and only if all unsaturated vertices with respect to D are non-adjacent. The next proposition gives us a basic inequality.

Proposition 1 *Let G be a graph with $n \geq 3$. Then $D_E(G \times K_n) \geq \frac{v(G)\cdot(n-1)}{2}$.*

Proof: We can think of $G \times K_n$ as $v(G)$ copies of K_n. For any edge dominating set D of $G \times K_n$, each copy of K_n can contain at most one unsaturated vertex with respect to D. This means that the number of saturated vertices with respect to D is at least $v(G) \cdot (n-1)$. If D is a minimum edge dominating set of disjoint edges then we obtain the inequality $D_E(G \times K_n) \geq \frac{v(G)(n-1)}{2}$. □

The following two theorems investigate when the lower bound in Propostion 1 can be achieved.

Theorem 1 *Let n be odd with $n \geq 3$. If G is an n-colorable graph then $D_E(G \times K_n) = \frac{v(G)\cdot(n-1)}{2}$.*

Proof: To prove this theorem, using Proposition 1, all that remains is to construct an edge dominating set D for $G \times K_n$ with $\frac{v(G)(n-1)}{2}$ edges. Fix an n-coloring α of G using the colors $\{1, 2, ..., n\}$. Let $\{u_1, u_2, ..., u_t\}$ be the vertices of G and let $\{v_1, v_2, ..., v_n\}$ be the vertices of K_n. Again we can think of $G \times K_n$ as t copies of K_n. Construct a subset of edges D as follows: for each $i = 1$ to t take $\frac{n-1}{2}$ disjoint edges from the i^{th} copy of K_n so that the vertex $\{u_i, v_{\alpha(u_i)}\}$ is not incident to any of the edges. For an example of this see Figure 1. Clearly, D contains $\frac{v(G)(n-1)}{2}$ edges. Consider any two vertices $\{u_i, v_{\alpha(u_i)}\}$ and $\{u_j, v_{\alpha(u_j)}\}$ that are unsaturated by D. Since $\alpha(u_i) = \alpha(u_j)$ implies that u_i and u_j are not adjacent in G we see that any two unsaturated vertices with respect to D are not adjacent in $G \times K_n$. Thus D is an edge dominating set of the required size. □

Theorem 2 *Let n be even with $n \geq 4$. If G is an n-colorable graph with a full matching then $D_E(G \times K_n) = \lceil\frac{v(G)\cdot(n-1)}{2}\rceil$.*

Proof: By the above proposition all that is needed is to construct an edge dominating set D for $G \times K_n$ that has $\lceil\frac{v(G)(n-1)}{2}\rceil$ edges. Fix an n-coloring α of G using the colors $\{1, 2, ..., n\}$. Let $\{u_1, u_2, ..., u_t\}$ be the vertices of G and let $\{v_1, v_2, ..., v_n\}$ be the vertices of K_n. Also, let M be a full matching for G. Again we can think of $G \times K_n$ as t copies of K_n. We construct a subset of edges D in two steps. For each edge $\{u_i, u_j\}$ in M find an l that is not equal to either $\alpha(u_i)$ or $\alpha(u_j)$ and place the edge $\{(u_i, v_l), (u_j, v_l)\}$ in D. At this point one vertex in each copy of K_n is saturated with respect to D except in the case when $v(G)$ is odd in which case some copy K_n still has only unsaturated vertices (without loss of generality we will assume that this is the 1^{st} copy of K_n). Now, if $v(G)$ is even then for each $i = 1$ to t take $\frac{n-2}{2}$ disjoint edges from the i^{th} copy of each K_n so that the vertices $(u_i, v_{\alpha(u_i)})$ and (u_i, v_l) are not on any of the edges. If $v(G)$ is odd then for each $i = 2$ to t take $\frac{n-2}{2}$ disjoint edges from the i^{th} copy of each K_n so that the vertices $(u_i, v_{\alpha(u_i)})$ and (u_i, v_l) are not on any of the edges. Also, in the 1^{st} copy of K_n take $\frac{n}{2}$ disjoint edges. In either case we have a set D with $\lceil \frac{v(G)\cdot(n-1)}{2} \rceil$ edges. For an example of this see Figure 2. As in the proof of Theorem 1 we see that all unsaturated vertices with respect to D are non-adjacent and D is an edge dominating set of the required size. □

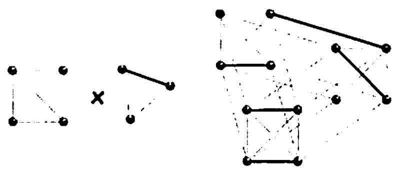

Figure 2: K_4, K_3, and $K_4 \times K_3$

In [8] Vizing shows that $\chi(G \times H) = max\{\chi(G), \chi(H)\}$. This implies inductively that the chromatic number of $K_{n_1} \times K_{n_2} \times ... \times K_{n_m}$ is n_1 assuming without loss of generality that n_1 is the largest subscript. Combining this result with Theorems 1 and 2 we have the following corollary.

Corollary 1 *Let $G = K_{n_1} \times K_{n_2} \times ... \times K_{n_m}$ with at least one $n_i \geq 3$. Assume that $n_1 = max\{n_1, n_2, ..., n_m\}$. Then $D_E(G) = \lceil \frac{(n_1-1)\cdot n_2 \cdot n_3 \cdot ... \cdot n_m}{2} \rceil$.*

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