On regular-stable graphs

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Abstract

We introduce graphs G, with at least one maximum independent set of vertices, I, such that $\forall v \in V(G) \setminus I$, the number of vertices in $N_G(v) \cap I$ is constant. When this number of vertices is equal to λ we say that I has the λ-property and that G is λ-regular-stable. Furthermore we extend the study of this property to the well-covered graphs (that is, graphs where all maximal independent sets of vertices have the same cardinality). In this study we consider well-covered graphs for which all maximal independent sets of vertices have the λ-property, herein called well-covered λ-regular-stable graphs.

Keywords: Graph Theory, well-covered graphs, regular-stable graphs.

1 Introduction

Let $G = (V, E)$ be a finite connected undirected graph. We denote by $V(G)$ the set of vertices and $E(G)$ the set of edges of G. G is a graph without loops and multiple edges. Here we give some results for graphs with maximum independent sets with a special property which is designated by the λ property.

Let I be a maximum independent set of vertices in G. Then G is *λ-regular-stable* relatively to the set of vertices I, if $\forall v \in V(G)\setminus I$, the number of vertices in $N_G(v) \cap I$, is constant, and equal to λ. In this case, we say that I has the λ property. We also say that a graph is *λ-regular-stable* if it has at least one maximum independent set with the λ property. This concept is connected with the recognition of graphs for which the upper bound on the stability number, obtained by convex quadratic programming is attained [5, 6].

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The graphs C_{2n} are examples of 2-regular-stable graphs. The graph $K_{p,n}$, with $p \geq n$, is an example of a p-regular-stable graph.
Another interesting class of graphs which has an easy stability number to determine is the class of well-covered graphs introduced in [7]. It is a challenge to know among the graphs in this class which ones have all maximal independent sets with the λ-property.
The plan of the paper is as follows. In section 2 the main results obtained regarding λ-regular-stable graphs are introduced. In section 3 we give some results about well-covered graphs for which all maximal independent sets have the λ-property. Finally, in section 4, we propose some open questions.

2 λ-regular-stable graphs

Given a λ-regular-stable graph, G, we obtain the upper and lower bounds on the stability number of G, $\alpha(G)$, established in the next theorem.

Theorem 1 *If G be is a λ-regular-stable graph then*

$$\frac{n\lambda}{\Delta(G)+\lambda} \leq \alpha(G) \leq \frac{n\lambda}{\delta(G)+\lambda},$$

with $\delta(G)$ and $\Delta(G)$ denoting the minimum and maximum degree of G, respectively.

Proof: Let us denote by $d_G(i)$ the degree of the vertex i. Let I be a maximum independent set of vertices with the λ-property. Thus

$$\sum_{i\in I} d_G(i) = \lambda(n-\alpha(G)) \Leftrightarrow (\frac{1}{|I|}\sum_{i\in I} d_G(i))\alpha(G) = \lambda(n-\alpha(G))$$

and, if $\bar{d} = \frac{1}{|I|}\sum_{i\in I} d_G(i)$, then

$$\bar{d}\alpha(G) = \lambda(n-\alpha(G)) \Leftrightarrow \alpha(G) = \frac{n\lambda}{\lambda+\bar{d}}.$$

Since $\delta(G) \leq \bar{d} \leq \Delta(G)$, we have

$$\frac{n\lambda}{\lambda+\Delta(G)} \leq \frac{n\lambda}{\lambda+\bar{d}} = \alpha(G) \leq \frac{n\lambda}{\lambda+\delta(G)}.$$

As a consequence of this theorem, if G is a k-regular graph (that is, such that $\forall v \in V(G)\ \ d_G(v) = k$) which is λ-regular-stable then $\alpha(G) = \frac{n\lambda}{\lambda+k}$.

A connected graph, G, with order greater than one, such that $L(G)$ is not complete (where $L(G)$ denotes the line graph of G), has a perfect matching if and only if it belongs to the class of graphs with convex-QP stability number [2] (that is, the class of graphs for which the stability number is the optimal value of a convex quadratic programing problem). Now we prove a similar result.

Theorem 2 *Let G, $G \neq P_2$, be a graph with maximum matching M. Then the graph $H = L(G)$ is 2-regular-stable if and only if M is perfect.*

Proof: ($\Longrightarrow$)Let M be a maximum matching in G. There is a maximum independent set I in H formed by $L(M)$. If M is not perfect, there is a vertex $g \in V(G)$ not covered by M. Then the edge ug, for $u \in V(G)$ will be a vertex in the graph H with only one neighbour in the maximum set I. On the other hand, there is a path $uhxy$, such that uh and $xy \in M$. Then the vertex $L(xh)$ has $L(xy)$ and $L(uh)$ as neighbours in H. So, if M is not perfect H is not λ-regular-stable.
($\Longleftarrow$)If M is perfect, then $L(M)$ is a maximum independent set of vertices in H, and every vertex in $H \backslash L(M)$ has exactly two neighbours in $L(M)$. □

Note that $G = P_5$ does not have a perfect matching, but $L(G) = P_4$ is λ-regular-stable, with $\lambda = 1$. In the next theorem a complete characterization of 1-regular-stable graphs is given.

A vertex v of a graph is a *simplicial* vertex if it appears in exactly one clique of the graph. A clique of a graph G containing at least one simplicial vertex of G is called a *simplex* of G. A graph G is a *simplicial graph* if every vertex of G is a simplicial vertex of G or is adjacent to a simplicial vertex of G. A graph G is *chordal* if every cycle of G of length four or more has a chord.

Next, the 1-regular-stable graphs are characterized.

Theorem 3 *A graph G is 1-regular-stable iff each vertex belongs to exactly one simplex.*

Proof: Let us suppose that G is 1-regular-stable and S is a maximum stable set for G, such that $\forall v \in V(G) \setminus S$, $|N_G(v) \cap S| = 1$. Then $\forall i \in S$ i is a simplicial vertex. Otherwise, $N_G[i]$ is not a clique and then $\exists p, q \in N_G(i)$ such that $[p, q] \notin E(G)$ and

$$(N_G(p) \cup N_G(q)) \cap S = \{i\}.$$

Therefore $T = (S \setminus \{i\}) \cup \{p, q\}$ is a stable set for G such that $|T| > |S|$ which is a contradiction.
It is proven that each vertex $i \in S$ determines a simplex, S_i, and each vertex not in S belongs at least to one of these simplices. Let us suppose that $\exists i, j \in S$ such that $S_i \cap S_j \neq \emptyset$. Then $\forall v \in S_i \cap S_j$ $|N_G(v) \cap S| \geq 2$ and we again have a contradiction.
Conversely suppose that each vertex belongs to exactly one simplex. Then

$$V(G) = \cup_{i \in I} S_i, \text{ with } S_p \cap S_q = \emptyset \ \forall p \neq q,$$

where $\forall i \in I$ i is a simplicial vertex and S_i is a simplex. Therefore I is a maximum independent set such that $\forall v \in V(G) \setminus I$, $|N_G(v) \cap S| = 1$. □

There are graphs with more than one maximum independent set where some of them have the λ-property and others do not have it. As examples we have P_4 and the graph of figure 3.

However if $\lambda \geq -\lambda_{min}(A_G)$, where $\lambda_{min}(A_G)$ denotes the minimum eigenvalue of the adjacency matrix of G, A_G, (which it is not greater than -1 if G has at least one edge) and there is a maximum independent set with the λ-property, then all maximal independent sets have the λ-property (see [2]).

3 well-covered λ-regular-stable graphs

A graph G is *well-covered* if all maximal independent sets of vertices in G have the same cardinality. The recognition problem of well-covered graphs in general is Co-NP-complete [3, 9]. A recent survey paper about these graphs was given by Hartnell [4]. It is not true that if G is well-covered and 2-regular stable it must have all maximal independent sets of vertices with the λ-property (for instance, see figure 3).
We are interested in well-covered graphs for which all maximal independent

sets of vertices have the λ-property, for a given λ. From now on we call the graphs with this property, well-covered λ-regular-stable graphs.
For any λ, the graph $K_{\lambda,\lambda}$ is an example of a well-covered λ-regular-stable graph.
A graph G is *randomly matchable* if every maximal matching in G is perfect. This class of graphs was characterized in [10]. So, according to Theorem 2, if we take the line graph of a randomly matchable graph, we obtain a well-covered 2-regular-stable graph. As examples, we have the graphs of figures 1 and 2, where both are well-covered 2-regular-stable graphs.
It is not true that every well-covered 2-regular stable graph must be a line graph of some graph. Figure 4 gives an example of a well-covered 2-regular stable graph that are not line graph of any graph.
Given a subset S of $V(G)$, $G[S]$ denotes the graph induced by S and $\alpha(G[S])$ denotes the stability number of $G[S]$.

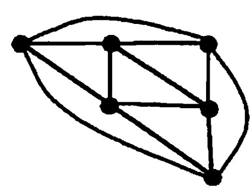

Figure 1: A well-covered, 2-regular-stable graph.

Theorem 4 *[8] Let G be a simplicial graph. Then G is well-covered if and only if every vertex in $V(G)$ belongs to exactly one simplex.*

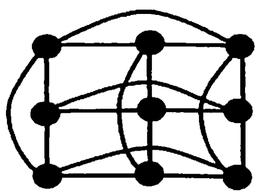

Figure 2: The graph $L(K_{3,3})$.

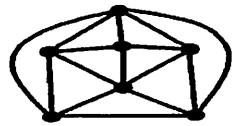

Figure 3: G is 2-regular-stable, but it has a maximal independent set without this property.

Theorem 5 *[8] Let G be a chordal graph. G is well-covered if and only if it is simplicial.*

As an immediate consequence of the above theorems and Theorem 3 we may conclude that a simplicial graph is 1-regular-stable if and only if it is well-covered. A chordal graph is 1-regular stable if and only if it is simplicial.

Theorem 6 *If G is a simplicial and well-covered λ-regular-stable graph, then $\lambda = 1$ and G is complete.*

Proof: Assuming that G is a well-covered simplicial graph, then by Theorem 4 each vertex belongs exactly to one simplex.
If G is complete then we obtain the result. Let us suppose that G is not a complete graph and let $I = \{s_1, s_2, ..., s_n\}$ be the maximal independent set of G built with the simplicial vertices, each one taken from a different simplex. Then I has the 1-property. Now, replace s_1 by a vertex $g \in N_G(s_1)$ with g not simplicial. Then g has a neighbour h, such that h is not a simplicial vertex (otherwise the graph must be complete) but, since G is well-covered, by Theorem 4, h must belong to only one simplex, say S_2 with a simplicial vertex s_2. Then $J = \{g, s_2, ..., s_3\}$ is a maximal independent set of vertices, where h has two neighbours (g and s_2) in J. Hence G must be the complete graph. □

Corollary 1 *Let G be a chordal well-covered graph with more than one maximal independent set of vertices. Then there is at least one maximal independent set of vertices in G without the λ-property.*

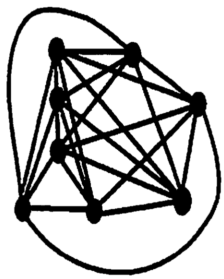

Figure 4: An example of a well-covered graph that is not line graph of any graph.

Proof: Since by Theorem 5, a well-covered chordal graph must be a simplicial graph, we apply the above theorem. □

Theorem 7 *Let G be a well-covered λ-regular-stable graph. Then $\forall v \in V(G)$ $\alpha(G[N_G(v)]) = \lambda$.*

Proof: If $\exists v \in V(G)$ with $\alpha(G[N_G(v)]) \neq \lambda$, taking a maximal independent set I that contains the maximum independent set of $G[N_G(v)]$, v will not have λ neighbours in I, which is a contradiction. □

As an immediate consequence of the above theorem we have the following corollary.

Corollary 2 *Let G be a well-covered 2-regular-stable graph. Then G does not have $K_{1,3}$ as an induced subgraph.*

Theorem 8 *Let G be a well-covered λ-regular-stable graph, not bipartite. Then G has triangles.*

Proof: Let I be a maximal independent set of vertices in G. Since G is not bipartite, there must exist x, $y \in V(G) \backslash I$ such that $x \sim y$. We now have to prove that $N_G(x) \cap N_G(y) \neq \emptyset$. If $N_G(x) \cap N_G(y) = \emptyset$, then there is a maximal independent set J which includes the maximum independent set of the induced subgraph $G[N_G(x) \cup \{y\}]$. Then x has more than λ neighbours in J, which is a contradiction. □

Furthermore the above proof allows to conclude that every edge of a well-covered λ-regular-stable graph with both ends out of some maximal independent set belongs to a triangle.

Theorem 9 *[10] Let G be a graph for which every maximal matching is perfect, then $G = K_{n,n}$ or $G = K_n$, for some $n \in N$.*

From this theorem we obtain the following corollary.

Corollary 3 *Let G be a graph such that exists a graph H with $G = L(H)$. Then G is a well-covered λ-regular-stable graph if and only if $H = K_n$ or $H = K_{n,n}$, for some n.*

For $n \geq 3$, the graphs $L(K_{n,n})$ have C_3 and all even cycles up to C_{2n} and for $n \geq 4$, the graphs $L(K_n)$ have all cycles from C_3 to C_{n+1}. Therefore, for every $n \geq 4$ it is possible to build well-covered λ-regular-stable graphs, with $\lambda = 2$, and with C_n as induced cycles.

Theorem 10 *The graphs given in figure 5 are forbidden induced subgraphs for well-covered graphs for which every maximal independent set of vertices has the λ property, with $\lambda = 2$.*

Proof:

1)See corollary 2

2)In this case, see figure 6, when we choose a maximal independent set I that contains $\{a, b\}$, c must have a vertex $w \in I$ as a neighbour. But, since w cannot be adjacent to z (as $\lambda = 2$), we would then have the induced $K_{1,3}$ formed by vertices $cwxz$, a contradiction.

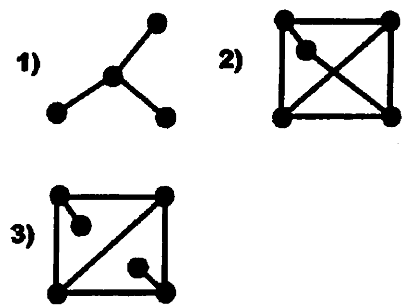

Figure 5: The forbidden induced subgraphs of Theorem 10

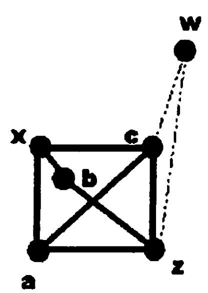

Figure 6: A forbidden induced subgraph.

3)In this case, see figure 7, when we choose a maximal independent set I that contains $\{c, d, e\}$, a must have a vertex $x \in I$ as a neighbour. But then, since x is not joined to z nor to y (as $\lambda = 2$), we would have the induced $K_{1,3}$ formed by vertices $xayz$, a contradiction. □

The graphs of the Theorem 10 belong to the set of nine forbidden graphs for the line graphs [1].

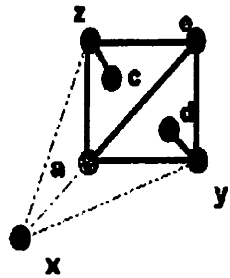

Figure 7: A forbidden induced subgraph.

4 Open problems

Theorem 10 gives some forbidden induced subgraphs of well-covered graphs, for which all maximal independent sets have the λ-property, with $\lambda = 2$. It is an open question to decide if these are the only forbidden subgraphs for the family of well-covered 2-regular-stable graphs.
Another natural question, not considered in this paper, is to determine the properties of well-covered λ-regular-stable graphs, with $\lambda > 2$.

References

[1] L.W. Beineke, *Derived graphs and digraphs*, In Beitrage Zur Graphentheorie, 17-33, Tubner 1968.

[2] D.M. Cardoso, *Convex Quadratic Programming Approach to the Maximum Matching Problem*, Journal of Global Optimization 19, 291-306, 2001.

[3] V. Chvatal and P.J. Slater, *A note on well-covered graphs, Quo Vadis, Graph Theory?*, Ann. Discrete Math. 55, 179-182, 1993.

[4] B. Hartnell, *Well-covered graphs*, Journal Comb. Math. Comb. Comp. 29, 107-115, 1999.

[5] C.J. Luz, *An upper bound on the independence number of a graph computable in polynomial-time*, Oper. Res. Lett. 18, No.3, 139-145, 1995.

[6] C.J. Luz and D.M. Cardoso, *A generalization of the Hoffman-Lovasz upper bound on the independence number of a regular graph*, Annals of operations Research 81, 307-309, 1998.

[7] M.D. Plummer, *Some covering concepts in graphs*, Journal Combin. Theory 8, 91-98, 1970.

[8] E. Prisner, J. Topp and P.D. Vestergaard, *Well-covered simplicial, chordal and circular arc graphs*, J. of Graph Theory 21, 113-119, 1996.

[9] R.S. Sankaranarayana and L.K. Stewart, *Complexity results for well-covered graphs*, Networks 22, 247-262, 1992.

[10] D. P. Sumner, *Randomly matchable graphs*, J. of Graph Theory 3, 183-186, 1979.