Circular Chromatic Numbers and Fractional Chromatic Numbers of Distance Graphs with Distance Sets Missing An Interval

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Abstract

Given positive integers m, k, and i. Let $D_{m,[k,k+i]}$ represent the set $\{1,2,\ldots,m\} - \{k,k+1,\ldots,k+i\}$. The distance graph $G(Z, D_{m,[k,k+i]})$ has vertex set all integers Z and edges connecting j and j' whenever $|j-j'| \in D_{m,[k,k+i]}$. The fractional chromatic number, the chromatic numbers, and the circular chromatic number of $G(Z, D_{m,[k,k+i]})$ are denoted by $\chi_f(Z, D_{m,[k,k+i]})$, $\chi(Z, D_{m,[k,k+i]})$, and $\chi_c(Z, D_{m,[k,k+i]})$, respectively. For $i = 0$, we simply denote $D_{m,[k,k+0]}$ by $D_{m,k}$. $\chi(Z, D_{m,k})$ was studied by Eggleton, Erdős and Skilton [5], Kemnitz and Kolberg [8], and Liu [9], and was completely solved by Chang, Liu and Zhu [1] who also determined $\chi_f(Z, D_{m,k})$ for any m and k. The value of $\chi_c(Z, D_{m,k})$ was studied by Chang, Huang and Zhu [2] who finally determined $\chi_c(Z, D_{m,k})$ for any m and k. This paper extends the study of $G(Z, D_{m,[k,k+i]})$ to values i with $1 \le i \le k-1$. We completely determine $\chi_f(Z, D_{m,[k,k+i]})$, and $\chi(Z, D_{m,[k,k+i]})$ for any m and k with $1 \le i \le k-1$. However, for $\chi_c(Z, D_{m,[k,k+i]})$, only some special cases are determined.

1. Introduction

The fractional chromatic of a graph is a well-known variation of the chromatic number. Let G be a graph. A *fractional coloring* of G is a mapping c from $\Gamma(G)$, the set of all independent sets of G, to the interval $[0,1]$ such that $\sum_{x \in I \in \Gamma(G)} c(I) \geq 1$ for all vertices x in G. The *fractional chromatic number* $\chi_f(G)$ of G is the infimum of the value $\sum_{I \in \Gamma(G)} c(I)$ of a fractional coloring c of G.

The circular chromatic number of a graph is a natural generalization of the chromatic number, introduced by Vince [11] under the name the "star chromatic number" of a graph. Given two positive integers p and q, such that $p \geq 2q$. A *(p,q)-coloring* of a graph $G = (V, E)$ is a mapping ϕ from V to $\{0, 1, \ldots, p-1\}$, such that $||\phi(x) - \phi(y)||_p \geq q$ for any edge $xy \in E$, where $||a||_p = min\{||a||, p - ||a||\}$. The *circular chromatic number* $\chi_c(G)$ of G is the infimum of the ratios $\frac{p}{q}$ for which there exist (p,q)-colorings of G. It is obvious that a $(p,1)$-coloring of a graph is simply an ordinary p-coloring of G. It was proved in [17] that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ for any graph G. Thus $\chi_c(G)$ can be viewed as a refinement of $\chi(G)$, and it contains more information about the structure of the graph. It is usually much more difficult to determine the circular chromatic number of a graph than to determine its chromatic number.

For any graph G, it is well-known that

$$max\{\omega(G), \frac{|V(G)|}{\alpha(G)}\} \leq \chi_f(G) \leq \chi_c(G) \leq \lceil \chi_c(G) \rceil = \chi(G) \qquad (1)$$

Given a set D of positive integers, the distance graph $G(Z, D)$ has vertices all integers Z, and two vertices j and j' in Z are adjacent if and only if $|j - j'| \in D$. We call D the *distance set*. The fractional chromatic number, the circular chromatic number, the chromatic number, the clique number of a distance graph $G(Z, D)$ are denoted by $\chi_f(Z, D)$, $\chi_c(Z, D)$, $\chi(Z, D)$, and $\omega(Z, D)$, respectively.

The problem of determining $\chi(Z, D)$ for different types of distance sets D has been studied extensively. (See [1, 3, 4, 5, 6, 8, 9, 10, 12, 13]). Recently $\chi_c(Z, D)$ for several types of distance sets D has also been investigated. (See ([1, 2, 7, 15, 16]).

Given integers m, k, and i, let $D_{m,[k,k+i]}$ denote the set $\{1, 2, \ldots, m\} - \{k, k+1, \ldots, k+i\}$. This paper discusses the graph $G(Z, D_{m,[k,k+i]})$. For

$i = 0$, we simply denote $D_{m,[k,k+0]}$ by $D_{m,k}$. The following results are from [1] and [2].

Theorem 1.1. If $m < 2k$, then

$$\omega(Z, D_{m,k}) = \chi_f(Z, D_{m,k}) = \chi_c(Z, D_{m,k}) = k.$$

Theorem 1.2. If $m \geq 2k$, then $\chi_f(Z, D_{m,k}) = \frac{m+k+1}{2}$.

Theorem 1.3. Suppose $m \geq 2k$. Let $m + k + 1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers and m' and k' are odd integers. If $r \leq s$ and $gcd(m + k + 1, k) \neq 1$, then $\chi_c(Z, D_{m,k}) = \frac{m+k+2}{2}$; otherwise $\chi_c(Z, D_{m,k}) = \frac{m+k+1}{2}$.

This paper extends the study of $\chi_f(Z, D_{m,[k,k+i]})$ and $\chi_c(Z, D_{m,[k,k+i]})$ to values i with $1 \leq i \leq k - 1$. We completely determine the fractional chromatic number and the chromatic number of $G(Z, D_{m,[k,k+i]})$ for any m, k, and i with $1 \leq i \leq k-1$. For some cases, the circular chromatic number of $G(Z, D_{m,[k,k+i]})$ are determined, and, of other cases, lower bounds and upper bounds of $\chi_c(Z, D_{m,[k,k+i]})$ are given.

2. $\chi_f(Z, D_{m,[k,k+i]})$ and $\chi_c(Z, D_{m,[k,k+i]})$

Since $G(Z, D_{m,[k,k+i]})$ is a subgraph of $G(Z, D_{m,k})$, it is obvious that $\chi_f(Z, D_{m,[k,k+i]}) \leq \chi_f(Z, D_{m,k})$ and $\chi_c(Z, D_{m,[k,k+i]}) \leq \chi_c(Z, D_{m,k})$ for any m, k, and i. Note that $\omega(Z, D_{m,[k,k+i]}) \geq k$, the following theorem follows immediately from Theorem 1.1.

Theorem 2.1. If $m < 2k$, then

$$\omega(Z, D_{m,[k,k+i]}) = \chi_f(Z, D_{m,[k,k+i]}) = \chi_c(Z, D_{m,[k,k+i]}) = k.$$

In order to prove Theorem 2.2 below, we need a lemma in [2].

Lemma 2.2. [2] Suppose D is a set of positive integers, and that p and q are positive integers. Let $d_D(p,q) = min\{\| qj \ \ mod \ \ p \|_p\colon j \in D\}$. If $d_D(p,q) \geq 1$, then $\chi_c(Z, D) \leq \frac{p}{d_D(p,q)}$.

For brevity, we denote the subgraph of $G(Z, D_{m,[k,k+i]})$ induced by $\{0, 1, 2, \ldots, j\}$ as G_j.

Theorem 2.3. If $2k \leq m < 2k + 2i$, and $1 \leq i \leq k - 1$, then $\chi_f(Z, D_{m,[k,k+i]}) = \chi_c(Z, D_{m,[k,k+i]}) = \frac{m+1}{2}$.

Proof. Since $i \leq k - 1$, it is not hard to see that $\alpha(G_m) = 2$. It follows that $\chi_f(Z, D_{m,[k,k+i]}) \geq \frac{m+1}{2}$. By (1), it suffices to show that $\chi_c(Z, D_{m,[k,k+i]}) \leq \frac{m+1}{2}$. According to Lemma 2.1, it suffices to prove that $d_{D_{m,[k,k+i]}}(m + 1, 2) = 2$. For $j \in \{1, 2, \ldots, k - 1\}$, $2 \leq 2j \leq 2k - 2 \leq m + 1 - 2$, and for $j \in \{k + i + 1, k + i + 2, \ldots, m\}$, $m + 1 + 2 \leq 2k + 2i + 2 \leq 2j \leq 2m = 2(m + 1) - 2$. Thus $d_{D_{m,[k,k+i]}}(m + 1, 2) = 2$ and $\chi_c(Z, D_{m,[k,k+i]}) \leq \frac{m+1}{2}$. Theorem 2.3 follows from (1) immediately. ■

Theorem 2.4. If $m \geq 2k+2i$ and $1 \leq i \leq k-1$, then $\chi_f(Z, D_{m,[k,k+i]}) = \frac{m+k+1}{2}$.

Proof. We first prove $\alpha(G_{m+k}) = 2$. Let S be a maximum independent set of G_{m+k}. Without loss of generality, assume $0 \in S$. Clearly, $S \setminus \{0\} \subseteq \{k, k + 1, \ldots, k + i\} \cup \{m + 1, m + 2, \ldots, m + k\}$. Note that $m \geq 2k + 2i$ and $i \leq k - 1$, it is easy to check that the subgraph of G_{m+k} induced by $\{k, k + 1, \ldots, k + i\} \cup \{m + 1, m + 2, \ldots, m + k\}$ is a complete graph. Therefore, $\alpha(G_{m+k}) = 2$. By (*), $\chi_f(Z, D_{m,[k,k+i]}) \geq \chi_f(G_{m+k}) \geq \frac{m+k+1}{2}$. Recall that $G(Z, D_{m,[k,k+i]})$ is a subgraph of $G(Z, D_{m,k})$, by Theorem 1.2, Theorem 2.4 holds. ■

Lemma 2.5. [14] If G has a circular chromatic number p/q (where p and q are relatively prime), then $p \leq |V(G)|$, and any (p, q)-coloring c of G is an onto mapping from $V(G)$ to $\{0, 1, \cdots, p - 1\}$.

In order to obtain the counterpart of Theorem 1.3 for $G(Z, D_{m,[k,k+i]})$ with $m \geq 2k + 2i$ and $1 \leq i \leq k - 1$, we first prove the following two lemmas.

Lemma 2.6. Suppose $m \geq 2k + 2i$ and $1 \leq i \leq k - 1$. Let $m + k + 1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers and m' and k' are odd integers. If $1 \leq r \leq s$, then $\chi_c(G_{2m+2k+1}) \in \{\frac{m+k+1}{2} + \frac{1}{3}, \frac{m+k+2}{2}\}$.

Proof. We first show that $\chi_c(G_{2m+2k+1}) > \frac{m+k+1}{2}$. Since $\chi_c(G_{2m+2k+1}) > \chi(G_{2m+2k+1}) - 1$ and $\frac{m+k+1}{2}$ is an integer, it suffices to show that $\chi(G_{2m+2k+1}) > \frac{m+k+1}{2}$. Assume to the contrary that $\chi(G_{2m+2k+1}) \leq \frac{m+k+1}{2}$. Let c be a $\frac{m+k+1}{2}$-coloring of $G_{2m+2k+1}$.

For $0 \leq j \leq m + k + 1$, let H_j denote the subgraph of $G_{2m+2k+1}$ induced by the $m + k + 1$ vertices $\{j, j + 1, \ldots, j + m + k\}$. By the proof of Theorem 2.4, $\alpha(H_j) = 2$ for $j = 0, 1, \ldots, m + k$. Therefore, each of the $\frac{m+k+1}{2}$ colors is used at most, and thus exactly twice in each H_j $(j = 0, 1, \ldots, m + k + 1)$.

It follows that $c(j) = c(j+m+k+1)$ for $j = 0, 1, \ldots, m+k$. We now prove that for each $j \in S = \{0, 1, \ldots, m+k\}$, the only possible vertices in S having the same color as j are $j+k$ and $j-k$, where addition and minus are taken under modulo $m+k+1$. Looking at H_j for any $j = 0, 1, \cdots, m+k$, because $c(j) = c(j+m+k+1)$, the only possible vertices that may be colored by the same color as j are $j+k$ and $m+1+j$. Because $m+1+j = j-k(mod\ m+k+1)$, we conclude that the only possible vertices that can be colored by the same color as j are $j+k$ and $j-k(mod\ m+k+1)$.

Consider the circulant graph $C(m+k+1, k)$ with vertex set S, and in which vertex j is adjacent to vertex j' if and only if $j' \equiv j+k$ *or* $j-k(mod\ \ m+k+1)$. It follows from the discussion in preceding paragraph that two vertices x and y in S have the same color only if xy is an edge of the circulant graph $C(m+k+1, k)$. Since the intersection of each color class with S contains exactly two vertices, the coloring induces a perfect matching of $C(m+k+1, k)$. However, $C(m+k+1, k)$ is the disjoint union of d cycles of length $\frac{m+k+1}{d}$, where $d = gcd(m+k+1, k)$. Since $C(m+k+1, k)$ has a perfect matching, each cycle has an even length. This implies that $r > s$, contrary to the assumption $r \leq s$. Hence $\chi_c(G_{2m+2k+1}) > \frac{m+k+1}{2}$.

Suppose $\chi_c(G_{2m+2k+1}) = \frac{p}{q}$, where p and q are relatively prime. Then, by Lemma 2.5, $p \leq |V(G_{2m+2k+1})| = 2m+2k+2$, and $\frac{p}{q} > \frac{m+k+1}{2}$. If $q \geq 4$, then $p > 2m+2k+2$, a contradiction. Hence $q \leq 3$. Recall that $G(Z, D_{m,[k,k+i]})$ is a subgraph of $G(Z, D_{m,k})$, by Theorem 1.3, $\chi_c(Z, D_{m,[k,k+i]}) \leq \chi_c(Z, D_{m,k}) \leq \frac{m+k+2}{2}$. If $q = 2$, then $\frac{p}{q} = \frac{m+k+2}{2} = \chi_c(G_{2m+2k+1}) = \chi_c(Z, D_{m,[k,k+i]})$. If $q = 3$, then $\frac{m+k+1}{2} < \frac{p}{3} < \frac{m+k+2}{2}$. Thus, $\frac{3(m+k+1)}{2} < p < \frac{3(m+k+1)}{2} + \frac{3}{2}$. Since p is an integer, $p = \frac{3(m+k+1)}{2} + 1$ and $\frac{p}{q} = \chi_c(G_{2m+2k+1}) = \frac{m+k+1}{2} + \frac{1}{3}$. Concluding the above discussion, the only possible values of $\chi_c(G_{2m+2k+1})$ are $\frac{m+k+1}{2} + \frac{1}{3}$ and $\frac{m+k+2}{2}$. This completes the proof of Lemma 2.6. ■

Lemma 2.7. Suppose $m \geq 2k+2i$ and $1 \leq i \leq k-1$. If $m+k+1$ is odd and $gcd(m+k+1, k) \neq 1$, then $\chi_c(G_{2m+2k+1}) \in \{\frac{m+k}{2} + \frac{2}{3}, \frac{m+k}{2} + 1\}$.

Proof. We first prove that $\chi_c(G_{2m+2k+1}) > \frac{m+k+1}{2}$. Since $\alpha(G_{m+k}) = 2$, $\chi_c(G_{2m+2k+1}) \geq \chi_c(G_{m+k}) \geq \frac{m+k+1}{2}$. Suppose $\chi_c(G_{2m+2k+1}) = \frac{m+k+1}{2}$.

Same as in the proof of Lemma 2.6, let H_j denote the subgraph of $G_{2m+2k+1}$ induced by $\{j, j+1, \ldots, j+m+k\}(0 \leq j \leq m+k+1)$. Clearly, $\alpha(H_j) = 2$ and $\chi_c(H_j) \geq \frac{m+k+1}{2}(j = 0, 1, \ldots, m+k+1)$. Let c be a $(m+k+1, 2)$-coloring of $G_{2m+2k+1}$. Then for each $j \in S = \{0, 1, \ldots, m+k\}$, $c(H_j)$ is a $(m+k+1, 2)$-coloring of H_j. Since $m+k+1$ and 2 are relatively prime, by Lemma 2.5, every $(m+k+1, 2)$-coloring of $H_j(0 \leq j \leq m+k+1)$

is onto and hence one-to-one. Consequently, $c(j) = c(j+m+k+1)$ for each $j \in S$. Similar to the proof of Lemma 2.6, it can be proved that, for each $j \in S$, the only possible vertices in S having the colors $c(j)+1$ or $c(j)-1$ are $j+k$ and $j-k$(where addition and minus are taken under modulo $m+k+1$). Define a circulant graph $C(m+k+1,k)$ as in the proof of Lemma 2.6. Since $c(G_{m+k})$ is a $(m+k+1,2)$-coloring of G_{m+k}, there exists an ordering $x_0, x_1, x_2, \ldots, x_{m+k}$ of $V(G_{m+k})$ such that $c(x_i) = i$ for $0 \le i \le m+k$. Therefore , $X = (x_0, x_1, x_2, \ldots, x_{m+k}, x_0)$ is a cycle in $C(m+k+1,k)$. However, since $gcd(m+k+1,k) = d \ne 1$, $C(m+k+1,k)$ is the disjoint union of d cycles of length $\frac{m+k+1}{d}$. This is a contradiction. Thus $\chi_c(G_{2m+2k+1}) > \frac{m+k+1}{2}$.

Suppose $\chi_c(G_{2m+2k+1}) = \frac{p}{q}$, where p and q are relatively prime. Then $p \le 2m+2k+2$. Since $\frac{p}{q} > \frac{m+k+1}{2}$, if $q \ge 4$ then $p > 2m+2k+2$, a contradiction. Thus $q \le 3$. Recall that $G(Z, D_{m,[k,k+i]})$ is a subgraph of $G(Z, D_{m,k})$, by Theorem 1.3, $\chi_c(Z, D_{m,[k,k+i]}) \le \chi_c(Z, D_{m,k}) \le \frac{m+k+2}{2}$. If $q = 2$, then $\frac{p}{q}$ must be $\frac{m+k+2}{2}$. If $q = 3$, then $\frac{m+k+1}{2} < \frac{p}{3} < \frac{m+k+2}{2}$. Thus, $\frac{3(m+k)}{2} + \frac{3}{2} < p < \frac{3(m+k)}{2} + 3$. Since p is an integer, $p = \frac{3(m+k)}{2} + 2$ and $\frac{p}{q} = \chi_c(G_{2m+2k+1}) = \frac{m+k}{2} + \frac{2}{3}$. We conclude that, $\chi_c(G_{2m+2k+1}) \in \{\frac{m+k}{2} + \frac{2}{3}, \frac{m+k+2}{2}\}$. This completes the proof of Lemma 2.7. ■

Theorem 2.8 follows immediately from Theorem 1.3, Theorem 2.4 and Lemmas 2.6, 2.7.

Theorem 2.8. Suppose $m \ge 2k$ and $1 \le i \le k-1$. Let $m+k+1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers and m' and k' are odd integers.

(1) If $1 \le r \le s$, then $\frac{m+k+1}{2} + \frac{1}{3} \le \chi_c(Z, D_{m,[k,k+i]}) \le \frac{m+k+2}{2}$;

(2) If $r = 0$ and $gcd(m+k+1,k) \ne 1$, then $\frac{m+k}{2} + \frac{2}{3} \le \chi_c(Z, D_{m,[k,k+i]}) \le \frac{m+k+2}{2}$;

(3) Otherwise, $\chi_c(Z, D_{m,[k,k+i]}) = \frac{m+k+1}{2}$.

It seems very difficult to determine the exact values of $\chi_c(Z, D_{m,[k,k+i]})$ under the conditions of Theorem 2.8 (1) and (2).

Corollary 2.9. Suppose $1 \le i \le k-1$. Let $m+k+1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers and m' and k' are odd integers. Let $d = gcd(m+k+1, k)$. Then

$$\chi(Z, D_{m,[k,k+i]}) = \begin{cases} k, & \text{if } m < 2k \\ \lceil \frac{m+1}{2} \rceil, & \text{if } 2k \leq m < 2k+2i \\ \lceil \frac{m+k+2}{2} \rceil, & \text{if } m \geq 2k+2i,\ r \leq s, \text{ and } d \neq 1 \\ \frac{m+k+1}{2}, & \text{otherwise} \end{cases}$$

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