On inequivalent Hadamard matrices of order 44

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Abstract

In this paper we construct many Hadamard matrices of order 44 and we use a new efficient algorithm to investigate the lower bound of inequivalent Hadamard matrices of order 44. Using four $(1, -1)$ circulant matrices of order 11 in the Goethals - Seidel array we obtain many new Hadamard matrices of order 44 and we show that there are at least 6018 inequivalent Hadamard matrices for this order. Moreover, we use a known method to investigate the existence of double even self-dual codes $[88, 44, d]$ over $GF(2)$ constructed from these Hadamard matrices.

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1 Introduction

A Hadamard matrix of order n is an $n \times n$ $(1, -1)$-matrix satisfying $HH^T = nI_n$. A Hadamard matrix is normalized if all entries in its first row and column are equal to 1. Two Hadamard matrices are equivalent if one can be transformed into the other by a series of row or column permutations and negations. It is well known that if n is the order of a Hadamard matrix, then n is necessarily 1, 2 or a multiple of 4.

The discussion of Hadamard equivalence is quite difficult, principally because of the lack of a good canonical form. The exact results which have been discovered are as follows : Hadamard matrices of orders less than 16 are unique up to equivalence. There are precisely five equivalence classes of order 16, and three equivalence classes of order 20, see [7, 8]. There are precisely 60 equivalence classes of order 24, see [9, 11]. There are precisely 487 equivalence classes of order 28, see [12, 13]. The classification of Hadamard matrices of orders $n \geq 32$ is still remains an open and difficult problem since an algorithmic approach of an exhaustive search is an NP hard problem. In particular for $n = 32$, Lin, Wallis and Lie [15] found 66104 inequivalent Hadamard matrices of order 32. Extensive results appear in [16] and [17]. Thus the lower bound for inequivalent Hadamard matrices of order 32 is 66104.

There are at least 1036 inequivalent Hadamard matrices of order 36. In fact this number is obtained as follows: Seberry's home page ("http://www.uow.edu.au/~jennie") gives 192 inequivalent Hadamard matrices of order 36. These are supplied by E. Spence (180 matrices) see [19], Z. Janko, (1 matrix of Bush-type) see [10] and V. D. Tonchev (11 matrices) see [20]. Using an efficient algorithm and the Magma software, Georgiou and Koukouvinos [5] found that 172 of their transposes, are inequivalent to these. They also in [5] improved further this bound to 1036 by constructing 672 new Hadamard matrices of order 36.

Lam, Lam and Tonchev [14] showed that the lower bound for inequivalent Hadamard matrices of order 40 is 3.66×10^{11}.

Recently Topalova [21] classified the Hadamard matrices of order 44 with an automorphism of order 7, and found 384 inequivalent Hadamard matrices of this order. In this paper using an efficient algorithm and the Magma software [1] we found that 6 of their transposes, are inequivalent to these. Two more Hadamard matrices were given in Sloane's web page "http://www.research.att.com/~njas/hadamard/" (one is the Williamson type Hadamard matrix and the other is the Paley type Hadamard matrix first given in [18]). In this paper we show that the transposes of these two matrices are inequivalent to all known Hadamard matrices of order 44. Moreover, we further improve this lower bound to 6018 by constructing 5624 new Hadamard matrices.

Before we give a brief description of our algorithm we need the following notations and definitions. Let $A = \{A_j : A_j = \{a_{j1}, a_{j2}, ..., a_{jn}\}, j = 1, \ldots, \ell\}$, be a set of ℓ sequences of length n. The *non-periodic autocorrelation function* $N_A(s)$ of the above sequences is defined as

$$N_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-s} a_{ji} a_{j,i+s}, \qquad s = 0, 1, ..., n-1. \tag{1}$$

If $A_j(z) = a_{j1} + a_{j2}z + \ldots + a_{jn}z^{n-1}$ is the associated polynomial of the sequence A_j, then

$$A(z)A(z^{-1}) = \sum_{j=1}^{\ell}\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ji}a_{jk}z^{i-k} = N_A(0) + \sum_{j=1}^{\ell}\sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}). \tag{2}$$

Given A_ℓ, as above, of length n the *periodic autocorrelation function* $P_A(s)$ is defined, reducing $i+s$ modulo n, as

$$P_A(s) = \sum_{j=1}^{\ell}\sum_{i=1}^{n} a_{ji}a_{j,i+s}, \qquad s = 0,1,...,n-1. \tag{3}$$

For the results of this paper generally PAF is sufficient. However NPAF sequences imply PAF sequences exist.
The following theorem which uses four circulant matrices is very useful in our construction for Hadamard matrices.

Theorem 1 *[3, Theorem 4.49] or [6]. Suppose there exist four circulant matrices A, B, C, D of order n satisfying*

$$AA^T + BB^T + CC^T + DD^T = nI_n$$

Let R be the back diagonal matrix. Then

$$GS = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{pmatrix}$$

is a Hadamard matrix of order $4n$.

Corollary 1 *If there are four sequences A, B, C, D of length n with entries from $\{\pm 1\}$ with zero periodic or non-periodic autocorrelation function, then these sequences can be used as the first rows of circulant matrices which can be used in the Goethals-Seidel array to form a Hadamard matrix of order $4n$.* □

In this paper we use a simple algorithm to find four $(1,-1)$ sequences A,B,C,D of length 11, which have zero PAF, i.e. $P_A(s) + P_B(s) + P_C(s) + P_D(s) = 0$, $s = 1,2,3,4$, and are given in Table 1. From these sequences we can construct the corresponding circulant matrices A, B, C, D of order 11, which are used in theorem 1, for the construction of new inequivalent Hadamard matrices of order 44. The inequivalence of the Hadamard matrices was checked by the aim of an algorithm, which is given in section 2, and by the help of the Magma software [1].

2 The algorithm

The following algorithm was first given in [4]. In the same paper the authors prove that this algorithm can be used as necessary and sufficient criterion to check equivalence of Hadamard matrices. This algorithm has been already used in [5] to investigate the inequivalent Hadamard matrices of order 36.

The *Hamming distance distribution* ($W(x)$) and the *symmetric Hamming distance distribution* ($SW(x)$), of a projection of a Hadamard matrix of order n in k columns, is defined to be

$$W_k(x) = a_0 + a_1x^1 + \ldots + a_kx^k \quad \text{and}$$
$$SW_k(x) = \begin{cases} \sum_{i=0}^{(k-1)/2} (a_i + a_{k-i})x^i, & \text{when k is odd} \\ \sum_{i=0}^{(k-2)/2} (a_i + a_{k-i})x^i + a_{\frac{k}{2}}x^{\frac{k}{2}}, & \text{when k is even} \end{cases}$$

respectively, where a_m is the number describing how many pairs of rows of the projection have distance m.

Example 1 Consider the projections for $k = 3$ and $n = 8$. A Hadamard matrix of order 8 is

$$\begin{array}{rrrrrrrr} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \end{array}$$

Since $k = 3$ the projections are all possible 3-sets of columns. We will just illustrate with the sets of columns 2, 3, 4 and 2, 3, 5.

$$\begin{array}{rrr} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{array} \quad \text{and} \quad \begin{array}{rrr} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{array}$$

We now consider the distance between all pairs of rows of these 8×3 matrices. The first set has distance 3 (4 times), 2 (12 times) and 1 (12 times) so its Hamming distance distribution and its symmetric Hamming distance distribution is

$$W_3(x) = 0 + 12x + 12x^2 + 4x^3, \quad SW_3(x) = 4 + 24x$$

respectively, while the second set has 0 (4 times) and 2 (24 times) so its Hamming distance distribution and its symmetric Hamming distance distribution is

$$W_3(x) = 4 + 24x^2, \quad SW_3(x) = 4 + 24x$$

respectively. □

The Hamming distance distribution $W_k(x)$ is invariant only to permutations of columns or rows, or negations of columns while the symmetric Hamming distance distribution $SW_k(x)$ is invariant to permutations and negations of both rows and columns.

Lemma 1 *Two equivalent projections have the same symmetric Hamming distance distribution.*

Lemma 2 *All projections of two Hadamard matrices H_1, H_2 of order n in $k = 1, 2$ columns are the same (actually these give only one inequivalent projection) even thought the Hadamard matrices are inequivalent.*

Lemma 3 *Let H be a Hadamard matrix of order n. Any two rows of the Hadamard matrix have Hamming distance distribution and symmetric Hamming distance distribution $W_n(x) = SW_n(x) = x^{n/2}$.*

Definition 1 Let H be a Hadamard matrix of order n and P_k a set of k columns of H. We define the *complementary projection* of P_k to be the set of the columns of H which are not contained in P_k. Obviously the complementary projection of P_k consist of $n - k$ columns.

Remark 1 Let H_1, H_2 be two Hadamard matrices of order n. Suppose $r = \{r_1, r_2, \ldots, r_k\}$ and $p = \{p_1, p_2, \ldots, p_k\}$ be two rows of a projection of H_1 and $q = \{q_1, q_2, \ldots, q_k\}$ and $s = \{s_1, s_2, \ldots, s_k\}$ be two rows of a projection of H_2. Then $SW(x)$ of rows r, p is equal to $SW(x)$ of rows q, s if and only if the symmetric Hamming distance distribution of the corresponding rows of their complementary projections is equal.

Example 2 The complementary projections of the projections given in example 1 are

$$\begin{array}{rrrrr} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \end{array} \quad \text{and} \quad \begin{array}{rrrrr} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{array}$$

with symmetric Hamming distance distribution $SW_{8-3}(x) = SW_5(x) = 4 + 24x$.

From Lemmas 1, 2 and 3 it is obvious that:

Corollary 2 *All projections of two Hadamard matrices H_1, H_2 of order n in $k = 1, 2$ and $k = n$ columns have the same symmetric Hamming distance distribution.*

Using Remark 1 and the above lemmas we can conclude:

Corollary 3 *Let H_1, H_2 be two Hadamard matrices of order n. We need only to check the symmetric Hamming distance distribution of projections for $k = 3, 4, \dots, n/2$ because if these have the same symmetric Hamming distance distribution, then the corresponding complementary projections will have the same symmetric Hamming distance distribution as well.*

The Symmetric Hamming distance distribution algorithm:

(i) Set $k = 3$.

(ii) Find all projections for each Hadamard matrix of a given order n and k columns by taking all possible k columns of the entire $n \times n$ Hadamard matrix. These are $\binom{n}{k}$ projections in total.

(iii) In the projections found in step (ii) calculate the symmetric Hamming distance distributions for any two rows of the projection. These are $\binom{n}{2}$ symmetric Hamming distance distributions and save different symmetric Hamming distance distributions and how many times each of them appear.

(iv) Check if the set of all different symmetric Hamming distance distributions of the first Hadamard matrix is the same with the set of all different symmetric Hamming distance distribution of the second Hadamard matrix.

(v) If the answer in step (iv) is false, then stop and say that these two Hadamard matrices are inequivalent, otherwise increase k by 1.

(vi) If now $k < n/2$ then go to step (ii) and continue, otherwise stop and say that this algorithm can not decide for the equivalence of these Hadamard matrices.

This algorithm can perform a fast search and provide partial results in all cases of Hadamard matrices. In the case of Hadamard matrices of order 44 the lower bound is $k \geq 6$. But to obtain some partial results faster, smaller values of k are possible.

We tested the algorithm in the already known 384 Hadamard matrices given by Topalova [21]. The results are as follows:

- For $k = 3$ we obtain only one different Hamming distance distribution and thus we can distinct only one from the 384 inequivalent matrices.
- For $k = 4$ we obtain 289 different Hamming distance distributions and thus we can distinct only 289 from the 384 inequivalent matrices.
- Finally for $k = 5$ we obtain 384 different Hamming distance distributions and thus we can distinct all the 384 inequivalent matrices.

As we can see even for a k which is less than the theoretical lower bound, this algorithm can give us some results. If we are only interesting in finding few inequivalent Hadamard matrices, from a large set of Hadamard matrices, this algorithm seems to be quite fast but when we wish to obtain all the inequivalent Hadamard matrices, from a large set of Hadamard matrices, this algorithm fails.

Since we are interesting in improving the lower bound of the known Hadamard matrices of order 44, we use a combination of this algorithm and the Magma software to obtain the results given in the next section. We totally searched 5000000 Hadamard matrices constructed from four circulant matrices in the Goethals-Seidel array as indicated by corollary 1.

3 The new results

To save space, in this section we present 1000 of the 5624 new Hadamard matrices of order 44, we have found. In Table 1 we give the first row of the corresponding circulant $(1, -1)$ matrices of order 11, in Hexadecimal (Hex) form, which can be used in the Goethals-Seidel array to obtain the 1000 of the 6018 new inequivalent Hadamard matrices of order 44. All 6018 inequivalent Hadamard matrices of order 44 are available on request.

In the next table we present the results of all four sequences in Hex form. To convert to (1,-1) sequences we transform each digit from Hex form into four digits in binary form. Thus we obtain a binary sequence $E = \{e_1, e_2, \ldots, e_{44}\}$ of 44 binary digits. Then we replace 0 by -1 and set $A = \{e_1, \ldots, e_{11}\}$, $B = \{e_{12}, \ldots, e_{22}\}$, $C = \{e_{23}, \ldots, e_{33}\}$, $D = \{e_{34}, \ldots, e_{44}\}$. These are four (1,-1) sequences of lenght 11 with zero PAF and can be used in Theorem 1 to obtain the desirable Hadamard matrices of order 44.

3FA7239A095	5EC38065F6A	6F47F865F6A	6F5C0065F6A	4D1E3C65F6A
4D00E065F6A	59100C65F6A	590FF065F6A	788FD469189	788A8069189
771ABC69189	77150069189	8C0B43EA197	8C12FBEA197	C41A5FEA197
C41A0BEA197	3EE5F7EA197	3EE5A3EA197	7CFA5FEA197	7CFA0BEA197
3EE6A27434F	3EFEA67434F	3EE57E7434F	3EE5627434F	7CF5827434F
7CFCAE7434F	7CFA9E7434F	7CFA827434F	046B067434F	04706A7434F
047F527434F	047F2A7434F	106AC27434F	10706A7434F	107E567434F
107D4E7434F	5A0F133B3AF	5A0E1F3B3AF	E96E1F3B3AF	E97C473B3AF
F1786B3B3AF	F1794F3B3AF	8161AF3B3AF	817E533B3AF	F4761B3B3AF
F47CA73B3AF	A0786B3B3AF	A06F2B3B3AF	57CE20C826D	57D8F4C826D
7D4E20C826D	7D5EE0C826D	779A9CC826D	779A80C826D	779828C826D
7797C8C826D	7B9054C826D	7B80D4C826D	7B9D4CC826D	7B9CACC826D
471068C826D	471054C826D	470FA8C826D	470F94C826D	711A80C826D
7115F0C826D	711580C826D	71153CC826D	5601C4C826D	561EE0C826D
6A0E3CC826D	6A0E20C826D	6FC6F5E68AE	6FC5EDE68AE	6FC4D1E68AE
6FC42DE68AE	7EC6F5E68AE	7EC685E68AE	7EC5EDE68AE	7EC591E68AE
4C06F5E68AE	4C05EDE68AE	4C0165E68AE	4C0135E68AE	6406F5E68AE
6405EDE68AE	640165E68AE	640135E68AE	4D1B07ACBD7	4D1C4FACBD7
4D191FACBD7	4D1C1BACBD7	59113FACBD7	591077ACBD7	59106FACBD7
590F93ACBD7	6C03B7ACBD7	6C03A7ACBD7	6C02DFACBD7	6C025FACBD7
74049FACBD7	74036FACBD7	E4EE93ACBD7	E4EE27ACBD7	E4F6E3ACBD7
E4E3B7ACBD7	E8EE4BACBD7	E8ED8FACBD7	D36F93ACBD7	D36F8BACBD7
D36DC3ACBD7	D36C1FACBD7	D9791FACBD7	D9789FACBD7	D9783BACBD7
D97837ACBD7	F16727ACBD7	F166DBACBD7	817C93ACBD7	817B63ACBD7
88BE754D503	88B1814D503	A22C114D503	E5C2314D503	E5CEF94D503
E9D8214D503	E9D7CD4D503	88DE754D503	7CC2F7D66FA	7CC177D66FA
5F8677D66FA	5F85CFD66FA	5F8337D66FA	5F82CFD66FA	7E8677D66FA
7E85CFD66FA	7E8337D66FA	7E82CFD66FA	5E8CFBD66FA	5E8C1BD66FA
5D1983D66FA	5D133FD66FA	6602F7D66FA	660177D66FA	5C0677D66FA
5C05CFD66FA	5C0337D66FA	5C02CFD66FA	740677D66FA	7405CFD66FA
740337D66FA	7402CFD66FA	7B4F1225130	7B570E25130	0E453E25130
0E41AE25130	1A50EE25130	1A4F1225130	58423E25130	585DC225130
139E5225130	1381AE25130	8DF61A25130	8DF29E25130	B1E53E25130
B1E1AE25130	96F70E25130	96EF1225130	C23BA28242C	C23A268242C
C23D8A8242C	C2322E8242C	C431768242C	C431728242C	C4346E8242C
C433A28242C	E3C3768242C	E3C2EE8242C	E3C2E68242C	E3C2768242C
F1D46E8242C	F1D62E8242C	F1D13A8242C	F1D22E8242C	7BD92112E17
61123112E17	6BDCCDA0456	6BD065A0456	6DC709A0456	6DCE3DA0456
76C11DA0456	76D8F5A0456	7ADCCDA0456	7AC0CDA0456	779921A0456
7796CDA0456	7B9921A0456	7B9B39A0456	499DC5A0456	498471A0456
6498F5A0456	648239A0456	471191A0456	470D9DA0456	53067DA0456
5300CDA0456	65199DA0456	651831A0456	711921A0456	711675A0456
6FA57CD1907	6FA0D4D1907	7DB418D1907	7DB7C8D1907	2826A0D1907
282BE4D1907	48257CD1907	4820D4D1907	5F5BACD1907	5F5420D1907
0B55ECD1907	0B5420D1907	4A11771DC26	4A113B1DC26	520BB31DC26
5207471DC26	5806AF1DC26	58054F1DC26	6806AF1DC26	68054F1DC26
E5E6AF1DC26	E5E54F1DC26	E9F3571DC26	E9F2A71DC26	D6E2EF1DC26
D6E2771DC26	DAE2771DC26	DAFB171DC26	7B63D622839	7B62BE22839
7B61D622839	7B615E22839	6FA3B622839	6FA3A622839	6FA2DE22839
6FA25E22839	7DBD2222839	7DA25E22839	7DBB4622839	7DBC5A22839
243C5622839	243C2A22839	243A8E22839	243A8622839	4823B622839
4823A622839	4822DE22839	48225E22839	095A4622839	0956E222839
095B4622839	0944BA22839	5D8BE94C3BE	5D8B414C3BE	6E90594C3BE
4E97D14C3BE	4E9A094C3BE	6E9D2D4C3BE	5C85F54C3BE	5C80B54C3BE
4D02F54C3BE	4D01754C3BE	5911794C3BE	5910B94C3BE	A6FD154C3BE
A6FD094C3BE	B2E2F54C3BE	B2E1754C3BE	4515848CB81	45129C8CB81
5115848CB81	5114F08CB81	75B8FA03334	57C19603334	57C14E03334
57DEB203334	57DE6A03334	6BDF5203334	6BD82A03334	7AC7D603334
7AC0AE03334	7D4D7A03334	7D4CF603334	7D4CA203334	7D4C2A03334
529D8E03334	5307D603334	5300AE03334	65158203334	6514FA03334
5607AE03334	5606BE03334	56019603334	56014E03334	6A0D7A03334
6A0CF603334	6A0CA203334	6A0C2A03334	8429748CB81	8428AC8CB81
F3C5D48CB81	F3C2B48CB81	EF5AC08CB81	EF594C8CB81	F75E508CB81
F75AC08CB81	F2BD220C17B	F2ADC60C17B	82A91E0C17B	82BC5A0C17B
D0B5E60C17B	D0B4320C17B	F122960C17B	F13AD60C17B	F225160C17B
F236BA0C17B	B82B5E0C17B	B822960C17B	D8314A0C17B	D82EB60C17B
A823A60C17B	A822DE0C17B	A7C3A60C17B	A7C2DE0C17B	8BD0CA0C17B

Table 1: First rows of circulant matrices of order 11 (in Hex form).

8BCD7A0C17B	75BB8D019B1	779451019B1	779759019B1	7B9289019B1
7B9D69019B1	5298F5019B1	471149019B1	470EB5019B1	7114A1019B1
7115AD019B1	70E29503268	70FAD503268	67B5E103268	67BC2903268
63B41903268	63A7D503268	71BF5103268	71A0D503268	4734F903268
472A0D03268	43385503268	43378903268	4E341903268	4E33E903268
4C38AD03268	4C385503268	6BC2BD03268	6BC15D03268	23C29503268
23D5AD03268	4A136F8BDC6	4A11938BDC6	520DBB8BDC6	5208CB8BDC6
BEA6DF8BDC6	BEA3278BDC6	F6B1938BDC6	F6AE6F8BDC6	91053D9EA07
9101AD9EA07	8C06D59EA07	8C056D9EA07	8C04D59EA07	8C04AD9EA07
9405CD9EA07	9402CD9EA07	A405CD9EA07	A402CD9EA07	C41A999EA07
C41A919EA07	C419599EA07	C419299EA07	3EE6D59EA07	3EE6A59EA07
3EE56D9EA07	3EE5659EA07	7CF9299EA07	7CFB299EA07	7CFA999EA07
7CFA919EA07	5F65CD9EA07	5F62CD9EA07	776B0D9EA07	7769E59EA07
7D65CD9EA07	7D62CD9EA07	6568C19EA07	2564619EA07	2E75BD9EA07
2E74819EA07	1676F99EA07	1668299EA07	3A74819EA07	3A6DF59EA07
6A6C119EA07	5261199EA07	346B7D9EA07	3461499EA07	5BC777B38AE
5BC22FB38AE	6BC5EFB38AE	6BC137B38AE	5DC1A7B38AE	7AC5EFB38AE
7AC137B38AE	77486BB38AE	7B5623B38AE	7B4EEBB38AE	5881A7B38AE
6881A7B38AE	4B1D3BB38AE	4B022FB38AE	5305EFB38AE	530137B38AE
651A13B38AE	65165FB38AE	6913BBB38AE	691117B38AE	88B85360F6B
8A25EF60F6B	8A216760F6B	E9D85360F6B	88C14F60F6B	E745EF60F6B
E7416760F6B	EE414F60F6B	8346777C20A	8342CF7C20A	AF86777C20A
AF82CF7C20A	D799177C20A	D79D8B7C20A	BB83B77C20A	BB83A77C20A
BB82DF7C20A	BB825F7C20A	8386AF7C20A	83854F7C20A	F591737C20A
F59D8B7C20A	F985D77C20A	F985777C20A	F982D77C20A	F982B77C20A
EE83B77C20A	EE83A77C20A	EE82DF7C20A	EE825F7C20A	AE999F7C20A
AE98337C20A	FA99D37C20A	FA96637C20A	BA867F7C20A	BA80CF7C20A
DF13577C20A	DF12A77C20A	B7111B7C20A	97111B7C20A	BB01977C20A
FB06AF7C20A	FB054F7C20A	BB1ACF7C20A	9D14337C20A	DD07AF7C20A
DD01977C20A	AD06637C20A	ED02377C20A	9D1EB37C20A	AD1F337C20A
B510677C20A	B50F9B7C20A	B908677C20A	B907AF7C20A	E9111B7C20A
D11B8B7C20A	D11A3B7C20A	D1123B7C20A	D1112F7C20A	B103B77C20A
B103A77C20A	B102DF7C20A	B1025F7C20A	A619177C20A	A60BB37C20A
CA0BB37C20A	CA0B937C20A	9C06AF7C20A	9C054F7C20A	AC06777C20A
AC02CF7C20A	CC15D37C20A	CC15A37C20A	CC152F7C20A	CC15177C20A
CF47AEEC01A	8653D6EC01A	9847AEEC01A	D79EB2EC01A	6DC6BF158B0
76C6BF158B0	49979B158B0	649CD7158B0	699B2E117E2	698136117E2
4981AE117E2	499E52117E2	64853E117E2	6481AE117E2	927852117E2
B1B0A6117E2	8CA5EE117E2	8CA136117E2	B7253E117E2	B721AE117E2
9725BE117E2	97212E117E2	A73A12117E2	A737B2117E2	DB253E117E2
DB21AE117E2	CB2F66117E2	CB3A12117E2	8B2B7A117E2	8B2962117E2
A33096117E2	D332DE117E2	A33BCA117E2	D33096117E2	77A9B670641
77A9AA70641	7BA6D670641	7BA4D670641	36A14670641	36BAEE70641
35285270641	3537AE70641	143A9A70641	143A9270641	4426D670641
4424D670641	56CBBE70641	56C14670641	4AD7AE70641	4AD0A270641
2BF2B4C8443	2BF268C8443	2F62B4C8443	056AD8C8443	056A4CC8443
2162B4C8443	37A2B4C8443	2BBA10C8443	15A4D0C8443	09B454C8443
41BA90C8443	41B958C8443	3EA56CC8443	3EA4D4C8443	2EB098C8443
16A684C8443	E9D5A4460FD	9EC6AC460FD	9EC54C460FD	D8CB5C460FD
D8C294460FD	88C6B4460FD	90C6AC460FD	90C54C460FD	9B59E8460FD
935D64460FD	CD41A4460FD	A64D20460FD	865368460FD	8652B4460FD
865268460FD	865254460FD	B247AC460FD	C246AC460FD	C2454C460FD
C454D4460FD	9846D4460FD	9846A4460FD	98456C460FD	984564460FD
D786D4460FD	D7856C460FD	D784D4460FD	D784AC460FD	B786AC460FD
B7854C460FD	BB86B4460FD	938B5C460FD	938294460FD	AD8F58460FD
F58AD8460FD	F58AC8460FD	F58A6C460FD	F58A4C460FD	5EDE3C65F6A
3FA7DB9A095	37A1C39A095	37BC7F9A095	2FFB1F9A095	2FFC839A095
2F78079A095	2F63FF9A095	1F7F2F9A095	1BF7F39A095	0B21C39A095
0B3F1F9A095	09AFF39A095	09B00F9A095	0737F39A095	04E5FF9A095
01A3939A095	01BF639A095	017F639A095	016E439A095	3FA23DDB2EB
3FB0EDDB2EB	2FEE1DDB2EB	2FE8F1DB2EB	11E0B1DB2EB	11FF4DDB2EB
0EF601DB2EB	0EFA7DDB2EB	01BDC1DB2EB	01BE21DB2EB	017DC1DB2EB
017E21DB2EB	C207E852CD8	C2181452CD8	C21A0452CD8	C202FC52CD8
9EE5C052CD8	9EE5F852CD8	9EE7E852CD8	9EE81C52CD8	1DE2E159921
1DE2FD59921	1DE3A159921	1DE3F559921	08E5C159921	08E5F959921
08E81D59921	08E8FD59921	27167D6F395	1F767D6F395	1C87ED6F395
1BE7ED6F395	C40FB1E9D68	9FB10DE9D68	9FB1DDE9D68	9F64E1E9D68
9F71F5E9D68	9DE139E9D68	9DE3EDE9D68	9C08C1E9D68	9BFC75E9D68
9BE711E9D68	980F1DE9D68	98110DE9D68	907181E9D68	90310DE9D68
9031DDE9D68	8FA08DE9D68	8F6819E9D68	8EF901E9D68	8E0819E9D68
8DE8C1E9D68	8C0C7DE9D68	8C0271E9D68	886819E9D68	8829C1E9D68
0F27DE84D12	0F28C284D12	0DE7DE84D12	0DE8C284D12	09EFBA84D12
09F18284D12	086C1E84D12	063E1284D12	063E1A84D12	063E4284D12
063EC284D12	047D8684D12	047E1284D12	047E1A84D12	047E4284D12
00F96284D12	00FA6284D12	00FCB284D12	00FCD284D12	7E80CD34F5E
7E933D34F5E	7E266134F5E	7E27CD34F5E	74019934F5E	74199D34F5E

Table 1: (cont.)

7020CD34F5E	703F3134F5E	67CE0534F5E	67D17D34F5E	6617E134F5E
66005D34F5E	27EF1FDD6CA	1F704FDD6CA	1F7C6FDD6CA	1F7D8FDD6CA
1F641FDD6CA	1EE6C3DD6CA	1EE7E7DD6CA	1DFE4FDD6CA	1DE363DD6CA
1BF63FDD6CA	1BE1CBDD6CA	1BF7C7DD6CA	1BF827DD6CA	0E3037DD6CA
0E327FDD6CA	08E6C3DD6CA	08E9FBDD6CA	0729C3DD6CA	072E43DD6CA
0731BFDD6CA	0723EFDD6CA	04F1BFDD6CA	04F20FDD6CA	04E13BDD6CA
04F63FDD6CA	036EF3DD6CA	F9CF0E84D12	F9D21E84D12	F8E65A84D12
F8E69A84D12	F8E72E84D12	F8E73684D12	F730F684D12	F73C2684D12
F72F2284D12	F73C3684D12	F6173E84D12	F6030684D12	F3CC1E84D12
F3A3DA84D12	F3B09E84D12	1FE2954C64E	1FFD694C64E	00F4A14C64E
00FD694C64E	1FF512670D2	1FFD5A670D2	1FBEEA670D2	1FA286670D2
17EA12670D2	17FEEA670D2	03A452670D2	03BEEA670D2	02FEEA670D2
02E116670D2	00F512670D2	00FD5A670D2	57E3D99459D	57E43D9459D
57E6F19459D	57E7859459D	3FA3A99459D	3FA3D59459D	3FAAE19459D
3FAAF19459D	2FF1E99459D	2FF0E99459D	2FEF159459D	2FE5719459D
1FFA299459D	1EBF959459D	1EA1A19459D	1D4BF99459D	1D480D9459D
1C0AD19459D	1A1A8D9459D	1A0AF19459D	1A015D9459D	1A01D59459D
160A1D9459D	160A3D9459D	160BC59459D	15E1A19459D	160E159459D
15FF959459D	15C5FD9459D	15C0D19459D	15186D9459D	1518799459D
1502799459D	1508799459D	0925C211D59	0928FE11D59	01B90A11D59
01BE9A11D59	01790A11D59	017E9A11D59	1FFAA396E76	11F02B96E76
11F57F96E76	0EE54396E76	0EEAFF96E76	02BE2396E76	02A1DF96E76
00FAA396E76	E5E42E11D59	E5F37A11D59	E4EA4211D59	E4FADE11D59
DB617211D59	DB6FC611D59	DB6FD211D59	DB681E11D59	D81BEA11D59
D8120A11D59	D3E42E11D59	D3FA6E11D59	03382918F25	033F2918F25
02782918F25	027F2918F25	6EE3F5679E5	6EE5C1679E5	11347D679E5
1120B9679E5	6EE87F9D769	6EE1EB9D769	3FBD139D769	3FA3779D769
3FA5CB9D769	3FA5DB9D769	3F7A179D769	3F7A2F9D769	37FA179D769
37FA2F9D769	2FFC8B9D769	2FFD139D769	2FFD1B9D769	2FFD8B9D769
1BA05B9D769	1BB2FF9D769	177CBF9D769	1721639D769	1725FF9D769
1762C39D769	13B6039D769	13B97F9D769	1130FB9D769	112F439D769
033D0B9D769	033E8B9D769	027D0B9D769	027E8B9D769	01BC8B9D769
01BD139D769	01BD1B9D769	01BD8B9D769	017C8B9D769	017D139D769
017D1B9D769	017D8B9D769	FD982918F25	FD9F2918F25	FCD95D18F25
FCC0AD18F25	8641A015E68	85340415E68	8314BC15E68	830D0415E68
829F6015E68	82841C15E68	825F6015E68	8240E415E68	8225A015E68
8225F415E68	7AC03415E68	77DA0815E68	77DA5C15E68	73E5A015E68
73E5F415E68	6FA83815E68	6FBF6015E68	6BF07015E68	6BF63C15E68
6BC03415E68	C8FEB534451	C8E52134451	C7F52534451	C7F6A534451
C7F92934451	C7F95934451	C4FF5934451	C4E90534451	C026D534451
C02AD934451	C024AD34451	C024D534451	B83F5934451	B8252134451
B07ADD34451	B0652134451	1FBA42446B8	1FBED2446B8	17FA42446B8
17FED2446B8	03BA42446B8	03BED2446B8	02FA42446B8	02FED2446B8
77D95D8BCB0	77C1598BCB0	77DA818BCB0	77DA9D8BCB0	73EA7D8BCB0
73E3518BCB0	73E5618BCB0	73E57D8BCB0	73ABF98BCB0	720F5D8BCB0
7208518BCB0	7055DD8BCB0	7045118BCB0	6F97798BCB0	6F84298BCB0
63F4418BCB0	63F7AD8BCB0	6292FD8BCB0	6214F98BCB0	620F958BCB0
6200AD8BCB0	6200D58BCB0	60953D8BCB0	608F958BCB0	6081A98BCB0
6082B18BCB0	3BB17E5AEF9	3BB7025AEF9	2EF8165AEF9	2EF8BE5AEF9
11A5C25AEF9	11A5FA5AEF9	1165C25AEF9	117C5E5AEF9	C026AE984D0
C02656984D0	AF63B2984D0	AF647E984D0	AF64FA984D0	AF66E2984D0
AEF23E984D0	AEEF8A984D0	AEE0DE984D0	AEE762984D0	A213E2984D0
A2160E984D0	A207CA984D0	A21836984D0	A17E76984D0	A16312984D0
A1027E984D0	A11B06984D0	A11C8E984D0	A10DC2984D0	9FB0AE984D0
9FB15E984D0	9BE15E984D0	9BFC56984D0	9810AE984D0	98115E984D0
9030AE984D0	90315E984D0	8FF532984D0	8FF66A984D0	8BC77E984D0
8BC832984D0	0A287ACBB44	0A2C3ECBB44	08A7B2CBB44	08A87ACBB44

Table 1: (cont.)

We will explain the notation in Table 1 by the next example.

Example 3 The first solution in Table 1 is $3FA7239A095$. This is a sequence of length 11 with digits from Hexadecimal arithmetic system. We convert each Hexadecimal digit into four binary digits. Thus we obtain 0011 1111 1010 0111 0010 0011 1001 1010 0000 1001 0101. Then we replace zero by -1 and we have $E = \{-1,-1,1,1,1,1,1,1,1,-1,1,-1,-1,1,1,1,-1,-1,1,-1,-1,-1,1,1,1,-1,-1,1,1,-1,1,-1,-1,-1,-1,-1,1,-1,-1,1,-1,1,-1,1\}$. Now set $A = \{e_1,\dots,e_{11}\} = \{-1,-1,1,1,1,1,1,1,1,-1,1\}$, $B = \{e_{12},\dots,e_{22}\} = \{-1,-1,1,1,1,-1,-1,1,-1,-1,-1\}$, $C =$

$\{e_{23}, \ldots, e_{33}\} = \{1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1\}$, $D = \{e_{34}, \ldots, e_{44}\} = \{-1, -1, -1, 1, -1, -1, 1, -1, 1, -1, 1\}$. These are four $(1, -1)$ sequences of order 11 with zero PAF and can be used in Theorem 1 to obtain the desirable Hadamard matrix of order 44.

4 Self-dual codes of length 88

It is known, [22, pp. 279-280] that every Hadamard matrix H of order $4t$ is associated in a natural way with a SBIBD with parameters $(4t-1, 2t-1, t-1)$, and with its complement, a $(4t-1, 2t, t)$ configuration. Let A be the icidence matrix of the dual SBIBD with parameters $(43, 22, 11)$, which is obtained from a Hadamard matrix of order 44.

Let $A^+ = \begin{pmatrix} A & e^t \\ e & 0 \end{pmatrix}$, where e is the all-one vector of dimension 43. A generator matrix of a self-dual code of length 88 can be obtained as $(A^+ \; I_{44})$.

There are three inequivalent extremal binary linear self-dual codes of length 88 given in [2] and [21].

By searching all 6018 known matrices of order 44 we obtain the following results.

Binary double even self-dual codes		
Hamming distance 8	Hamming distance 12	Hamming distance 16
3296	2717	5

Table 2: Binary double even self-dual codes from Hadamard matrices of order 44

These 5 extremal binary self-dual codes [88,44,16] we have found can be constructed using the following Hadamard matrices. The first is the Hadamard matrix given by Paley [18] and the other two are the Hadamard matrices numbered 170 and 246 given by Topalova [21]. In this paper we show that the tanspose of the matrix 246 given by Topalova [21] and the tanspose of the Paley type Hadamard matrix, found in [18] and in Sloane's home page "http://www.research.att.com/~njas/hadamard/", also give extremal binary self-dual codes with parameters [88,44,16].

References

[1] J. J. Cannon and C. Playoust, *An Introduction to Algebraic programming with Magma*, University of Sydney, 1996.

[2] S. Dougherty, T. A. Gulliver, and M. Harada, Extremal binary self-dual codes, *IEEE Trans. Inf. Theory*, 43 (1997), 2036–2047.

[3] A.V.Geramita, and J.Seberry, *Orthogonal designs: Quadratic forms and Hadamard matrices*, Marcel Dekker, New York-Basel, 1979.

[4] S. Georgiou and C.Koukouvinos, On equivalence of Hadamard matrices and projection properties, *Ars Combin.*, (to appear).

[5] S. Georgiou and C.Koukouvinos, On inequivalent Hadamard matrices of order 36, *Ars Combin.*, (to appear).

[6] J. M. Goethals and J. J. Seidel, A skew Hadamard matrix of order 36, *J. Austral. Math. Soc.*, 11 (1970), 343–344.

[7] M. Hall Jr., Hadamard matrices of order 16, *JPL Research Summary* No. 36-10, Vol. 1 (1961), 21–26.

[8] M. Hall Jr., Hadamard matrices of order 20, *JPL Technical Report* No. 32-76, Vol.1 (1965).

[9] N. Ito, J. S. Leon and J. Q. Longyear, Classification of $3-(24,12,5)$ designs and 24-dimensional Hadamard matrices, *J. Combin. Theory Ser. A*, 31 (1981), 66–93.

[10] Z. Janko, The existence of a Bush-type Hadamard matrix of order 36 and two new infinite classes of symmetric designs, *J. Combin. Theory Ser. A*, 95 (2001), 360–364.

[11] H. Kimura, New Hadamard matrices of order 24, *Graphs Combin.*, 5 (1989), 236–242.

[12] H. Kimura, Classification of Hadamard matrices of order 28 with Hall sets, *Discrete Math.*, 128 (1994), 257–268.

[13] H. Kimura, Classification of Hadamard matrices of order 28, *Discrete Math.*, 133 (1994), 171–180.

[14] C. Lam, S. Lam and V. D. Tonchev, Bounds on the number of affine, symmetric, and Hadamard designs and matrices, *J. Combin. Theory Ser. A*, 92 (2000), 186–196.

[15] C. Lin, W. D. Wallis, Z. Lie, Equivalence classes of Hadamard matrices of order 32, *Congressus Numerantium*, 95 (1993), 179–182.

[16] C. Lin, W. D. Wallis, Z. Lie, Hadamard matrices of order 32, Preprint #92-20, Department of Mathematical Science, University of Nevada, Las Vegas, Nevada.

[17] C. Lin, W. D. Wallis, Z. Lie, Hadamard matrices of order 32 II, Preprint #93-05, Department of Mathematical Science, University of Nevada, Las Vegas, Nevada.

[18] R. E. A. C. Paley, On orthogonal matrices, *J. Math. Phys.*, MIT 12 (1933), 311–320.

[19] E. Spence, Regular two-graphs on 36 vertices, *Linear Alg. Appl.*, 226-228 (1995), 459–497.

[20] V. D. Tonchev, Hadamard matrices of order 36 with automorphism of order 17, *Nagoya Math. J.*, 104 (1986), 163–174.

[21] S. Topalova, Hadamard matrices of order 44 with automorphisms of order 7, *Discrete Math.*, (to appear).

[22] W. D. Wallis, A. P. Street, and J. Seberry Wallis, *Combinatorics: Room squares, sum-free sets and Hadamard matrices*, Lecture Notes in Mathematics, Vol. 292, Springer-Verlag, Berlin, Heidelberg, New York, 1972.