Edge-critical isometric subgraphs of hypercubes

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Abstract

Isometric subgraphs of hypercubes are known as partial cubes. Edge-critical partial cubes are introduced as the partial cubes G for which $G - e$ is not a partial cube for any edge e of G. An expansion theorem is proved by means of which one can generate many edge-critical partial cubes. Edge-critical partial cubes are characterized among the Cartesian product graphs. We also show that the 3-cube and the subdivision graph of K_4 are the only edge-critical partial cubes on at most 10 vertices.

1 Introduction

Graphs that admit isometric embeddings into hypercubes are known as partial cubes and have been intensively studied by now. They were introduced by Graham and Pollak [8] and soon after characterized by Djoković [5]. For additional characterizations of partial cubes see [2, 3, 19, 20], for different applications of these graphs consult [4, 6, 8, 12, 16], and for the algorithmic point of view we refer to [1, 10]. Partial cubes are presented in detail in the book [11].

Posing some additional condition(s) on partial cubes, one may ask several interesting questions. For instance, which are planar partial cubes and which are regular partial cubes? These two questions are open at the present, in particular the second one seems to be quite difficult. On the other hand, Weichsel [18] succeeded to determine all distance regular partial cubes. In [13] partial cubes are characterized among the subdivision graphs and in [9] it is proved that partial cubes different from cycles are 3-connected provided that $|W_{ab}| = |W_{ba}|$ holds for all edges ab. (For the definition of W_{ab} see below.)

In this note we introduce the following concept. A partial cube G is called *edge-critical* if for any edge e of G, $G - e$ is not a partial cube. In the rest of this section we give necessary definitions and preliminary observations. We follow by an expansion theorem which, from an edge-critical partial cube, produces another such (bigger) graph by means of a peripheral expansion obeying an additional condition. We also characterize edge-critical partial cubes

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among the Cartesian product graphs. For instance, the Cartesian product of an edge-critical partial cube with an arbitrary partial cube is edge-critical. We conclude by showing that the 3-cube Q_3 and the subdivision graph of K_4, $S(K_4)$, are the only edge-critical partial cubes on at most 10 vertices.

For a graph G, the *distance* $d_G(u,v)$ (or briefly $d(u,v)$) between vertices u and v is defined as the number of edges on a shortest u,v-path. A subgraph H of G is called *isometric* if $d_G(u,v) = d_H(u,v)$ for all $u,v \in V(H)$. Isometric subgraphs of hypercubes are called *partial cubes*. A set X in $V(G)$ is called *convex* if for all $u,v \in X$ the vertices of any shortest u,v-path belong to X. A subgraph H in G is *convex* if its vertex set is convex.

Let $G = (V,E)$ be a connected, bipartite graph and let ab be an edge of G. Set

$$W_{ab} = \{x \in V(G) \mid d(x,a) < d(x,b)\}.$$

Djoković [5] proved that a graph is a partial cube if and only if it is bipartite and the sets W_{ab} are convex. Two edges xy and uv of G are in the Djoković-Winkler [5, 20] relation Θ if

$$d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u).$$

If graph G is bipartite, then the edges xy and uv are in relation Θ precisely when $d(x,u) = d(y,v)$ and $d(x,v) = d(y,u)$. Winkler [20] proved that a bipartite graph is a partial cube if and only if Θ is transitive.

Let G' be a connected graph. A *proper cover* G'_1, G'_2 consists of two isometric subgraphs G'_1, G'_2 of G' such that $G' = G'_1 \cup G'_2$ and $G'_0 = G'_1 \cap G'_2$ is a nonempty subgraph, called the *intersection* of the cover. Additionally there are no edges between $G'_1 \setminus G'_2$ and $G'_2 \setminus G'_1$. The *expansion* of G' with respect to G'_1, G'_2 is the graph G constructed as follows. Let G_i be an isomorphic copy of G'_i, for $i = 1,2$, and, for any vertex u' in G'_0, let u_i be the corresponding vertex in G_i, for $i = 1,2$. Then G is obtained from the disjoint union $G_1 \cup G_2$, where for each u' in G'_0 the vertices u_1 and u_2 are joined by an edge. We denote the copy of G'_0 in G_i by G_{0i}, for $i = 1,2$. Expansion is called *peripheral* if at least one of the graphs G'_1 or G'_2 is equal to G. Note that then the other graph equals the intersection that is thus necessarily isometric in G.

Chepoi [3] proved that a graph is a partial cube if and only if it can be obtained from K_1 by a sequence of expansions. This result was later independently obtained by Fukuda and Handa [6] and is analogous to the convex expansion theorem for median graphs [17].

The *Cartesian product* $G \Box H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph with vertex set $V(G) \times V(H)$ and vertex (a,x) is adjacent to vertex (b,y) in $E(G \Box H)$ whenever $ab \in E(G)$ and $x = y$, or, if $a = b$ and $xy \in E(H)$. For a fixed vertex a of G, the vertices $\{(a,x) \mid x \in V(H)\}$ induce a subgraph isomorphic to H. We call it an *H-layer*. Analogously we define G-layers. The *subdivision graph* $S(G)$ of a graph G is obtained from G by subdividing every edge of G.

Let G and H be connected graphs. Then $G \Box H$ is a partial cube if and only if G and H are partial cubes. This observation follows from the fact that the layers of the Cartesian product are convex, which is in turn implied by the fact that the distance function of the product is the sum of distance functions of the factors, cf. [11].

The following observation (probably part of the folklore) will be used in the sequel.

Lemma 1.1 *If an edge e of a graph G lies in a cycle, then e also lies in an isometric cycle of G.*

Proof. Let $e = uv$ and let P be a shortest path connecting the endpoints of e. Such a path exists since e lies in a cycle. But then $C = u \to \ldots P \ldots \to v \to u$ is an isometric cycle containing e, for otherwise P would not be shortest. □

2 Generating edge-critical partial cubes

Recall that G is an edge-critical partial cube if for any edge e of G, $G - e$ is not a partial cube. Clearly, an edge-critical partial cube is 2-edge-connected.

One can verify that the generalized Petersen graph $P(10,3)$ and a graph of Gedeonova from [7], see Fig. 2.4 in [11]), are edge-critical partial cubes. Two more examples of such graphs are shown in Fig. 1; the graph G was observed to be edge-critical in [15].

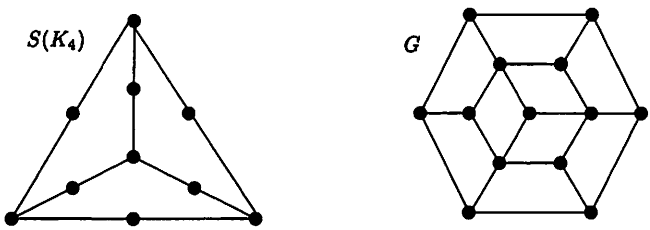

Figure 1: Two edge-critical partial cubes

In order to obtain more such graphs we are going to present an expansion procedure that preserves the property. First a lemma.

Lemma 2.1 *Let G be a bipartite graph, $H = C_{2n} \square K_2$ ($n \geq 2$) an isometric subgraph of G, and e an edge of H. Then $G - e$ is not a partial cube.*

Proof. Let $v_1, v_2, \ldots, v_{2n}$ be the consecutive vertices of the cycle C_{2n} and let $V(K_2) = \{1,2\}$. Thus vertices of H are of the form (v_i, j), $1 \leq i \leq 2n$, $1 \leq j \leq 2$.

Let e be an edge of a K_2-layer of H. Then we may without loss of generality assume that $e = (v_2,1)(v_2,2)$. As H is isometric, we infer that $(v_1,1)(v_2,1)$ is in relation Θ in $G - e$ with $(v_{n+1},1)(v_{n+2},1)$. Clearly, $(v_{n+1},1)(v_{n+2},1)$ is in relation Θ with $(v_{n+1},2)(v_{n+2},2)$ and using isometry again we infer that $(v_{n+1},2)(v_{n+2},2)$ is in relation Θ with $(v_1,2)(v_2,2)$. But as G is bipartite, $(v_1,2)(v_2,2)$ is not in relation Θ with $(v_1,1)(v_2,1)$ in $G - e$. Hence Θ is not transitive in $G - e$ and so $G - e$ cannot be a partial cube.

Let e be an edge of a C_{2n}-layer. We may assume that $e = (v_1,1)(v_2,1)$. Then in $G - e$ we have

$$
\begin{aligned}
&(v_2,1)(v_2,2)\Theta(v_3,1)(v_3,2), \\
&(v_3,1)(v_3,2)\Theta(v_4,1)(v_4,2), \\
&\qquad \vdots \\
&(v_{2n},1)(v_{2n},2)\Theta(v_1,1)(v_1,2),
\end{aligned}
$$

however, $(v_2,1)(v_2,2)$ is not in relation Θ with $(v_1,1)(v_1,2)$ in $G - e$. Hence Θ is again not transitive in $G - e$. □

Theorem 2.2 *Let G' be an edge-critical partial cube and let G be a peripheral expansion of G' with respect to G'_1, G'_2, where $G'_1 = G'$. Then G is an edge-critical partial cube if and only if every edge of G'_2 lies in an isometric cycle of G'_2.*

Proof. Suppose that every edge of G'_2 lies in an isometric cycle of G'_2 and let e be an edge of G. We wish to show that $G - e$ is not a partial cube.

Let e be an edge with at least one endpoint from $G_1 \setminus G_{01}$. We claim that $G_1 - e$ is an isometric subgraph of $G - e$. Let u and v be any vertices of G_1 and let P be a shortest u,v-path in $G - e$. Suppose that $P \cap G_{02} \neq \emptyset$. Let x be the first vertex of P in G_{02} and y the last such vertex. Let R be the x,y-subpath of P. We may without loss of generality assume that $P \cap G_{02} = R$. Let x', y' be the neighbors of x, y in G_{01} and let R' be the copy of R in G_{01}. Then we can replace the $x' \to x \to \cdots R \cdots \to y \to y'$-subpath of P with $x' \to \cdots R' \cdots \to y'$. But this new subpath is shorter and does not contain e, which proves the claim. (In fact, we have proved even more, the convexity of $G_1 - e$ in $G - e$.) Now suppose that $G - e$ is a partial cube. Then $G_1 - e$ would also be such. Since $G_1 - e$ is isomorphic to $G' - e$, this contradicts the assumption that G' is edge-critical.

In the second case, e is an edge of the subgraph of G induced by $G_{01} \cup G_{02}$. In the first subcase, let $e \in G_{01}$. (The case when e belongs to G_{02} is treated analogously.) Let C be an isometric cycle of G_{01} containing e and consider the subgraph $H = C \Box K_2$ of $G_{01} \cup G_{02}$. Note that H is an isometric subgraph of G. Then, using Lemma 2.1 we infer that $G - e$ is not a partial cube. In the second subcase, e is an edge between G_{01} and G_{02}. Let g be an edge of G_{01} adjacent to e. The edge g exists since G is 2-edge-connected and as G_{01} is isometric in G_1. Let C be an isometric cycle of G_{01} containing g. Then we have again an isometric subgraph $H = C \Box K_2$ of $G_{01} \cup G_{02}$ and $G - e$ is not a partial cube.

For the converse suppose that G is an edge-critical partial cube and assume that there is an edge $u'v'$ in G'_2 that does not lie in an isometric cycle of G'_2. (Note that it may lie in an isometric cycle of G'.) Let u and v be the copies of u' and v' in G_{01} and let a and b be the adjacent vertices of u and v in G_{02}, respectively. Let $e = ab$. The proof will be complete by showing that $G - e$ is a partial cube.

Let E_1^G and E_2^G be the Θ-classes in G with representatives uv and ua, respectively. Let $f = x_1x_2$ and $g = y_1y_2$ be edges of an arbitrary Θ-class E^G of G.

Suppose that E^G is different from E_1^G and E_2^G. We claim that f is in relation Θ with g also in $G - e$. Note first that if f and g belong to G_1, the conclusion follows from the fact that G_1 is convex. Suppose next that $f \in G_1$ and $g \in G_{02}$. Let P be an x_1, y_1-geodesic in G and assume that $e \in P$. Let w_1 be the last vertex of P in G_1 and w_2 its neighbor on P in G_{02}. Path P is of the form $x_1 \to \cdots \to w_1 \to w_2 \to \cdots R \cdots \to y_1$, where R contains e. Let w_3 be the neighbor of y_1 in G_{01}. Then the subpath $w_1 \to w_2 \to \cdots R \cdots \to y_1$ can be replaced by a path $w_1 \to \cdots R' \cdots \to w_3 \to y_1$ of the same length, where R' is a copy of R lying in G_{01}. Hence, $d_G(x_1, y_1) = d_{G-e}(x_1, y_1)$. Analogously we get $d_G(x_2, y_2) = d_{G-e}(x_2, y_2)$. Hence also in this case f is in relation Θ with g in $G - e$. The last case is when both f and g lie in G_{02}. If f is not in relation Θ with g in $G - e$ then we may assume that $d_G(x_1, y_1) < d_{G-e}(x_1, y_1)$ and $d_G(x_2, y_2) = d_{G-e}(x_2, y_2)$. Thus, there is an x_1, y_1-geodesic in G_{02} containing e and an x_2, y_2-geodesic in G_{02} not containing e. Hence there exists a cycle in G_{02} containing e. As G_{02} is convex, by Lemma 1.1 we get an isometric cycle C in G_{02} containing e. Considering the copy C' of C in G_{01} that contains uv we get a contradiction.

Assume next that f and g are edges of E_1^G different from e. Then we observe that f and g are both in G_1 for otherwise we would find (similarly as above) an isometric cycle in G_{02} containing e. Hence by the convexity of G_1, f is in relation Θ with g in $G - e$.

Consider finally the class E_2^G. Let $x_1, y_1 \in G_{01}$ and $x_2, y_2 \in G_{02}$. Set $E_{21}^{G-e} = E_2^G \cap W_{ab}$ and $E_{22}^{G-e} = E_2^G \cap W_{ba}$ and assume $f, g \in E_{21}^{G-e}$. Then an x_2, y_2-geodesic does not contain e (since $x_2, y_2 \in W_{ab}$). Clearly, as G_1 is convex, an x_1, y_1-geodesic also does not contain e. Therefore f

is in relation Θ with g also in $G - e$. Analogously we see that if the edges f and g lie in E_{22}^{G-e}, then they are in relation Θ in $G - e$. Let $f \in E_{21}^{G-e}$ and $g \in E_{22}^{G-e}$. We claim that f is not in relation Θ with g. Let $k = d_G(x_1, y_1)$. As G_{01} is isometric in G_1, $d_{G-e}(x_1, y_1) = k$. Let P be a shortest x_2, y_2-path. Note that P lies completely in G_{02}. If e is not on P, then ab lies on a cycle and thus also on an isometric cycle. Therefore, uv belongs to an isometric cycle of G_{01}, which is not possible. Hence, ab is on every x_2, y_2-geodesic and consequently $d_{G-e}(x_2, y_2) = k + 2$. This proves the claim.

In conclusion, $G-e$ is a partial cube with the following Θ-classes: $E_1^G - e$ (obtained from E_1^G), E_{21}^{G-e} and E_{22}^{G-e} (obtained from E_2^G), while all the other classes coincide with the remaining classes of G. □

Theorem 2.2 can be applied as follows. Take any edge-critical partial cube G' and an isometric cycle C of G'. Then expand G' with respect to G', C in order to obtain another edge-critical partial cube G. Note also that as Q_n is a peripheral expansion of Q_{n-1} with respect to Q_{n-1}, Q_{n-1}, Theorem 2.2 also implies that all hypercubes Q_n, $n \geq 3$, are edge-critical. (That the 3-cube Q_3 is edge-critical is clear.)

Another example of edge-critical partial cubes are the subdivision graphs of complete graphs $S(K_n)$, $n \geq 4$. We refer to [13] that they are indeed partial cubes. To see that they are edge-critical consider an arbitrary edge e. Then e belongs to an isometric subgraph H isomorphic to $S(K_4)$. Then in $H - e$ we can find two isometric 6-cycles sharing two edges such that their union is isometric. Now consider these two cycles in $G - e$ to find out that Θ is not transitive. However $S(K_4)$ can only be obtained by expansion from Q_3^- which is not an edge-critical partial cube. Thus not all edge-critical partial cubes can be generated using Theorem 2.2.

We next consider the question which Cartesian products of partial cubes are edge-critical.

Proposition 2.3 *Let G and H be partial cubes. Then $G \Box H$ is an edge-critical partial cube if and only if for any pair of edges $f \in E(G)$, $g \in E(H)$, f or g lies in a cycle of G or in cycle of H.*

Proof. Assume first that $G \Box H$ is an edge-critical partial cube and there are edges $f \in E(G)$ and $g \in E(H)$ such that neither f is in a cycle of G nor g lies in a cycle of H. Let $f = uv$ and $g = xy$. Clearly, the edge $e = (u,x)(v,x)$ lies in exactly one (isometric) square, that is, in $(u,x)(v,x)(v,y)(u,y)$. We claim that $(G \Box H) - e$ is a partial cube. Let $E_1^{G \Box H}$ and $E_2^{G \Box H}$ be the Θ-classes of $G \Box H$ with representatives $(u,x)(v,x)$ and $(u,x)(u,y)$, respectively. We now argue similarly as in the proof of Theorem 2.2 to infer that $(G \Box H) - e$ is a partial cube with the Θ-classes: $E_1^{G \Box H - e}$ (obtained from $E_1^{G \Box H}$), $E_{21}^{(G \Box H)-e}$ and $E_{22}^{(G \Box H)-e}$ (obtained from $E_2^{G \Box H}$), and the remaining classes that coincide with the remaining Θ-classes of $G \Box H$.

For the converse assume that for any pair of edges $f \in E(G)$, $g \in E(H)$, at least one of f or g lies in a cycle of G or in cycle of H. Let e be an arbitrary edge of $G \Box H$. We need to show that $(G \Box H) - e$ is not a partial cube. By the definition of the Cartesian product, e lies in a square S of $G \Box H$. Let p_G denotes the projection onto G and let $p_G(S) = h$. We may without loss of generality assume that h lies in a cycle C. Moreover, by Lemma 1.1 we may in addition assume that C is isometric. Suppose first that $p_G(e) = h$. Then e lies in subgraph $C \Box K_2$ of $G \Box H$ and by Lemma 2.1 we infer that $(G \Box H) - e$ is not a partial cube. In the other case, $p_G(e)$ is a vertex. Then e is a neighbor of an edge f such that $p_G(f)$ is an edge lying in an isometric cycle C of G. Hence also in this case e lies in subgraph isomorphic to $C \Box K_2$ and we get the same conclusion. □

Let G be 2-edge-connected partial cube and H an arbitrary partial cube. Then Proposition 2.3 implies that $G \Box H$ is an edge-critical partial cube. For example, the Cartesian products $C_{2k} \Box P_n$, $k, n \geq 2$, are edge-critical partial cubes.

3 Small edge-critical partial cubes

As we already mentioned, Handa [9] proved that partial cubes G with $|W_{ab}| = |W_{ba}|$ for any edge ab are 3-connected if G has at least two edges and G is not a cycle. For edge-critical partial cubes we cannot prove neither 3-connectivity nor 3-edge-connectivity, as the example of $S(K_n)$, $n \geq 4$, demonstrates. On the other hand, for Θ-classes that also form edge cutsets, we can show:

Lemma 3.1 *Let G be an edge-critical partial cube and F a Θ-class of G. Then $|F| \geq 3$.*

Proof. By 2-edge-connectivity, $|F| \geq 2$. Suppose on the contrary that $F = \{e, f\}$. Let $e = uv$ and consider the graph $G - f$. Recall that W_{uv} and W_{vu} induce convex subgraphs of G thus they induce partial cubes. Since e is a cut-edge of $G - f$ between the blocks induced by W_{uv} and W_{vu}, $G - f$ is a partial cube and so G cannot be edge-critical. □

Proposition 3.2 *Let G be an edge-critical partial cube on at most 10 vertices. Then G is either Q_3 or $S(K_4)$.*

Proof. Let F be a Θ-class of G of maximum cardinality. Let $x_i y_i \in F$, $i = 1, \ldots, |F|$. Let G_1 and G_2 be subgraphs of G induced by $W_{x_1 y_1}$ and $W_{y_1 x_1}$, and let H_1 and H_2 be subgraphs of G induced by $\{x_1, \ldots, x_{|F|}\}$ and $\{y_1, \ldots, y_{|F|}\}$.

Case 1: $|F| = 5$.
In this case $V(G) = V(H_1) \cup V(H_2)$. Hence G is obtained by an expansion from G' with respect to $G'_1 = G'_2 = G'$ which implies that $G = H_1 \Box K_2$. As there is no partial cube on 5 vertices with the property that every edge would lie in a cycle, if follows from Proposition 2.3 that we get no edge-critical partial cube in this case.

Case 2. $|F| = 4$.
Assume first that H_1 contains at most two edges. Then H_1 and H_2 are not connected and thus there exist vertices $x_5 \in G_1 \setminus H_1$ and $y_5 \in G_2 \setminus H_2$. It is now easy to see that in no way we can add edges between x_5 and x_1, x_2, x_3, x_4 and between y_5 and y_1, y_2, y_3, y_4 to obtain an edge-critical partial cube.

Let there be three edges in H_1. Suppose $H_1 = K_{1,3}$ and let x_1 be the vertex of H_1 of degree 3. As $K_{1,3} \Box K_2$ is not edge-critical, there must be another vertex $x_5 \in G_2 \setminus H_2$. Assume $|G| = 9$. By 2-edge-connectivity we may assume that x_5 is adjacent to x_3 and x_4. If x_5 is adjacent also to x_2, then G is not a partial cube, otherwise G is a partial cube that is not edge-critical. Hence $|G| = 10$ and let the tenth vertex x_6 belong to $G_1 \setminus H_1$. Then in any case $G - x_2 y_2$ is not a partial cube. Thus, $x_6 \in G_2 \setminus H_2$. Again x_5 must be adjacent to precisely two vertices, say x_2 and x_3. If x_6 is adjacent to y_2 and y_3, then G is not a partial cube, otherwise x_6 must be adjacent to, say, y_2 and y_4, in which case G is not edge-critical.

Suppose H_1 is the path on four vertices $x_1 x_2 x_3 x_4$. If any of the vertices x_1, x_4, y_1, or y_4 is of degree 2, then $G - x_1 y_1$ or $G - x_4 y_4$ is not a partial cube. But we can avoid this only by introducing two new vertices and creating odd cycles.

Finally, assume that there are 4 edges in H_1. Then $H_1 = C_4$. If $|G| = 8$, then G is Q_3 which is edge-critical. We wish to show that there is no other edge-critical partial cube. Assume $|G| = 9$ and let $x_5 \in G_1 \setminus H_1$. By 2-edge-connectivity we may without loss of generality assume that x_5 is adjacent to x_1 and x_3. As this gives an induced $K_{2,3}$, $|G| = 10$. Then we have two adjacent vertices $x_5, x_6 \in G_1 \setminus H_1$. Let $x_5x_1 \in E(G)$. If $x_6x_2 \in E(G)$ we would have a Θ-class with at least 5 edges. Hence $x_6x_2 \notin E(G)$ and $x_6x_4 \in E(G)$ which yields the same conclusion.

Case 3: $|F| = 3$.
In this case Lemma 3.1 implies that all Θ-classes contain 3 edges. Suppose first that there are two edges in H_1. We may without loss of generality assume that they are x_1x_2 and x_2x_3. Then y_1y_2 and y_2y_3 are edges of H_2. None of vertices x_1, x_3, y_1, and y_3 is of degree 2 in G for otherwise G would not be edge-critical. Therefore we have a vertex x_4 in $G_1 \setminus H_1$ and a vertex vertex x_5 in $G_2 \setminus H_2$. By 2-edge-connectivity x_4 is then adjacent to x_1 and x_3, while x_5 is adjacent to y_1 and y_3. This gives an induced Q_3 minus an edge in G that is not a partial cube.

Suppose next that there is only one edge in H_1, say x_1x_2. Since G_1 is isometric there should be a geodesic P_1 in G_1 from x_3 to x_1. Likewise, we have a y_3, y_1-geodesic P_2 in G_2. Moreover, $|P_1| = |P_2|$. As we consider graphs on at most 10 vertices, $|P_1| < 4$. Let $|P_1| = 3$ and denote the inner vertices on P_1 by x_4 and x_5, where x_4 is adjacent to x_1. Let the inner vertices of P_2 be y_4 and y_5, where y_4 is adjacent to y_1. Observe that the cycle $x_1x_4x_5x_3y_3y_5y_4y_1$ is isometric and consider the Θ-class of x_1x_2. It has to have at least three edges, so we may without loss of generality assume that x_5 is adjacent to x_2. Then $R = y_1y_4y_5y_3x_3x_5x_2y_2$ is a walk connecting the endpoints of the edge y_1y_2. Hence R contains an edge g that is in relation Θ to y_1y_2, cf. [11, Lemma 2.4]. But now x_1x_2, x_4x_5, y_1y_2, and g are different edges belonging to the same Θ-class. We conclude that $|P_1| = |P_2| = 2$. Let x_4 be the vertex of P_1 adjacent to x_1 and x_3 and y_4 the corresponding vertex of P_2. Observe that the cycle $x_1x_4x_3y_3y_4y_1x_1$ is convex. To see this it suffices to observe that P_1 is the unique x_1, x_3-geodesic, as well as is P_2 the unique y_1, y_3-geodesic, which in turn holds because otherwise Θ would not be transitive. As G is edge-critical, there is another vertex $x_5 \in G_1$ adjacent to x_2 and a vertex $y_5 \in G_2$ adjacent to y_2. Clearly, x_5 is adjacent neither to x_1 not to x_3, so it must be adjacent to x_4. Similarly, y_5 must be adjacent to y_4. But then the edges x_4x_5, x_1x_2, y_1y_2, and y_4y_5 belong to the same Θ-class.

Finally, we need to consider the case when there are no edges between vertices in H_1. Suppose that $d(x_1, x_2) = 3$. Let P_1 be a x_1, x_2-geodesic, x_4 and x_5 its inner vertices and x_4 adjacent to x_1. Similarly, there is a y_1, y_2-geodesic P_2 in G_2 with inner vertices y_4 and y_5. To make G_1 connected, we must either add the edge x_4x_3 (and y_4y_3) or the edge x_5x_3 (and y_5y_3). However, in both cases G is not a partial cube. Therefore, $d(x_i, x_j) = d(y_i, y_j) = 2$ for any pair i, j, $i \neq j$. The conditions $d(x_i, x_j) = 2$ can be achieved by either adding one or three vertices in $G_1 \setminus H_1$, the same holds for the conditions $d(y_i, y_j) = 2$. Adding three vertices to $G_1 \setminus H_1$ and three to $G_2 \setminus H_2$ gives a graph on 12 vertices, while adding one vertex on each side we get a graph that is not a partial cube. Hence we must add, say, three vertices of degree two to $G_1 \setminus H_1$ and a vertex of degree three to $G_2 \setminus H_2$. The obtained graph is $S(K_4)$ and is edge-critical. To complete the case observe that adding edges to $S(K_4)$ yields no further (edge-critical) partial cube. □

The graphs $C_4 \Box P_3$ and $C_6 \Box P_2$ are edge-critical partial cubes on 12 vertices, while the graph G from Fig. 1 is an example with 13 vertices. We do not know any such graph on 11 vertices.

4 Concluding remark

We have studied partial cubes G with the property that removing any edge of G destroys a possibility of its isometric embedding into a hypercubes. A more general question is the following. Let e be an edge of a partial cube G. Under which conditions is $G - e$ a partial cube? This problems seems to be rather involved, but the following concept might be useful in its attack.

Let G be a connected graph with at least one cycle and let $\mathcal{C}(G) = \{C^1, C^2, \ldots, C^r\}$ be the set of isometric cycles of G. Let $G(\mathcal{C})$ be the intersection graph of $\mathcal{C}(G)$. More precisely, the vertex set of $G(\mathcal{C})$ is $\mathcal{C}(G)$, and two vertices are adjacent if the corresponding cycles intersect in at least one edge. Label the edges C^iC^j of $G(\mathcal{C})$ with $C^i \cap C^j$ where the cycles are considered as sets of edges and denote the obtained edge-labeled graph $G_\ell(\mathcal{C})$. For an edge $e \in G_\ell(\mathcal{C})$, let $l(e)$ denote its label.

We thus pose the question if the structure of $G_\ell(\mathcal{C})$ suffices to find out whether $G - e$ a partial cube. For example, one can show:

Proposition 4.1 *Let G be a partial cube and $G(\mathcal{C})$ a tree. Then $G - e$ is a partial cube for any edge e of G.*

Proof. Since $G(\mathcal{C})$ a tree, two isometric cycles of G intersect in at most one edge, for otherwise G would not be a partial cube. Moreover, if C and C' are isometric cycles of G sharing an edge, then $C_1 \cup C_2$ is isometric subgraph of G. Now it is straightforward to verify the assertion. □

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