Connected colorings of graphs

Yair Caro *   Yehuda Roditty †

Abstract

Let $H = K_{k_1,k_2,\cdots,k_t}$, be a complete multipartite graph having $t \geq 3$ parts. Extending the well-known result that a simple graph G or its complement, $\bar{G}$, is connected, it is proved that in any coloring of the edges of H with two colors, blue and red, at least one of the subgraphs of H induced by the blue edges or by the red edges, is connected.

1 Introduction

A folklore (see [1] - [8]) in Graph Theory is the following result:

Result 1: *Let G be a graph. Then, either G or its complement, $\bar{G}$, is connected.*

A slightly stronger result is (see [4]):

Result 2: *If $diam(G) \geq 3$ then $diam(G) \leq 3$, and if $diam(G) \geq 4$ then $diam(G) \leq 2$, where $diam(G)$ is the diameter of G.*

Rephrasing Result 1 in terms of coloring of the edges of the complete graph K_n, we obtain:

Proposition 1.1 *In any coloring of the edges of the complete graph K_n, by the colors, say, blue and red, either the subgraph induced by the blue edges or the subgraph induced by the red edges, is connected.*

Bialostocki, Dierker and Voxman [1] suggested several generalization of these results and asked for further possible directions. Here we suggest a new direction concerning connected colorings of not necessarily complete graphs.

*Department of Mathematics School of Education University of Haifa - ORANIM Tivon ISRAEL 36910

†School of Computer Sciences, Tel Aviv University, Tel Aviv 69978, Israel and Department of Computer Science, The Academic College of Tel-Aviv-Yaffo, Tel-Aviv 61161, Israel. email: jr@post.tau.ac.il

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Definition 1.2 *A graph G is called* **robust** *if in any coloring of its edges by two colors, say, blue and red, either the subgraph induced by the blue edges or the subgraph induced by the red edges, is connected.*

We notice here that a connected subgraph in a robust graph need not be a spanning subgraph.

Thus, Proposition 1.1 says that K_n is robust and so, for example, is any graph having an edge incident with all other edges (e.g., stars, double stars etc.).

The aim of this paper is to prove the following two theorems:

Theorem 1.3 *Let $H = K_{k_1,k_2,\ldots,k_t}$, $t \geq 3$ be a complete multipartite graph. Then, H is robust.*

Theorem 1.4 *Let G be a graph on $n \geq 6$ vertices obtained by deleting at most three edges from K_n. Then, G is robust.*

We emphasize that the requirement in Theorem 1.3 that $t \geq 3$ is essential since in the case of $t = 2$, namely, a complete bipartite graph, which is not a star, $K_{1,n}$, the statement of Theorem 1.3 does not hold. Indeed, consider $K_{m,n}$, where, $m = a + b \geq 2$, $n = x + y \geq 2$, $a, b, x, y \geq 1$, and the partite sets are $|A| = m$, $|B| = n$. We put $A = A_1 \cup A_2$, where, $|A_1| = a$, $|A_2| = b$ and $B = B_1 \cup B_2$, where, $|B_1| = x$, $|B_2| = y$. Color the edges of $< A_1 \cup B_1 >$ and $< A_2 \cup B_2 >$ with the blue color, and the edges of $< A_1 \cup B_2 >$ and $< A_2 \cup B_1 >$ with the red color. One can easily observe that each of the color classes induces a disconnected subgraph.

Finally, we observe that the result of Theorem 1.4 is best possible. Indeed, there is, for $n \geq 6$, a graph G on n vertices and $\binom{n}{2} - 4$ edges, which is non-robust. Let $G = G(V, E) = K_n \setminus C_4$ with $V = A \cup B$, $|A| = n - 4$, $|B| = 4$, $B = \{a, b, x, y\}$, where the edges $(a, b), (x, y)$ are the only edges left from the induced complete graph on the vertex set B. Now color the edges of the graph $K_{n-2} \setminus \{(x, y)\}$. (where, the $n - 2$ vertices are $A \cup \{x, y\}$). by the blue color as well as the edge (a, b). The rest of the edges of G, namely, the complete bipartite graph $K_{2,n-4}$. (where, the two vertices are a, b and the $n - 4$ vertices are the set A) together with the edge (x, y), are colored red. Again, one can find that both colored subgraphs are disconnected. In addition there is a graph G on 5 vertices and 7 edges which is not robust. Indeed, let $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = E(K_5) \setminus \{(1, 2), (1, 5), (3, 4)\}$. Now define, $G_B = K_3 \cup K_2$, where, $V(K_3 \cup K_2) = \{2, 3, 5\} \cup \{1, 4\}$, and $G_R = P_3 \cup K_2$, where, $V(P_3 \cup K_2) = \{2, 4, 5\} \cup \{1, 3\}$. One can see that both G_B and G_R are disconnected, where, G_B, G_R are the induced subgraphs on the blue and red edges, respectively.

Hence, both Theorem 1.3 and Theorem 1.4 are best possible.

2 Proofs

To the sequel we assume that $H = K_{k_1,k_2,\ldots,k_t}$, $t \geq 3$ is a complete multipartite graph. In any two coloring of the edges of H by the colors blue and red we denote by H_B, H_R the subgraphs of H induced by the blue edges and the red edges, respectively. Let $v \in V(H)$ by $d_B(v), d_R(v)$ we denote the degree of v in H_B and H_R, respectively. The partite sets of H are denoted by $A_1, A_2, \cdots, A_t$ and $< A_i, A_j >, i \neq j$, is the complete bipartite subgraph of H induced by $A_i \cup A_j$. Finally, we denote by $d_B(u, v)$ the distance between u and v in H_B.

Proof of Theorem 1.3

If H_R is connected we are done. Hence, we may assume that H_R is disconnected with $R_1, R_2, \cdots, R_s$, $s \geq 2$ as its connected components. Denote by $R_{ij} = R_i \cap A_j$ the set of vertices at the red component R_i in the j-th part of H. The following is a simple but crucial observation:

Observation 2.1 *The subgraph $< R_{ij}, R_{xy} >$, $i \neq x, j \neq y$, is a complete bipartite subgraph of H_B.*

We will show that H_B is connected by considering several cases.

Case 1: $A_i \setminus V(H_R) \neq \emptyset$.

In this case there is a vertex $u \in A_i \setminus V(H_R)$ and hence $d_B(u) = \sum_{j=1, j \neq i}^{t} k_j$. But then any other possible blue edge is incident with some vertex of $V(H) \setminus A_i$ which is, in turn, adjacent to u. Hence, H_B is connected and in fact $diam(H_B) \leq 4$, since for every vertex v in H_B, $d_B(u, v) \leq 2$.

Case 2: $A_i \setminus V(H_R) = \emptyset$.

In this case we show that for any vertex $u \in A_i$ and any vertex $v \in A_j$, there is a path in H_B between them.

Sub-case 2.1: $j \neq i$.

Assume first that u and v are in distinct components of H_R. Then by observation 2.1 u and v are connected by a blue edge.

So let now u and v be in the same component of H_R, say, R_1. That is $u \in R_{1i}, v \in R_{1j}$. Recall that H_R has at least two components and H has at least three parts. If $A_x \setminus R_1 \neq \emptyset$ for some $x \neq i, j$, then we have, without lost of generality, $R_{2x} \neq \emptyset$. But then applying twice observation 2.1 on $< R_{1i}, R_{2x} >$ and $< R_{2x}, R_{1j} >$, we obtain that $d_B(u, v) \leq 2$.

If $A_x \setminus R_1 = \emptyset$ for any $x \neq i, j$ then since H_R is not connected it follows that ,w.l.o.g. , R_{2i} and R_{2j} are not empty.

Applying, again, observation 2.1 on $< R_{1i}, R_{2j} >$, $< R_{2j}, R_{1x} >$, $< R_{1x}, R_{2i} >$, $< R_{2i}, R_{1j} >$, we find $d_B(u, v) \leq 4$.

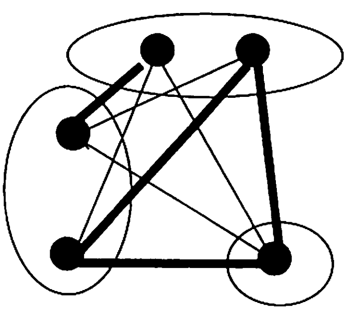
Sub-case 2.2: $j = i$.

In that case $u, v \in A_i$. If u and v are in the same component, say, R_s of H_R, then for some $x \neq s, y \neq i, R_{xy}$ is not empty, and hence by observation 2.1 applying on $< R_{si}, R_{xy} >$ it follows that $d_B(u, v) \leq 2$.

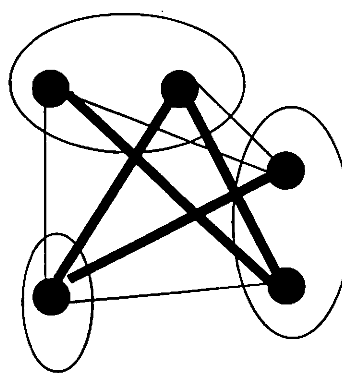
If u and v are in distinct components of H_R, we may assume that $u \in R_1, v \in R_2$. If for some $r \neq i$ and $x \neq 1, 2, R_{xr} \neq \emptyset$. Then by applying observation 2.1 on $< R_{1i}, R_{xr} >, < R_{xr}, R_{2i} >$ it follows that $d_B(u, v) \leq 2$. Hence, H_R has only two components R_1, R_2 and since H has at least three parts, it follows that there is some k such that $R_{2k} \neq \emptyset$. If for some $x \neq i, k, R_{1x} \neq \emptyset$ we may apply observation 2.1 on $< R_{1i}, R_{2k} >, < R_{2k}, R_{1x} >, < R_{1x}, R_{2i} >$ and thus, $d_B(u, v) \leq 3$.

It remains only the case that $R_{1i}, R_{2i}, R_{1k}, R_{2k} \neq \emptyset$ and $A_x = R_{2x}$ for $x \neq i, k$. Hence, applying again observation 2.1 on $< R_{1i}, R_{2x} >, < R_{2x}, R_{1k} >, < R_{1k}, R_{2i} >$ it follows that $d_B(u, v) \leq 3$, completing the proof. ■

Remark 2.2 *The proof of Theorem 1.3 contains a bit more then stated in the theorem. It says that if H_R is not connected then $diam(H_B) \leq 4$. In the following example we give two possible colorings of $K_{2,2,1}$ where in the first $diam(H_R) = \infty$ and $diam(H_B) = 4$ (Fig. 1(a)), while in the second coloring $diam(H_R) = diam(H_B) = 4$ (Fig. 1(b)). (The red edges are presented by the bold lines).*



a: first coloring



b: second coloring

Figure 1: 2–coloring of $K_{2,2,1}$ in two ways

Before proving Theorem 1.4 we need some further simple observations.

Observation 2.3 1. *The only graphs on six vertices and five edges which are either disconnected or contain only one non-trivial component on at most four vertices are:*

$$A = \{(K_4 \setminus \{e\}) \cup E_2,\ C_4 \cup K_2,\ K_3 \cup P_3,\ TR_3 \cup K_2\},$$

where, E_2 is the empty graph on two vertices and TR_3 is a triangle with an edge incident to one of its vertices.

2. *The only graphs on six vertices and six edges which are either disconnected or contain only one non-trivial component on at most four vertices are:*

$$B = \{K_4 \cup E_2,\ 2K_3,\ (K_4 \setminus \{e\}) \cup K_2\}.$$

3. *The only graph on six vertices and seven edges which is disconnected without a trivial component is $K_4 \cup K_2$.*

Observation 2.4 1. *If G has six vertices and five edges and $G \notin A$, then the edges of G induced a connected subgraph on at least five vertices.*

2. *If G has six vertices and six edges and $G \notin B$, then the edges of G induced a connected subgraph on at least five vertices.*

3. *If G has six vertices and seven edges and $G \neq K_4 \cup K_2$, then the edges of G induced a connected subgraph on at least five vertices.*

4. *If G has six vertices and at least eight edges then the edges of G induced a connected subgraph on at least five vertices.*

Now we are ready to the proof of Theorem 1.4

Proof of Theorem 1.4

The proof is by induction on n. We start with the case when G is a graph on six vertices. We denote by G_B, G_R the subgraphs induced by the blue and red edges, respectively. By $e(G)$ we denote the number of edges of G. We assume that $e(G_R) \geq e(G_B)$.

Case 1: $e(G) = 15$.

The result of the theorem follows immediatly from Proposition 1.1.

Case 2: $e(G) = 14$.

If $e(G_R) \geq 8$ then we are done by Observation 2.4 (4). Otherwise $e(G_R) = e(G_B) = 7$ then we are done by Observation 2.4 (3) unless $G_B = G_R = K_4 \cup K_2$. But it is impossible to pack two copies of $K_4 \cup K_2$ into K_6, and we are done again.

Case 3: $e(G) = 13$.

If $e(G_R) \geq 8$ then we are done by Observation 2.4 (4). So $e(G_R) = 7$ and $e(G_B) = 6$. But, as one can check, it is impossible to pack $K_4 \cup K_2$ with a member of B into K_6. So that we are done again.

Case 4: $e(G) = 12$.

If $e(G_R) \geq 8$ then we are done by Observation 2.4 (4). If $e(G_R) = 7$ then $e(G_B) = 5$, but once again it is easy to check that it is impossible to pack $K_4 \cup K_2$ with a member of A into K_6. Now if $e(G_R) = e(G_B) = 6$ then one can check that it is impossible to pack any member of B with any member of B into K_6.

Hence, it follows that in any coloring of a graph G on six vertices and at least 12 edges, either G_R or G_B is connected on at least five vertices.

So we may assume that for $n-1 \geq 5$ either G_B or G_R is connected and of order at least $n-2$. Consider a two coloring of the edges of a G. $|G| = n$ and G is obtained from K_n by deleting at most three edges. Since $|G| \geq 7$ there is a vertex $v \in G$ such that $deg(v) = n-1$. Define $G^* = G \setminus \{v\}$. Then by the induction hypothesis as G^* is missing at most three edges, either G^*_B or G^*_R is connected of order at least $n-2$. Assume $|G^*_R| \geq |G^*_B|$.

Case 5: $|G^*_R| = n-1$.

In this case we are done since either there was a red edge from v to G^* and thus, $|G_R| = n$ and G_R is connected or else, $|G_B| = n$ and G_B is connected.

Case 6: $|G^*_R| = n-2$.

If every edge from v to G^* is blue then G_B is connected of order n. If there are at least two red edges from v to G^* then $|G_R| \geq n-1$ and G_R is connected. If there is a red edge from v to G^*_R then again $|G_R| = n-1$ and G_R is connected. Hence we are left with the following possibility that there is a vertex $z \in G^*$ but $z \notin G^*_R$ and v is connected to all $n-2$ vertices of $G^* \setminus \{z\}$ by blue edges and (v, z) is red. However any blue edge in G has an end-vertex in G^*, and is incident with the blue edges emanating from v, so that G_B is connected with $|G_B| = n$ since all edges incident with z in G^* are blue edges (and $deg_B z > 0$).

This completes the proof. ■

References

[1] A. Bialostocki, P. Dierker and W. Voxman, Either a graph or its complement is connected: A continuing saga. *preprint*

[2] J. Bierbrauer and A Gyarfas, On (n,k)-colorings of complete graphs, Eighteenth Southeastren International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, Fla. 1987). *Congr. Numer. 58(1987), 123-139.*

[3] F. Harary, Graph Theory,*Addison-Wesley, 1969.*

[4] F. Harary and R.W. Robinson, The diameter of a graph and its complement, *Amer. Math. Monthly 92(1985), 211-212.*

[5] G. Ringel, Selbstkomplementare Graphen. (German). *Arch. Math. 14(1963), 354-358.*

[6] H. Sachs, Uber selstkomplementare Graphen. (German). *Publ. Math. Debrecen 9(1962), 270-288.*

[7] P.D. Straffin, Letter to the editor: The diameter of a graph and its complement. *Amer. Math. Monthly 93(1986).76.*

[8] D. West, Introduction to Graph Theory, *Simon & Schuster A Viacom Company, Upper Saddle River, NJ 07458. (1996).*

Integral Sum Graphs from a Class of Trees

Wenjie He

Department of Applied Mathematics

Hebei University of Technology, Tianjin, 300130, P.R.China

Lixin Wang

Department of Management Engineering

Mechanical Engineering College, Shijiazhuang, 050010, P.R.China

Honghai Mi, Yufa Shen and Xinkai Yu

Department of Applied Mathematics

Hebei University of Technology, Tianjin, 300130, P.R.China

e-mail: he_ wenjie@eyou.com

ABSTRACT. A graph $G = (V, E)$ is said to be an *integral sum graph* (respectively, *sum graph*) if there is a labeling f of its vertices with distinct integers (respectively, positive integers) , so that for any two vertices u and v, uv is an edge of G if and only if $f(u) + f(v) = f(w)$ for some other vertex w. For a given graph G, the *integral sum number* $\zeta = \zeta(G)$ (respectively, *sum number* $\sigma = \sigma(G)$) is defined to be the smallest number of isolated vertices which when added to G result in an integral sum graph (respectively, sum graph). In a graph G, a vertex $v \in V(G)$ is said to a *hanging vertex* if the degree of it $d(v) = 1$. A path $P \subseteq G$, $P = x_0x_1x_2\cdots x_t$, is said to be a *hanging path* if its two end vertices are respectively a hanging vertex x_0 and a vertex x_t whose degree $d(x_t) \neq 2$ where $d(x_j) = 2$ $(j = 1, 2, \cdots, t-1)$ for every other vertex of P. A hanging path P is said to be a *tail* of G, denoted by $t(G)$, if its length $|t(G)|$ is a maximum among all hanging paths of G. In this paper, we prove $\zeta(T_3) = 0$, where T_3 is any tree with $|t(T_3)| \geq 3$. The result improves a previous result for integral sum trees from identification of Chen(1998).

1. Introduction

All graphs in this paper are finite and have no loops or multiple edges. We follow in general the graph-theoretic notation and terminology of [1] unless otherwise specified.

F.Harary introduced the idea of sum graphs and integral sum graphs [2] [3]. At first, let N denote the set of positive integers. *The sum graph* $G^+(S)$ of a finite subset $S \subset N$ is the graph (S, E) with $uv \in E$ if and only if $u+v \in S$. A graph G is said to be a *sum graph* if it is isomorphic to the sum graph of some $S \subset N$. *The sum number* $\sigma(G)$ is the smallest nonnegative m such that $G \cup mK_1$, the union of G and m isolated vertices , is a sum graph. In the above definition by using the set Z of all integers instead of N we obtain the definition of the integral sum graph. Analogously, *the integral sum number* $\zeta(G)$ is the smallest nonnegative m such that $G \cup mK_1$ is an integral sum graph. It is easy to see that the graph G is an *integral sum graph* if and only if $\zeta(G) = 0$. It is obvious that $\zeta(G) \leq \sigma(G)$. Although some results on sum graphs and integral sum graphs were presented [2-12], but a considerable number of unsolved problems were remained. One of them is the conjecture: "Every tree is an integral sum graph", which was proposed by Zhibo Chen in 1998 [10]. In order to discuss this problem, here and now, we briefly summarize some results on tree graph. F.Harary [2] has conjectured that any tree can be made into a sum graph with the addition of a single isolated vertex in 1990. This conjecture was proved by Ellingham [5] in 1993. F.Harary [3] found that all paths and stars are integral sum graphs and conjectured that every integral sum tree is a caterpillar in 1994. This conjecture was disproved by Zhibo Chen [4] in 1996. Zhibo Chen [10] has also shown that every generalized star and tree with all forks at least distance 4 apart are integral sum graphs in 1998.

In a graph G, a vertex $v \in V(G)$ is said to be a *hanging vertex* if its vertex degree $d(v) = 1$. A path $P \subseteq G$, $P = x_0x_1x_2 \cdots x_t$, is said to be a *hanging path* if its two end vertices are respectively a hanging vertex x_0 and a vertex x_t with vertex degree $d(x_t) \neq 2$ and $d(x_j) = 2(j = 1, 2, \cdots, t-1)$ for every other vertex of P. A hanging path P is said to be a *tail* of G, denoted by $t(G)$, if its length $|t(G)|$ is maximum one among all hanging paths of G. In this paper, we shall prove $\zeta(T_3) = 0$, where T_3 is any tree with $|t(T_3)| \geq 3$. A tree is said to be a *caterpillar* C, if it consists of a path $s_0s_1 \cdots s_l$, called the *spine* of C, with some hanging vertices known as *feet* attached to the inner vertices (an ***inner vertex*** is a vertex with at least two adjacent vertices which are not the hanging vertices) of the spine by edges known as *legs* . Then $s_i(i = 1, 2, \cdots, l-1)$ was called as the *spine vertex* of C, s_0 as ***tail*** , s_l as ***head*** and s_{l-2} as ***heart*** and s_{l-1} as ***neck*** (see Figure 1). The result improves the previous result of integral sum trees from identification [10].

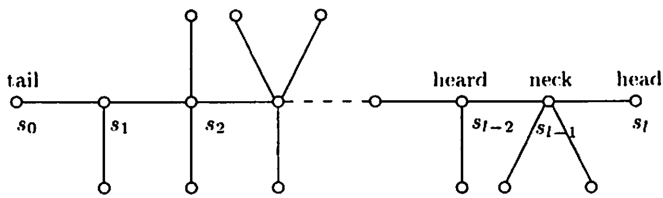


Figure 1. A caterpillar C

To prove that $\zeta(T_3) = 0$ for any tree whose tail length is not less than 3, we use a labelling algorithm. The labelling algorithm has two stages. And the first stage has to depend on the Ellingham's labelling algorithm [5] . Therefore, in the next section, we shall briefly introduce it.

Since it can easily be shown that $\zeta(T) = 0$ for $|T| < 6$, from now on, we assume $|T| \geq 6$.

2. Ellingham labelling algorithm [5]

Suppose that T is a tree with $|T| = n$ and z is an isolated vertex. And define a *shrub* S which is a special class of trees with at most one inner vertex. Then, using the Ellingham labelling algorithm, we can construct a sequence of caterpillars $C_1, C_2, \cdots, C_m$ and obtain two different types of decomposition of T.

Type 1. T is completely decomposed into some caterpillars, that is $T = C_1 \cup C_2 \cup \cdots \cup C_m$.

Type 2. T is decomposed into some caterpillars and a shrub S . Thus $T = C_1 \cup C_2 \cup \cdots \cup C_m \cup S$ $(m \geq 1)$.

Applying the Ellingham labelling algorithm, we can give a sum labelling f for the graph $T \cup \{z\}$, no matter what happens. We suppose the vertices $V(T) = \{v_1, v_2, \cdots v_n\}$ to be ordered such that $0 < f(v_1) < f(v_2) < \cdots < f(v_n) < f(z)$.

For type 1, the vertex v_n is the head of the last caterpillar C_m. If v_r and v_{n-k} are the heart and the neck of C_m, respectively, then we have that $f(v_{n-i}) = f(v_r) + (k-i)f(v_{n-k})(i = 0, 1, \cdots, k-1)$ and $f(z) = f(v_r) + (k+1)f(v_{n-k})$.

For type 2, without loss of generality, we assume that the shrub S has k hanging paths with length 2 and its root is v_r, of course v_r is also the heart of the last caterpillar C_m. Then we have that $f(z) = f(v_{n-k-i+1}) + f(v_{n-k+i})(i = 1, 2, \cdots, k)$, $f(v_r) < f(v_{n-2k})$ and $f(v_{n-k+i}) = f(v_r) + f(v_{n-2k+i})$ $(i = 1, 2, \cdots, k)$. If $v_d v_{n-2k} \in E(T)$, then we have that

$$f(v_{n-k+i}) = f(v_d) + f(v_{n-2k}) + (i-1) = f(v_{n-k+1}) + (i-1)$$

$(i = 1, 2, \cdots, k)$, $f(v_1) \geq k$ and $|f(v_i) - f(v_j)| \geq k$ for any $v_i, v_j \in V(T), i \neq j$ and $i \leq n - 2k$.

3.The integral sum labelling of T_3

Now we denote the tree whose tail length is at least 3 by T_3. In this section, we will give an integral sum labelling φ of T_3. Suppose that $|t(T_3)| = b = 3$. Therefore we can decompose T_3 into $T \cup P$, where P is a tail of T_3, $P = x_0 x_1 x_2 x_3$, x_3 is a hanging vertex of T_3 and $V(T) = \{v_1, v_2, \cdots, v_{n-3}\}$. In order to give an integral sum labelling of T_3, using the Ellingham labelling algorithm above, we give first a sum labelling f of $T \cup \{z\}$, where z is a vertex which is not in $V(T)$. Then, we consider two cases according to the location of x_0 as follows.

Case 1. $x_0 = v_r$, in other words x_0 is the heart of the last caterpillar C_m.

In this case, let $z = x_2$ in the Ellingham labelling f above. Then we extend from sum labelling f of $T \cup \{x_2\}$ to a labelling φ of T_3 by the following algorithm. Let

$$\begin{aligned}\varphi(v) &= f(v) \text{ for } v \in V(T \cup \{x_2\})\\ \varphi(x_1) &= -\varphi(x_2) + \varphi(v_q)\\ \varphi(x_3) &= -\varphi(x_2) + \varphi(v_q) + \varphi(v_r),\end{aligned}$$

where $v_q \in V(T)$, $v_q v_r \in E(T)$ and $\varphi(v_q) < \varphi(v_r)$ (see Figure 2).

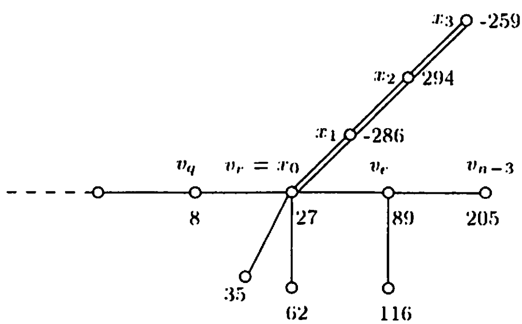


Figure 2. Illustration of Type 1 in Case 1

It is obvious that $\varphi(x_1) < \varphi(x_3) < 0 < \varphi(v_1) < \varphi(v_2) < \cdots < \varphi(v_{n-3}) < \varphi(x_2)$ and $\varphi(v_q) + \varphi(v_r) \leq \varphi(v_e)$, where v_e satisfies $v_e v_{n-3} \in E(T)$(The v_e is the neck of the last caterpillar C_m in type 1 and the v_e is equal to $v_{n-3-2k+1}$ in type 2). Now, we shall prove the labelling φ is an integral sum labelling of T_3. At first, for any $u, v \in V(T \cup \{x_2\})$, if $uv \in E(T)$, then we have $\varphi(u) + \varphi(v) = \varphi(w)$ for some $w \in V(T \cup \{x_2\})$ because $\varphi = f$ in $T \cup \{x_2\}$ and f is a sum labelling of $T \cup \{x_2\}$. In addition, we have that

$$\begin{aligned}\varphi(x_0) + \varphi(x_1) &= \varphi(v_r) - \varphi(x_2) + \varphi(v_q) = \varphi(x_3)\\ \varphi(x_1) + \varphi(x_2) &= \varphi(v_q) \text{ and}\\ \varphi(x_2) + \varphi(x_3) &= \varphi(v_q) + \varphi(v_r) = \varphi(y),\end{aligned}$$

for some $y \in V(T \cup \{x_2\})$ by $v_q v_r \in E(T)$. Therefore we have that for any $u, v \in V(T_3)$ and $uv \in E(T_3)$, $\varphi(u) + \varphi(v) = \varphi(w)$ for some $w \in V(T_3)$. Now, we just need to show that if any $u, v \in V(T_3)$, $u \neq v$ and $uv \notin E(T_3)$, then $\varphi(u)+\varphi(v)\neq\varphi(w)$ for any $w\in V(T_3)$. We may assume without loss of generality that $\varphi(u)<\varphi(v)$.

(1) For any $u,v\in V(T\cup\{x_2\})$, $u\neq v$ and $uv\notin E(T_3)$.

It is obvious that $\varphi(u)+\varphi(v)\neq\varphi(w)$ for any $w\in V(T\cup\{x_2\})$ because $\varphi=f$ in $T\cup\{x_2\}$. In addition, by the construction of the labeling φ, we know that $\varphi(x_1)<\varphi(x_3)<0$. Thus $\varphi(u)+\varphi(v)\neq\varphi(w)$ for any $w\in V(T_3)$.

(2) For any $v\in V(T_3)$ and $x_1v\notin E(T_3)$.

It is obvious that $v\neq x_0,x_2$, therefore

$$\begin{aligned}\varphi(x_1)+\varphi(v) &\leq -\varphi(x_2)+\varphi(v_q)+\varphi(v_{n-3})\\ &= -\varphi(v_{n-3})-\varphi(v_e)+\varphi(v_q)+\varphi(v_{n-3})\\ &= -\varphi(v_e)+\varphi(v_q)\\ &< -\varphi(v_r)+\varphi(v_q)<0.\end{aligned}$$

In addition, for $w=x_3$ we have only that $\varphi(x_0)+\varphi(x_1)=\varphi(x_3)$, but it is not in this case. Thus $\varphi(x_1)+\varphi(v)\neq\varphi(w)$ for any $w\in V(T_3)$ and $x_1v\notin E(T_3)$.

(3) For any $v\in V(T_3)\setminus\{x_1\}$ and $x_3v\notin E(T_3)$.

It is obvious that $v\neq x_2$, therefore

$$\begin{aligned}\varphi(x_1) &< \varphi(x_3)+\varphi(v)\leq\varphi(x_3)+\varphi(v_{n-3})\\ &= -\varphi(x_2)+\varphi(v_r)+\varphi(v_q)+\varphi(v_{n-3})\\ &= -\varphi(v_{n-3})-\varphi(v_e)+\varphi(v_r)+\varphi(v_q)+\varphi(v_{n-3})\\ &= -\varphi(v_e)+\varphi(v_r)+\varphi(v_q)\\ &\leq -\varphi(v_e)+\varphi(v_e)=0.\end{aligned}$$

Thus $\varphi(x_3)+\varphi(v)\neq\varphi(w)$ for any $w\in V(T_3)$ and $x_3v\notin E(T_3)$.

Summarizing the above mentions in Case 1, we obtain that labelling φ satisfies the condition of an integral sum labelling of T_3 when $x_0=v_r$.

Case 2. $x_0\neq v_r$.

At first, let $z=x_1$ in the Ellingham labelling f above. Then we can extend from the sum labelling f of $T\cup\{x_1\}$ to a labelling φ of T_3 by the following algorithm. Let

$$\begin{aligned}\varphi(v) &= f(v)\ \text{ for }\ v\in V(T\cup\{x_1\})\\ \varphi(x_3) &= \varphi(x_1)+\varphi(x_0)\\ \varphi(x_2) &= -\varphi(x_1)+\varphi(v_q),\end{aligned}$$

where $v_q \in V(T)$, $v_q x_0 \in E(T)$.

It is easy to see that $\varphi(x_2) < 0 < \varphi(v_1) < \varphi(v_2) < \cdots < \varphi(v_{n-3}) < \varphi(x_1) < \varphi(x_3)$.

It is easy to verify that if $u, v \in V(T_3), u \neq v$ and $uv \in E(T_3)$, then $\varphi(u) + \varphi(v) = \varphi(w)$ for some $w \in V(T_3)$. So we just need to show that for any $u, v \in V(T_3)$, if $uv \notin E(T_3)$, then there is no $w \in V(T_3)$ such that $\varphi(u) + \varphi(v) = \varphi(w)$. We may assume without loss of generality that $\varphi(u) < \varphi(v)$.

At first, if $u, v \in V(T \cup \{x_1\})$, and $uv \notin E(T_3)$, then $\varphi(u) + \varphi(v) \neq \varphi(w)$ for any $w \in V(T \cup \{x_1\})$ because $\varphi = f$ in $T \cup \{x_1\}$ and f is a sum labelling of $T \cup \{x_1\}$. Next, for $\varphi(u) + \varphi(v) = \varphi(x_3)$, $v = x_1$ if and only if $u = x_0$ according to the labeling φ. Therefore if $\varphi(u) + \varphi(v) = \varphi(x_3)$, then there must be that $\varphi(v) \leq \varphi(v_{n-3})$. We consider two subcases as follows.

Subcase 1. For $\varphi(u) \leq \varphi(v_e)$, where $v_e v_{n-3} \in E(T)$, we have that

$$\varphi(u) + \varphi(v) \leq \varphi(v_e) + \varphi(v_{n-3}) = \varphi(x_1) < \varphi(x_3)$$

Subcase 2. For $\varphi(u) > \varphi(v_e)$, where $v_e v_{n-3} \in E(T)$.

(1) If T is decomposed into type 1, namely $T = C_1 \cup C_2 \cup \cdots \cup C_m$, then

$$\begin{aligned}
\varphi(u) + \varphi(v) &= \varphi(v_{n-3-j}) + \varphi(v_{n-3-i}) \quad (i < j; i, j = 0, 1, 2, \cdots, k-1) \\
&= 2\varphi(v_{n-3}) - (i + j)\varphi(v_e) \\
&= \varphi(v_{n-3}) + \varphi(x_1) - (i + j + 1)\varphi(v_e) \\
&= \varphi(v_{n-3}) + \varphi(x_3) - \varphi(x_0) - (i + j + 1)\varphi(v_e) \\
&= \varphi(v_r) + \varphi(x_3) - \varphi(x_0) + (k - i - j - 1)\varphi(v_e).
\end{aligned}$$

Therefore now if $\varphi(u) + \varphi(v) = \varphi(x_3)$, then $\varphi(x_0) - \varphi(v_r) = (k - i - j - 1)\varphi(v_e)$. But since $x_0 \neq v_r$, we have $(k - i - j - 1) \neq 0$. Hence if $(k - i - j - 1) < 0$, then we obtain $\varphi(v_r) < \varphi(v_e)$, which is a contradiction with the Ellingham labelling of the last caterpillar C_m in type 1. If $(k - i - j - 1) > 0$, then we obtain $\varphi(x_0) \geq \varphi(v_r) + \varphi(v_e) = \varphi(v_{n-3-k+1})$, which is a contradiction with the supposition of $|t(T_3)| = 3$.

(2) If T is decomposed into type 2, namely $T = C_1 \cup C_2 \cup \cdots \cup C_m \cup S$, then

when $\varphi(u) \leq \varphi(v_{n-3-k})$,

$$\begin{aligned}\varphi(u)+\varphi(v) &\leq \varphi(v_{n-3-k})+\varphi(v_{n-3}) \\ &= \varphi(v_{n-3-k})+\varphi(v_{n-3-k+1})+(k-1) \\ &= \varphi(x_1)+(k-1) < \varphi(x_1)+\varphi(x_0)=\varphi(x_3);\end{aligned}$$

when $\varphi(u) > \varphi(v_{n-3-k})$,

$$\begin{aligned}\varphi(u)+\varphi(v) &= \varphi(v_{n-3-j})+\varphi(v_{n-3-i}) \\ &= 2\varphi(v_{n-3})-(i+j) \\ &= (\varphi(v_{n-3-k+1})+(k-1))+(\varphi(x_1)-\varphi(v_e))-(i+j) \\ &= \varphi(v_e)+\varphi(v_r)+\varphi(x_3)-\varphi(x_0)-\varphi(v_e)+(k-i-j-1) \\ &= \varphi(v_r)+\varphi(x_3)-\varphi(x_0)+(k-i-j-1).\end{aligned}$$

$(i<j; i,j=0,1,2,\cdots,k-1)$. Therefore now if $\varphi(u)+\varphi(v)=\varphi(x_3)$, then we have that $\varphi(x_0)=\varphi(v_r)+(k-i-j-1)$, that is $|\varphi(x_0)-\varphi(v_r)| = |k-i-j-1| \leq k-2$, which is a contradiction with the Ellingham labelling of the shrub S in type 2.

Summarizing the above mentions in Case 2, we obtain that the labelling φ satisfies the condition of an integral sum labelling of T_3 when $x_0 \neq v_r$.

Finally, since $\varphi(x_3)$ is the maximum label of T_3, for any $v \in V(T)$, we have that $\varphi(v)+\varphi(x_3) > \varphi(x_3)$. Hence, there is not $w \in V(T_3)$ such that $\varphi(v)+\varphi(x_3)=\varphi(w)$ for any $u,v \in V(T_3)$, $u \neq v$ and $uv \notin E(T_3)$.

Summarizing the above mentions, we can conclude that the labelling φ is an integral sum labelling of the tree T_3 with $|t(T_3)|=3$.

When $|t(T_3)|=b>3$, namely $P=x_0x_1\cdots x_b$ is a tail of T_3 , it need only to take $T=T_3\setminus\{x_b,x_{b-1},x_{b-2}\}$ and $P^*=x_{b-3}x_{b-2}x_{b-1}x_b$. And let $T_3=T\cup P^*$ and $\{x_{b-3}\}=T\cap P^*$. We choose the hanging vertex x_{b-3} to be the head of the first caterpillar C_1 of T (take the other end of the most longest path in T started from here as the tail of C_1) and then decompose T outright or partially into a sequence of caterpillars $C_1, C_2, \cdots, C_m$. Finally, using the above complete labeling algorithm, we can obtain an integral sum labeling φ of T_3 with the tail length more than 3. Thus we obtain the following result.

Theorem 1 *If T_3 is a tree with tail length at least 3, then* $\zeta(T_3)=0$.

4. Remarks

Recently, Zhibo Chen [10] proved that any tree T with all forks at least 4 apart is an integral sum graph. Although our result is not the final solution on integral sum trees, it improves the previous result and is very close to completion. We try hard to explore a method into the study of integral sum graphs in this paper. This method can connect the sum graph with the integral sum graph. That is, we extend from a sum labelling to the integral sum labelling. We believe that this method can be applied in elsewhere with similar problems, such as general graphs with tail.

References

[1] F. Harary, Graph Theory (Addison-Wesley. Reading. 1969).

[2] F. Harary, Sum graphs and difference graph, Conger. Numer. 72 (1990) 101-108.

[3] F. Harary, Sum graphs over all the integers, Discrete Math. 124 (1994) 99-105.

[4] Zhibo Chen, Harary's conjectures on integral sum graphs, Discrete Math., 160 (1996) 241-244.

[5] M. N. Ellingham, sum graphs from trees, Ars Combinatoria 35 (1993) 335-349.

[6] Wenjie He, Yufa Shen, Lixin Wang, Yanxun Chang, Qingde Kang, The integral sum number of the complete bipartite graphs $K_{r,s}$, Discrete Math., 239(2001)137-146.

[7] M. Miller, Slamin, J. Ryan and W. F. Smyth, Labelling wheels for Minimum sum number, JCMCC 28 (1998) 289-297.

[8] F. Harary, I. R. Hentzel and D. Jacobs, Digitizing sum graphs over the reals, Caribbean J. Math. Comput. Sci. 1 (1991) 1-4.

[9] N. Hartsfield and Smyth, The sum number of complete bipartite graphs, in: R.Rees ed., Graphs and matrices (Marcel Dekker, New York, 1992) 205-211.

[10] Zhibo Chen, Integral sum graphs from identification, Discrete Math., 181 (1998) 77-90.

[11] Wenjie He, Xinkai Yu, Honghai Mi, Yong Xu, Yufa Shen, Lixin Wang, The (integral) sum number of graph $K_n - E(K_r)$, Discrete Math., to appear.

[12] N. Hartsfield and Smyth, A family of sparse graphs of large sum number, Discrete Math., 141(1995)163-171.