Sphere-of-Influence Graphs on a Sphere

Ewa Kubicka and Grzegorz Kubicki
University of Louisville

Abstract. Given a collection of points in the plane, a circle is drawn around each point with radius equal to the smallest distance from that point to any other in the collection. The sphere-of-influence graph is the intersection graph of the open balls given by these circles. Any graph isomorphic to such a graph is a SIG realizable in a plane. Similarly, one can define a SIG realizable on a sphere by selecting a collection of points on a sphere. We show that K_9 is realizable as a SIG on a sphere and that the family of graphs realizable as SIGs on a sphere is at least as large as the family of SIGs in the plane.

1. Introduction

There are several families of graphs which can be associated with a given set of points in the euclidean plane. They capture some of the perceptual relevance of the original set of points and they are known as "proximity graphs". Toussaint [10-12] defined two families of proximity graphs as follows. Let X be a set of points in the plane. To each point of X assign an open ball centered at that point of radius equal to the smallest distance from that point to any other point of X. The sphere-of-influence graph $G(X)$ has X as the vertex set and two vertices from $G(X)$ are adjacent if their open balls intersect. We are going to use the abbreviation "SIG" for sphere-of-influence graphs. Similarly, closed sphere-of-influence graphs (or CSIGs) are defined like SIGs except that the balls assigned to points are closed. For all results, problems and questions discussed for SIGs there are corresponding ones formulated for CSIGs.

Which graphs are SIGs and which graphs are not ? The answer to this fundamental question is unknown and a full characterization seems to be difficult. The main obstacle is the non-hereditary property of sphere-of-influence graphs, namely, the fact that an induced subgraph of a SIG need not be a SIG (see [2], for example). Therefore, current research in this area is concentrated on obtaining partial information on sphere-of-influence graphs.

The characterization of trees that are sphere-of-influence graphs was done by Jacobson, Lipman and McMorris [3].

Theorem A. A tree is
(a) a CSIG if and only if it has a perfect matching,
(b) a SIG if and only if it has a $\{K_2, P_3\}$-factor.

Even the very special case of the fundamental problem for complete graphs is open. A construction of Malnic and Mohar [6] and Scheinerman in [2] shows that K_8 is a SIG. There is a strong belief that we cannot do better in the plane.

K_9 - conjecture. The complete graph K_9 is neither a SIG nor a CSIG.

The construction for K_8 together with the result of Kézdy and Kubicki [4] that the complete graph K_{12} is not a CSIG represent the current state of knowledge concerning the K_9-conjecture.

Edge density and clique size of SIGs were examined by Avis and Horton [1] and by Michael and Quint [7]. Lipman [5] proved that every SIG has a realization in which every vertex is an integer lattice point. Michael and Quint examined sphere-of-influence graphs in general metric spaces [8].

A survey of main results and conjectures in this area together with a complete bibliography is given by Michael and Quint [9].

The organization of this paper is the following. After introducing some terminology, notation, and concepts in Section 2, we present a realization of K_9 on the sphere in Section 3. The proof showing that every SIG in the plane is also a SIG on the sphere uses stereographic projection and is presented in Section 4. In Section 5 we shortly address an open problem, namely the existence of graphs that are SIGs on the sphere but not SIGs in the plane.

2. Terminology and notation

A graph G is a SIG if there exists a set X of points in the plane such that $G(X)$ is its sphere-of-influence graph. We say that X is a *realization* of G. We denote the vertices of G by $v_1, v_2, \ldots, v_n$, the corresponding points in the realization X of G by $x_1, x_2, \ldots, x_n$, the circles centered at these points by C_1, $C_2, \ldots, C_n$ and the open disks (balls) bounded by these circles by $B_1, B_2, \ldots, B_n$. As an example, in Figure 1, a realization of the graph C_5 is presented.

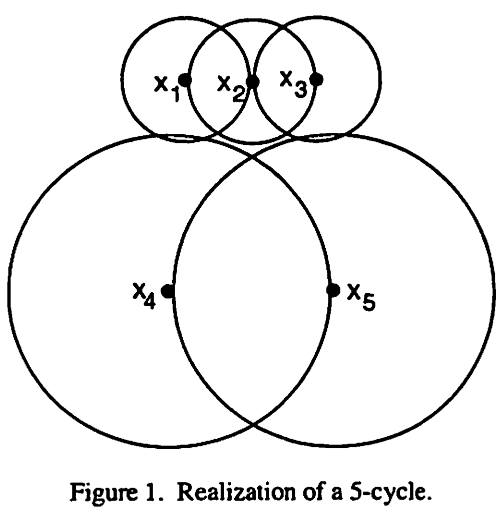


Figure 1. Realization of a 5-cycle.

In the proof of the main theorem in Section 4, we will use some concepts introduced by Lipman [5]. If x and y are two points in the realization X, then y *defines* x whenever y is a nearest point to x. In addition, we say that y *uniquely defines* x

whenever y is the only such point. According to [5], the realization X *has no accidental tangencies* if the following condition is fulfilled: two circles of X are tangent if and only if the point of tangency uniquely defines their centers. The realization of a 5-cycle in Figure 1 has no accidental tangencies. Even if two circles C_1 and C_3 are tangent at x_2, the point x_2 uniquely defines their centers x_1. and x_3. In fact, every realization of a 5-cycle (as well as an n-cycle for odd n, $n \geq 5$) has a pair of tangent circles. From [5, Corollary 2] we have:

Proposition B. If G is a SIG in a plane, then G has a realization X so that:
(a) every point in X is uniquely defined, and
(b) X has no accidental tangencies.

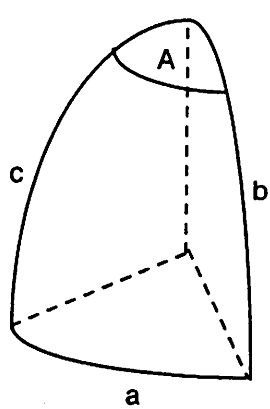
The realization of a 5-cycle in Figure 1 does not satisfy condition (a) of Proposition B, because the point x_2 is not defined uniquely. However, if we move point x_3 a little to the right, then the new realization will satisfy both conditions of Proposition B.

In a realization satisfying both conditions of Proposition B, all tangent circles (if they exist) can be partitioned into pairs. Two circles in each pair are tangent, and no other tangencies occur.

3. The complete graph K_9 is a SIG on a sphere

Before presenting the construction, we need some results for spherical triangles.

A spherical triangle is given by selecting three points on a sphere (here of radius 1). The sides of the triangle are formed by great circles passing through pairs of points. The length of a side is equal to the radian measure of the central angle determined by two endpoints of the side. The Law of Sines for spherical triangles is depicted in Figure 2.



$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

Figure 2. Law of Sines for spherical triangles

Theorem 1. The complete graph K_9 is a sphere-of-influence graph realizable on a sphere.

Proof. Consider the following nine points in spherical coordinates:

north pole $p = (1, 0, 0)$,

$$x=\left(1,\ \frac{\pi}{4},\ \frac{7\pi}{18}\right), y=\left(1,\ \frac{3\pi}{4},\ \frac{7\pi}{18}\right), z=\left(1,\ \frac{5\pi}{4},\ \frac{7\pi}{18}\right), w=\left(1,\ \frac{7\pi}{4},\ \frac{7\pi}{18}\right),$$

$$a=\left(1,\ 0,\ \frac{7\pi}{10}\right), b=\left(1,\ \frac{\pi}{2},\ \frac{7\pi}{10}\right), c=\left(1,\ \pi,\ \frac{7\pi}{10}\right), d=\left(1,\ \frac{3\pi}{2},\ \frac{7\pi}{10}\right).$$

The circles centered at these points have the following radii:
radius(p) = dist(p, x) = $7\pi/18$ ($\approx$ 1.22173), radius(x) = dist(x, p) = $7\pi/18$, because from the Law of Sines for spherical triangles

$$\text{dist}(x, a) = \cos^{-1}[\cos(7\pi/10)\cos(7\pi/18) + \sin(7\pi/10)\sin(7\pi/18)\cos(\pi/4)] \approx 1.22757 > 7\pi/18.$$

Similarly, radius(y) = radius(z) = radius(w) = $7\pi/18$.
Once again, from the Law of Sines,
radius(a) = dist(a, b) = $\cos^{-1}[\cos^2(7\pi/10) + \sin^2(7\pi/10)\cos(\pi/2)]$ =
$\cos^{-1}[\cos^2(7\pi/10)] \approx 1.218$, because dist($a, x$) > dist($a, b$).
Similarly, radius(b) = radius(c) = radius(d) = $\cos^{-1}[\cos^2(7\pi/10)] \approx 1.218$.

Of course, the circle centered at p intersects all remaining eight circles. The circle centered at a intersects the circles centered at p, b, c, d, x, and w. Its intersections with the remaining two circles centered at y and z are nonempty because

$$\text{dist}(a, y) = \text{dist}(a, z) = \cos^{-1}[\cos(7\pi/10)\cos(7\pi/18) + \sin(7\pi/10)\sin(7\pi/18)\cos(3\pi/4)] \approx 2.401782 < \text{radius}(a) + \text{radius}(y).$$

A similar situation occurs for the circles centered at y, z, and w. The circles centered at x and z are tangent at the north pole p. Therefore, the open disks determined by these circles do not intersect. We can make their overlap nonempty by slightly moving the point p, for example, to the position $p' = \left(1,\ \frac{\pi}{8},\ \frac{\pi}{10^6}\right)$.
Of course, replacing p by p' changes the radii of the circles centered at p (p'), x, y, z, and w, but these changes are so small that they do not affect intersections. All of the intersections will be still nonempty which shows that the sphere-of-influence graph for the set of points $\{p', x, y, z, w, a, b, c, d\}$ is isomorphic to K_9. □

As we mentioned in the introduction, the best known construction in the plane is for K_8 (Malnic and Mohar [6] and Scheinerman [2]), and there is a strong belief that we cannot do better.

4. Every SIG in a plane is also a SIG on a sphere

The proof showing that every SIG in the plane is also a SIG on the sphere uses stereographic projection. In the euclidean space consider the sphere S^2 of diameter r and the plane R^2 having exactly one point (0, 0, 0) in common. Let n = (0, 0, r) be the antipodal point on the sphere. Each line passing through the pole n and another point p on the sphere intersects R^2 at exactly one point q. The *stereographic projection* ϕ is a map from $S^2 - n$ to R^2 defined by $\phi(p) = q$ (see Figure 3). It is well known that if $p = (x, y, z)$, then

$$\phi(x, y, z) = (X, Y, 0), \text{ where } X = rx/(r-z) \text{ and } Y = ry/(r-z).$$

Also the inverse of the stereographic projection is given by

$$\phi^{-1}(X, Y, 0) = r/(r^2 + X^2 + Y^2)\,(Xr, Yr, X^2 + Y^2).$$

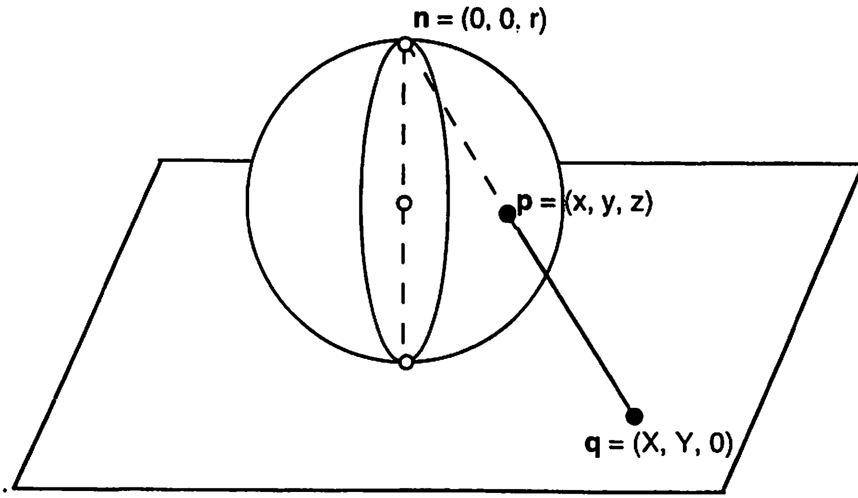


Figure 3. Stereographic projection

The idea of a proof of the main theorem is simple. We take the realization X of G in the plane 'near' the origin. We use a sphere with large diameter for the stereographic projection. Take the preimage of X. This set will be, after some modification, a realization of G on the sphere. We will use a well known property of preserving circles by the stereographic projection, namely, if C is a circle in the plane then its preimage $\phi^{-1}(C)$ is a circle on S^2. However, the centers of circles are not preserved. If a is the center of the circle C, then, in general, $\phi^{-1}(a)$ is not the center of the circle $\phi^{-1}(C)$. This fact causes some difficulties while working with sphere-of-influence graphs. In the remaining part of this section we take care of this problem. We will use the symbol ϕ_r to denote the stereographic projection with the sphere of diameter r.

Lemma 2. Let C be a circle in the plane with center a. For every $\varepsilon > 0$, there exists r, the diameter of the sphere in the stereographic projection ϕ_r, such that if b is the center of the circle $\phi_r^{-1}(C)$ on the sphere, then $\text{dist}(\phi_r^{-1}(a), b) < \varepsilon$.

Proof. Let us intersect a given circle C in the plane and the sphere in the stereographic projection by the vertical plane passing through the points n, a, and the origin O (see Figure 4). Denote the points of intersection of this plane with the circle C by a_1 and a_2. The points $b_1 = \phi_r^{-1}(a_1)$ and $b_2 = \phi_r^{-1}(a_2)$ are the intersection of the vertical plane with the circle $\phi_r^{-1}(C)$ on the sphere. The center b of the circle $\phi_r^{-1}(C)$ lies also in that vertical plane, From the formula for ϕ_r^{-1}, it is easy to see that $\phi_r^{-1}(a) \to a$, $\phi_r^{-1}(a_1) \to a_1$, and $\phi_r^{-1}(a_2) \to a_2$ as $r \to \infty$. Therefore, from the continuity of the euclidean metric,

$$\mathrm{dist}(\phi_r^{-1}(a_1), \phi_r^{-1}(a)) \to \mathrm{dist}(a_1, a) \text{ and}$$
$$\mathrm{dist}(\phi_r^{-1}(a_2), \phi_r^{-1}(a)) \to \mathrm{dist}(a_2, a).$$

Therefore, $|\mathrm{dist}(\phi_r^{-1}(a_1), \phi_r^{-1}(a)) - \mathrm{dist}(\phi_r^{-1}(a_2), \phi_r^{-1}(a))| \to 0$ as $r \to \infty$. Because the point b satisfies $|\mathrm{dist}(\phi_r^{-1}(a_1), b) - \mathrm{dist}(\phi_r^{-1}(a_2), b)| = 0$, we have $\phi_r^{-1}(a) \to b$ as $r \to \infty$, and the conclusion of the lemma follows. □

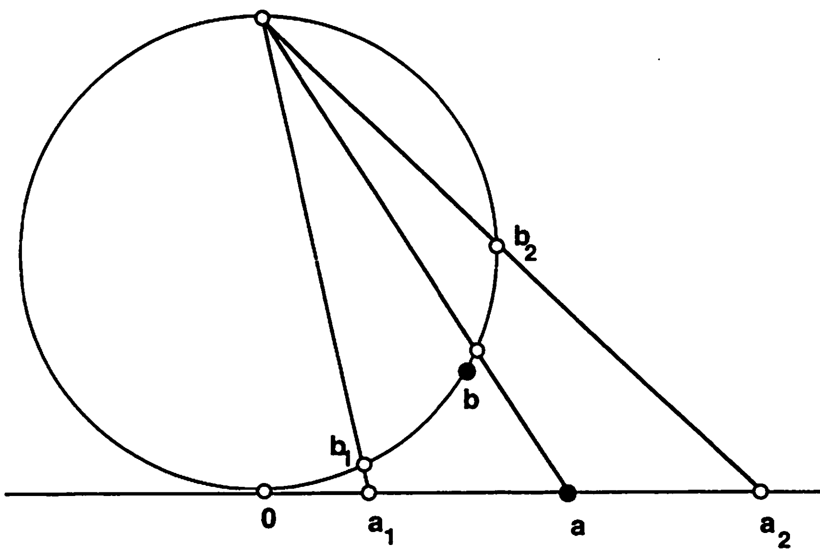


Figure 4. Relationship between centers of circles in the plane and on the sphere

Theorem 3. Every sphere-of-influence graph in a plane is also a sphere-of-influence graph on a sphere.

Proof. Let G be a SIG in a plane. Assume that the order of G is n. According to Proposition B, there is a realization X of G in the plane with no accidental tangencies and such that each point of X is uniquely defined. The set of all balls $B(x_i)$, $1 \le i \le n$, is bounded, say, included in some ball B.

Let ε represent the smallest of the following three numbers:
the minimum overlap between non-tangent circles in the realization X;
the minimum distance between non-tangent circles in the realization X;
the minimum distance between a point $x_i \in X$ and a circle C_j taken over all pairs i, j such that $i \neq j$ and x_i is not on the circle C_j.

Next, select r, the diameter of the sphere in stereographic projection, such that for all points a and b in the ball B, we have

$$\mathrm{dist}(\phi_r^{-1}(a), \phi_r^{-1}(b)) > 0.9\ \mathrm{dist}(a, b)$$

and such that for each i, $1 \le i \le n$, dist($\phi_r^{-1}(x_i), b_i) < \varepsilon/3$, where b_i is the center of the circle $\phi_r^{-1}(C_i)$ on the sphere. This can be achieved by Lemma 2 and by the fact that with $r \to \infty$ we have

$$\text{dist}(\phi_r^{-1}(a), \phi_r^{-1}(b)) \to \text{dist}(a, b)$$

for all points $a, b \in B$.

Consider the family of circles $\phi_r^{-1}(C_i)$, $1 \le i \le n$, on the sphere. If C_i and C_j are tangent in the plane, then $\phi_r^{-1}(C_i)$ and $\phi_r^{-1}(C_j)$ are tangent on the sphere. Let us partition all points $\phi_r^{-1}(x)$, $x \in X$, into two subsets: T - the set of all tangency points for the family $\phi_r^{-1}(C_i)$, $1 \le i \le n$, and S - the rest of points. If $\phi_r^{-1}(x_i) \in S$, then move it to the position y_i which is the center of the circle $\phi_r^{-1}(C_i)$. If $\phi_r^{-1}(x_i) \in T$, then define y_i to be $\phi_r^{-1}(x_i)$. We claim that the set $Y = \{y_i : 1 \le i \le n\}$ is a realization of the graph G on the sphere.

Suppose $v_i v_j$ is the edge of the graph G. Then the overlap of the circles C_i and C_j is at least ε, so the overlap of the circles $\phi_r^{-1}(C_i)$ and $\phi_r^{-1}(C_j)$ is at least 0.9ε. The circles centered at y_i and y_j in the realization Y might have different radii, because the points defining them could be moved. However, these new radii will not differ more than $\varepsilon/3$ from the radii of the circles $\phi_r^{-1}(C_i)$ and $\phi_r^{-1}(C_j)$ and, therefore, the circles centered at y_i and y_j in realization Y will still overlap.

If $v_i v_j$ is not an edge of the graph G, then there are two possibilities, the circles C_i and C_j are either disjoint or tangent. If they are disjoint, then the distance between them is at least ε, so the distance between their pre-images $\phi_r^{-1}(C_i)$ and $\phi_r^{-1}(C_j)$ is at least 0.9ε. By the same argument as above, in realization Y, the intersection of the circles centered at y_i and y_j is empty. If the circles C_i and C_j are tangent, then the circles $\phi_r^{-1}(C_i)$ and $\phi_r^{-1}(C_j)$ on the sphere are also tangent. The points x_i and x_j were uniquely defined at the point of tangency whose pre-image y is in the realization Y. Therefore, the radii of the circles centered at y_i and y_j in the realization Y are determined by their distance from y, so these circles are identical with $\phi_r^{-1}(C_i)$ and $\phi_r^{-1}(C_j)$. They are tangent and the corresponding open disks have empty intersection. □

5. Are there any SIGs on a sphere that are not realizable in the plane?

The complete graph K_9 would be an example of such a graph assuming that the K_9-conjecture is true. Another candidate, even more promising, might be a geodesic grid on a sphere. More precisely, consider all points on the sphere whose both angular spherical coordinates are whole multiples of $\pi/6$ (or 30 degrees). Let G be the sphere-of-influence graph (on the sphere) of this set of sixty two points. We belive that G is not realizable as a SIG in a plane.

We would like to finish the paper by stating the following conjecture.

Conjecture. There are graphs realizable as SIGs on the sphere but not realizable as SIGs in the plane.

References

1. D. Avis and J. Horton, Remarks on the sphere of influence graph, in *Discrete Geometry and Convexity*, J.E. Goodman et al (eds,), New York Academy of Sciences (1985), 323-327.

2. F. Harary, M. S. Jacobson, M. J. Lipman, and F. R. McMorris, On Abstract Sphere-of-Influence Graphs, *Math. Comput. Modelling* **17** (1993), 77-83.

3. M. S. Jacobson, M. J. Lipman, and F. R. McMorris, Trees that are Sphere-of-Influence Graphs, *Appl. Math. Letters* **8** (1993), 89-93.

4. A. E. Kézdy and G. Kubicki, K_{12} is not a closed sphere-of-influence graph, *Intuitive geometry* (Budapest, 1995) 383-397, Bolyai Soc. Math. Stud.,6, *Janos Bolyai Math.Soc., Budapest*, 1997.

5. M. J. Lipman, Integer realizations of sphere-of-influence graphs, *Congr. Numer.* **91** (1992), 63-70.

6. A. Malnic and B. Mohar, Two results on antisocial families of balls, in *Proc. of the Fourth Czechoslovak Symposium on Combinatorics, Graphs, Complexity*, Prachatice, 1990, M. Friedland and J. Nesetril, (eds.), 205-207.

7. T. S. Michael and T. Quint, Sphere of influence graphs, edge density and clique size, *Math. Comput. Modelling* **20** (1994), 19-24.

8. T. S. Michael and T. Quint, Sphere of influence graphs in general metric spaces, *Math. Comput. Modelling* **29** (1999), 45-53.

9. T. S. Michael and T. Quint, Sphere of influence graphs: a survey, *Congr. Numer.* **105** (1994), 153-160.

10. G. T. Toussaint, A Graph-Theoretical Primal Sketch, *Computational Morphology*, Elsevier, Amsterdam (1988), 229-260.

11. G. T. Toussaint, Pattern Recognition and Geometric Complexity, *Proceedings of the 5th International Conference on Pattern Recognition*, Miami Beach (1980), 1324-1347.

12. G. T. Toussaint, Computational Geometric Problems in Pattern Recognition, in *Pattern Recognition Theory and Application*, (J. Kittler, ed) NATO Advanced Study Institute, Oxford University (1981), 73-91.